An alternative interpretation of the Zeeman and Faraday laser

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In this paper we present a summary of the experimental observations of the polarization behavior of a Zeeman and Faraday laser and discuss difficulties with the usual interpretation in terms of two circularly polarized modes. To illustrate a single-mode view we solve analytically the problem of a laser with a weak dichroism and a weak Faraday rotation. The theory explains essentially all of the experimental observations. Equally important, we show that circularly polarized modes by themselves do not satisfy the accepted laser equations and consequently that a two-mode interpretation is not tenable. Our treatment is couched in the language of nonlinear dynamics. The transition from a fixed-point solution to a periodic solution that occurs at a critical field is identified as a saddle-point instability. In three separate appendixes, we consider: a polarization propagation picture that gives us physical insight into the behavior of a dichroic Zeeman or Faraday laser, possible extensions of our calculations to cavities with large anisotropies, and we discuss the merits of ''bare cavity'' laser models.

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INTRODUCTION

The Zeeman laser has been the object of numerous studies, both experimental and theoretical. There have also been several studies of the closely related Faraday laser. In the Zeeman laser, applying a longitudinal magnetic field to the gain medium creates a frequency-dependent Faraday rotation (circular birefringence) and a frequency-dependent differential gain between circular components of the field (circular dichroism). For the Faraday laser, applying a magnetic field to some intracavity element creates a Faraday rotation that is usually frequency independent. In both cases there is always some residual linear dichroism and linear birefringence due to other elements inside the cavity. Thus the two types of lasers have much in common and not surprisingly show similar behavior. Examination of the literature shows that the observations have been interpreted either in terms of single-mode or two-mode operation, with a heavy preference for the latter. Surprisingly, for both approaches, the theoretical formulation of the problem begins with what are the same equations computationally, but with a nomenclature that makes them appear to be different. It is the intent of this paper to (i) identify the point of divergence of the two interpretations, (ii) show that a single mode approach is tenable but that a two mode approach is not, (iii) present a mathematical and physical picture of the preferred interpretation for the particular case of a dichroic Faraday laser, and finally, (iv) show that a single-mode view is compatible with the general understanding of the vector nature of dual polarization lasers in the absence of a magnetic field.

This paper is not intended as a critical review of the very many papers on the polarization states of lasers. References [1–56] are selected experimental and theoretical papers that we feel are most relevant to the central question of the description of the polarization modes of Zeeman or Faraday gas lasers. We have restricted the treatment to lasers that can support only a single longitudinal and a single transverse mode but perhaps one or more polarization modes. As we limit ourselves to realistic models of real lasers, we rule out ring lasers because the number of polarization modes must presumably double if forward and backward waves are separable. We have also chosen to limit the presentation to ''static'' laser systems, i.e., to lasers where the various control parameters are not made time dependent. Furthermore, the anisotropies of the cavity optics and of the saturated gain medium are assumed to be weak. Our purpose is to provide an overview and a single-mode interpretation of the Zeeman and Faraday laser. To date a single-mode interpretation has not been accepted by the general laser community. A large part of this paper is concerned with convincing the reader that such an expression best describes the physics of a Zeeman or Faraday laser.

To begin, we specify what we mean by a single- versus two-mode model or interpretation. We are not concerned with spatial modes, longitudinal or transverse, but rather we are concerned only with the polarization aspect of laser modes. The modes we have in mind are modes of the entire laser system, not simply modes of the bare cavity. The well-known approach to establish the mathematical model of a laser is, first, to imagine a specific form of the field, second to solve the density matrix to determine the dipole moment per unit volume of the gain medium, and finally to insert the expression for the polarization into Maxwell's equation and demand that the predicted field is consistent with the field originally chosen. In this light, a *single* polarization mode means that the initial choice of field is characterized by just

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two complex components of the electric field, with slowly varying amplitudes and phases. The intensity of a mode is proportional to the sum of the squares of the amplitudes, the frequency is determined from the average or mean phase of the two components, while the polarization information is contained in the relative amplitude and difference in phase of the two components. The stable solutions to the self-consistent set of equations are the possible single modes of the model system. There may be more than one single-mode solution to the model equations. It is a separate question to ask whether or not the presence of one mode will suppress the operation of another, i.e., to ask if one can expect the laser in the laboratory to operate on a single mode. If mode suppression does *not* occur then one should reformulate the theory in terms of two (or more) modes for the initial field with two (or more) Fourier frequency components. In the multimode case, whether the modes are or are not independent is also a separate question. However, to be a true multimode case, to each mode there must also be a corresponding solution to the appropriate single-mode formulation of the problem. As a linearly polarized field can always be described as a linear combination of two circular components one is led (as we have done) to distinguish a mode from the components of that mode in a given basis. It is in the sense expressed in this paragraph that we use the expressions, single or two mode, in this paper.

As most of the experimental work on Zeeman or Faraday lasers deals with the He-Ne system, we first describe a generic gas laser and summarize the key experimental results [57]. Then we present the most common two-mode interpretation of the observations. This is followed by a summary of the general theory of single-mode quasi-isotropic lasers. We have chosen to solve the specific problem of the dichroic Faraday laser, rather than the more common Zeeman laser, because the solutions are analytic and thus more amenable to physical interpretation. We show, at least qualitatively, that the theory explains the existing experimental observations and is compatible with the interpretation of quasi-isotropic lasers in the absence of a Faraday rotation. We also show that a two-mode interpretation of the dichroic Faraday laser is not tenable. This completes the main objective of the paper. In Appendix A we use a beam propagation picture that gives us deeper physical insight into the operation of the Zeeman or Faraday laser. In Appendix B we consider possible extensions of the mean-field model to lasers with large optical anisotropies. In Appendix C we discuss non-mean-field models based on ''bare cavity'' modes. The appendixes serve to round out the overview.

ZEEMAN OR FARADAY GAS LASER

Figure 1 shows a typical experimental setup. It consists of a linear cavity containing a gain tube terminated with windows, W_1 and W_2. The strength of the Faraday rotation, γ, may be controlled by varying the magnetic field, H, applied either to an intracavity element F or directly to the gain medium Z. In the first case we have a Faraday laser and in the second a Zeeman laser. Technically it is very difficult to mount the windows normal to the optical axis; nor is it desirable to do so because of complications that arise from etalonning effects between the various cavity components [47,50,58]. For tilted windows, the cavity is always slightly dichroic because of the difference in transmission through windows for s and p polarization. Tilted windows or strained optics are also birefringent. However, for most of this paper we ignore the birefringence. In order to have a controllable dichroism, as opposed to a residual and unknown dichroism, one often adds to the cavity a tiltable plate (window), W_c. The dichroism may then be calculated using the well-known Fresnel coefficients of transmission [58,59]. The cavity is sufficiently short that it supports only one longitudinal mode within the gain curve. A diaphragm (not shown) limits the oscillation to a single transverse mode. For small anisotropies the order of the anisotropic elements in the cavity is unimportant [60].

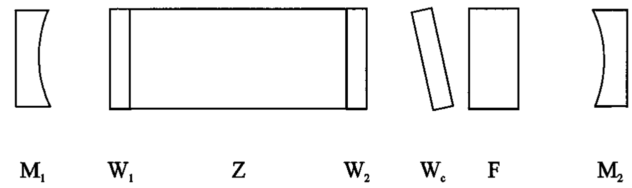

FIG. 1. Schematic drawing of a Zeeman or Faraday laser with an intracavity linear dichroism.

In such a He-Ne laser, with no Faraday rotation, $(H=0)$, and dominated by linear dichroism, one observes that the laser operates linearly polarized and aligned with the axis of minimum loss [61]. If ϕ_0 is the relative phase of the two circular components of the field, then the azimuth of the polarization ellipse is $\phi_0/2$. We define the origin $(\phi_0=0)$ as aligned with the polarization in the zero-field case, i.e., the low loss axis. This direction is taken as the x axis. For H fixed but weak, the laser continues to operate with a fixed linear polarization. However, the azimuth, while still constant in time, is rotated away from $\phi_0/2=0$, in the sense given by the Faraday rotation. The azimuth increases until a critical field, H_c, is reached, at which point the azimuth has a value of $\pi/4$. For larger values of the Faraday rotation $(H>H_c$ or equivalently, $\gamma>\gamma_c)$ the azimuth becomes periodic in time, i.e., it physically rotates. While always periodic, it is not harmonic. It becomes more harmonic as the magnetic field is increased further beyond H_c. Some examples of the above can be found in Refs. [8,15,20,36,38,55]. To date there has been only fragmentary and incomplete theoretical explanations of some of these characteristics.

The common experimental interpretation of these observations is that the laser operates in two circularly polarized modes above H_c and that the two-modes are locked together, below that field strength. We are now in a difficult position, if not with our understanding of the physics, at least with semantics [62]. A two-mode description does not join seamlessly onto the accepted zero-field understanding of such lasers. For zero magnetic field, there is excellent quantitative agreement between experiment and a single mode, vector theory of quasi-isotropic lasers [63,64]. For $H=0$, the present understanding is that there are at most two possible stable linearly polarized states (modes) of the laser. If both are stable, mode competition limits the operation of the laser to one of them.

A second difficulty with the common two-mode interpretation arises above the critical field where the laser field is not stationary. One is tempted to describe a time-dependent field as the superposition of two circularly polarized stationary modes of different frequency, as was done for the Zeeman and Faraday laser (see, for example, Refs. [1] and [36]). This view is not correct because it implies a constant beat frequency between the modes that is not seen experimentally. A way around both of these difficulties is to use the notion of a fixed point or static (vector) mode below the critical field and to use the notion of a limit cycle or a dynamic (vector) mode above the bifurcation point. Thus, at least conceptually, a single-mode theory should be sufficient to describe the Zeeman or the Faraday laser both above and below the critical field and would join seamlessly onto the accepted zero-field mono-mode theory.

A two-mode description of the Zeeman laser was suggested in the first experimental paper [1]. It is not surprising, therefore, to find a two-mode description of the polarization properties in the early theories [13,14,20–22,25–29]. To illustrate the origin of the two-mode picture and to set the scene for a single-mode interpretation, we begin with an overview of the general theory of quasi-isotropic lasers [24,65–68].

THEORY OF QUASI-ISOTROPIC LASERS

We take as given that a plane-wave, slowly varying amplitude, approximation provides a basis for the description of the properties of the gas lasers under consideration. For the He-Ne system one is also justified in adiabatically eliminating the dynamics of the gain medium. This means that one takes the steady-state solutions of the density matrix when calculating the electric-dipole moment per unit volume of the gain medium. The solution is always truncated at third order for gas lasers when polarization information must be retained. Thus, one is left simply with the equations for the electric field. In the mean-field approximation, ideal mirrors are assumed and all of the properties of the cavity and the gain medium are uniformly distributed along the axis of the cavity. This allows one to satisfy Maxwell's equations with a spatially uniform or mean field and finally leads to equations for the complex amplitude of the optical field that are functions of time alone. In a linear cavity, the mean fields are standing waves.

The time rate of change of the field has two contributions, one due to the gain medium and the other due to the cavity. For the gain medium, one must include in the calculation the spatial degeneracy, i.e., the angular momentum quantum numbers m_j of the Zeeman sublevels of the upper and lower atomic levels; it is in the m_j dependence of the matrix elements of the transition dipole that the saturated polarization properties are hidden [24,25,69]. The contribution from the cavity is established by using Jones matrices for the optical elements to write down the round-trip change in the field and then dividing by the round-trip time. The cavity contribution has the form

$$(\partial E/\partial t)_c=(M_{\mathrm{rt}}-U)E=(M_{\mathrm{ca}}-\mathscr{L}U)E \tag{1}$$

where in a circular basis, E is a 1×2 column matrix, $\left[\begin{smallmatrix} E_+ \\ E_- \end{smallmatrix}\right]$, or Jones vector, $\mathscr{L}$ is the isotropic loss, M_{rt} is the round-trip cavity matrix and U is the unit matrix. Time is measured in units of round-trip time, $2L/c$. The cavity matrix, M_{ca}, is a 2×2 matrix, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, containing the cavity anisotropies. When one adds the two rates, one due to the cavity, the other due to the gain medium, the resulting dynamical equations are

$$\dot{E}_+=[\alpha_+-\mathscr{L}-\beta_+I_+-\theta_+I_-]E_+ +[aE_+ +bE_-], \tag{2a}$$

$$\dot{E}_-=[\alpha_--\mathscr{L}-\beta_-I_--\theta_-I_+]E_- +[cE_+ +dE_-]. \tag{2b}$$

All quantities (except $\mathscr{L}$) are complex. In Eq. (2a), α_+ is the gain, β_+I_+ the direct saturation and θ_+I_- the cross saturation for the E_+ component of the field, all in units of per round trip. Here I_+ equals $\mathscr{E}_+^2$, and I_- equals $\mathscr{E}_-^2$. In each equation, the second term in square brackets contains contributions from the cavity anisotropies. The equations are written in a form directly applicable to a Zeeman laser, where the Faraday rotation arises from the difference between the imaginary parts of α_+ and α_-. In the Faraday laser, α_+ equals α_- and the Faraday rotation is contained in the imaginary parts of a and d. While the frequency-dependent coefficients, α, β, and θ are (complicated) functions of the homogeneous and inhomogeneous widths of the gain medium [67], for the purpose of this paper the reader may simply take them as given quantities. If the field components are written in terms of an amplitude and phase, $E_+=\mathscr{E}_+(t)\exp[i\phi_+(t)]$, $E_-=\mathscr{E}_-(t)\exp[i\phi_-(t)]$, and the two equations are separated into real and imaginary parts, the resulting equations have the form

$$\dot{\mathscr{E}}_+=f_1(\mathscr{E}_+,\mathscr{E}_-,\phi_+,\phi_-),\quad \dot{\mathscr{E}}_-=f_2(\mathscr{E}_+,\mathscr{E}_-,\phi_+,\phi_-),$$

$$\dot{\phi}_+=f_3(\mathscr{E}_+,\mathscr{E}_-,\phi_+,\phi_-),\quad \dot{\phi}_-=f_4(\mathscr{E}_+,\mathscr{E}_-,\phi_+,\phi_-). \tag{3}$$

In order to appreciate the difficulties associated with the usual two-mode interpretation of Eqs. (3), it is instructive to examine how the present vector formulation ties in with Lamb's scalar theory. In the *scalar* theory there is one dynamic equation for the field amplitude, $\mathscr{E}$, and one dynamic equation for the phase, ϕ. Only $\mathscr{E}$, $\dot{\mathscr{E}}$, and $\dot{\phi}$, but not ϕ, appear in the equations. As the net gain and the saturation parameters are both frequency dependent, one had to choose a frequency in order to set up the starting equations. Consequently, one interprets the equation for $\dot{\phi}$ as the correction to the frequency and iterates the solution until the condition $\dot{\phi}=0$ is satisfied. This is how the resonance condition [16,33] that the round-trip phase shift be an integral number times 2π appears in the mean-field treatment. Clearly there is but a single frequency in the single-mode scalar theory.

It is in applying the same approach to the *vector* case that a trap is laid. For example, Fork and Sargent [14] and many others, assign each *component* of the field its own frequency, ω_+ or ω_-. The right- and left-handed components are now called the right- and left-handed *modes* and we have slipped from a single-mode to a two-mode language. Below, we will try to convince the reader that the basic equations of all of the so-called two-mode Lamb type theories, as applied to Zeeman lasers, are in reality single-mode theories. Thus this paper is not about Eqs. (2), equations which are generally agreed upon as representing a reasonable plane-wave model for low gain, quasi-isotropic lasers. The paper is about the interpretation of the equations and their solutions; it is about the physics of the problem. While the specific form chosen to express the field is definitely a question of personal choice, the trap laid is in the semantics associated with the nomenclature. On one hand, the semantics may lead one to make a questionable approximation when trying to solve the equations analytically. (See, for example, Refs. [34] and [35] and particularly [21] and [29], where it was necessary to make "convenient" approximations to arrive at a "locking" equation.) On the other hand, if the full equations are solved numerically, different semantics will lead one to different physical interpretations of the same numerical results. The advantage of our treating a dichroic Faraday laser to advance our single-mode interpretation is that we can find analytical solutions to Eqs. (2) without approximations, thus removing at least one source of ambiguity. Furthermore, the theoretical results explain, at least qualitatively, all the existing experimental data.

Above we expressed the view that a mode is the operating point of the laser system as determined by solutions of the (model) laser equations. However, to clarify the situation, let us consider some general properties of laser fields inside cavities. For a *fixed time*, one can start at an arbitrary point inside the cavity and spatially integrate Maxwell's equations over a single round trip (with appropriate boundary conditions) to find the field at the starting point. However the field must be single valued. Thus both the intensity and the polarization must be reproduced after the round trip integration and the total phase accumulated on a single round trip must be an integral number times 2π. In this fixed-time or "snapshot" view, a mode is a possible "resonance" [16,33]. This suggests that the way to develop a single-mode theory is to write the four coupled mean-field equations, implied by Eqs. (2a) and (2b), in terms of intensity, polarization, and mean phase, *but only one frequency*, ω. The intensity and polarization are determined by $\mathcal{E}_+$, $\mathcal{E}_-$ and the relative phase $\phi_0=(\phi_+-\phi_-)$ while the overall phase is defined by the mean phase $\phi=(\phi_++\phi_-)/2$. In terms of these variables we find that the equations have the structure

$$\dot{\mathcal{E}}_+=f_1(\mathcal{E}_+,\mathcal{E}_-,\phi_0),\quad \dot{\mathcal{E}}_-=f_2(\mathcal{E}_+,E_-,\phi_0),$$

$$\dot{\phi}_0=f_3(\mathcal{E}_+,\mathcal{E}_-,\phi_0),\quad \dot{\phi}=f_4(\mathcal{E}_+,\mathcal{E}_-,\phi_0),\qquad (4)$$

where the f's are new functions. What is more important, the functions are independent of the mean phase ϕ. The first three equations determine the possible polarization states and field strengths that satisfy the resonance condition in our mean-field formulation of the problem. The last equation is to be used to satisfy the round-trip phase condition. The fact that Eqs. (2a) and (2b) as found in various forms in the literature have the structure outlined is the main basis of our claim that they are, in principle, single-mode equations. Writing the equations in terms of the intensity, polarization parameters, and a mean phase is to be preferred not only from a point of view of semantics but also from the point of view of the physics involved.

Often, the stationary solution(s) to Eqs. (2a) and (2b) can be found in the same manner as those of Lamb's scalar theory. The recipe, for solving the vector case follows. First, choose a frequency and solve the first three equations with the time derivatives set to zero to find the intensity and polarization state(s), and then solve the last equation to find a correction to the frequency. Iterate the process until $\dot{\phi}$ equals zero [70]. Setting the derivative equal to zero is a recipe for finding fixed-point solutions. For limit cycle solutions, the equation for $\dot{\phi}$ is the equation giving the frequency variation. Of course, this raises a subtle point; is the general approach self-consistent considering one has calculated the gain and saturation coefficients using a fixed frequency? We address this concern below, for the specific case of the dichroic Faraday laser.

THE DICHROIC FARADAY LASER

The general formulation of a mathematical model for single-mode quasi-isotropic lasers has been given in Refs. [24,64–68]. Here we present the formulation for the specific case of a dichroic Faraday laser. A dichroic cavity has been used experimentally for both the Faraday and the Zeeman laser [38,39]. The bare cavity has also been studied for this case [30]. We use a terminology similar to that of our earlier works [60,63,64,67,68]. We will give solutions for the mode(s) of such a laser for magnetic fields above and below the critical field H_c [57].

In a circular basis, and starting with the field incident on mirror 1 in Fig. 1, the round-trip Jones matrix for the cavity is

$$M_{\rm rt}=t\begin{bmatrix}1 & \epsilon\\ \epsilon & 1\end{bmatrix}\begin{bmatrix}e^{i\gamma} & 0\\ 0 & e^{-i\gamma}\end{bmatrix}r_2\begin{bmatrix}e^{i\gamma} & 0\\ 0 & e^{-i\gamma}\end{bmatrix}t\begin{bmatrix}1 & \epsilon\\ \epsilon & 1\end{bmatrix}r_1$$
$$=T\begin{bmatrix}e^{i2\gamma}+\epsilon^2e^{-i2\gamma} & 2\epsilon\cos2\gamma\\ 2\epsilon\cos2\gamma & e^{-i2\gamma}+\epsilon^2e^{i2\gamma}\end{bmatrix}.\qquad (5)$$

The mean transmission, t, and the transmission anisotropy, ϵ, are given by, $t=(t_x+t_y)/2$ and $\epsilon=(t_x-t_y)/(t_x+t_y)$, respectively. Here t_x and t_y are the transmissions of the Cartesian-field amplitudes through the tilted window. The single pass Faraday rotation is γ. The factor $T=t^2r_1r_2$ is the isotropic round trip "transmission" where r_1 and r_2 are the reflectance of the mirrors. From Eq. (1) we find $M_{\rm ca}$ equals

$$T\begin{bmatrix}i(1-\epsilon^2)\sin2\gamma & 2\epsilon\cos2\gamma\\ 2\epsilon\cos2\gamma & -i(1-\epsilon^2)\sin2\gamma\end{bmatrix},$$

and $\mathcal{L}$ equal to $1-T\cos2\gamma(1+\epsilon^2)$. In the following we will assume that the anisotropies are small, i.e., the laser is quasi-isotropic, so that $\epsilon^2\ll1$, $\cos2\gamma\simeq1$, $\sin2\gamma\simeq2\gamma$, and $\mathcal{L}\simeq1-T$. When written out in terms of their real and imaginary parts, Eqs. (2a) and (2b) become

$$\frac{d\mathcal{E}_+}{dt}=[(\alpha^r-\mathcal{L})-\beta^rI_+-\theta^rI_-]\mathcal{E}_++[2T\epsilon\cos\phi_0]\mathcal{E}_-,\qquad (6a)$$

$$\frac{d\mathscr{E}_-}{dt}=[(\alpha^r-\mathscr{L})-\beta^r I_- -\theta^r I_+]\mathscr{E}_- +[2T\epsilon\,\cos\phi_0]\mathscr{E}_+, \tag{6b}$$

$$\frac{d\phi_0}{dt}=4T\gamma-(\beta^i-\theta^i)(I_+-I_-)-2T\epsilon\,\sin\phi_0[(\mathscr{E}_+/\mathscr{E}_-)+(\mathscr{E}_-/\mathscr{E}_+)], \tag{6c}$$

$$2d\phi/dt=2\alpha^i-(\beta^i+\theta^i)(I_++I_-)+2T\epsilon\,\sin\phi_0[(\mathscr{E}_+/\mathscr{E}_-)-(\mathscr{E}_-/\mathscr{E}_+)]. \tag{6d}$$

To simplify the notation, in the following we absorb $\mathscr{L}$ into the isotropic gain, so that α^r is to be taken as the isotropic net gain. In contrast to the equations given earlier, the gain $(\alpha=\alpha^r+i\alpha^i)$ and saturation parameters $(\beta=\beta^r+i\beta^i, \theta=\theta^r+i\theta^i)$ are the same for the right- and left-handed circular components of the field.

The reader may easily manipulate Eqs. (6a) and (6b) to show that the equation for $S_z=(\mathscr{E}_+^2-\mathscr{E}_-^2)$, the Stokes parameter which describes the ellipticity, is given by

$$\frac{dS_z}{dt}=2[\alpha_r-\beta^r S_0]S_z, \tag{7}$$

where the total intensity, S_0, equals $(\mathscr{E}_+^2+\mathscr{E}_-^2)=(I_++I_-)$. In the neighborhood of linear polarization $(\mathscr{E}_+=\mathscr{E}_-)$ either Eq. (6a) or (6b) yields as a stationary solution

$$S_0=2[\alpha^r+2T\epsilon\,\cos\phi_0]/[\beta^r+\theta^r]. \tag{8}$$

Inserting Eq. (8) into Eq. (7) shows that S_z will decay to zero if β^r is greater than $\theta^r[\alpha^r/(\alpha^r+4T\epsilon\cos\phi_0)]\simeq\theta^r$. Thus we have verified the important and well-known result that a linearly polarized field is always a stable solution to the equations for this type of laser [57,74]. Having gained this physical insight we can now reduce the number of equations from four to three, one for the intensity, one for the orientation of the linearly polarized light, and one for the frequency or evolution of the mean phase. With a little manipulation the final dynamical equations can be written as

$$\frac{dS_0}{dt}=S_0[2\alpha^r+4\epsilon\,\cos\phi_0-(\beta^r+\theta^r)S_0], \tag{9a}$$

$$\frac{d\phi_0}{dt}=4\gamma-4\epsilon\,\sin\phi_0, \tag{9b}$$

$$\frac{2d\phi}{dt}=2\alpha^i-(\beta^i+\theta^i)S_0, \tag{9c}$$

where for compactness, we have defined a new ϵ and γ equal to the old value of ϵ and γ, multiplied by T. Equation (9b) is the Ricati equation, while Eq. (9a) is sometimes referred to as the Adler equation. We now see the advantage of choosing a dichroic Faraday laser as the example for illustrating our single-mode approach, namely, the equation for ϕ_0, Eq. (9b), giving the orientation of the linearly polarized field, may be solved analytically [75,76].

For $\gamma<\epsilon$, the solution can be written

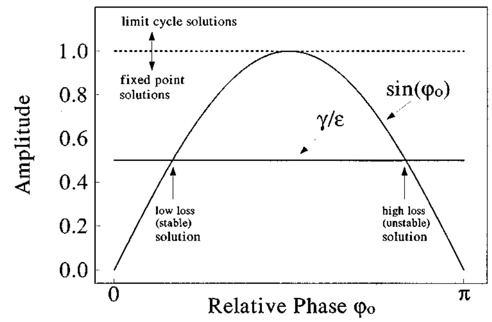

FIG. 2. A diagram illustrating, for $\gamma/\epsilon<1$, the two possible solutions to the equation, $\sin\phi_0=\gamma/\epsilon$. Limit cycle solutions exist for $\gamma/\epsilon>1$.

$$\tan(\phi_0/2)=[\epsilon(1-e^{4ut})+u(1+e^{4ut})]/\gamma(1-e^{4ut}) \tag{10}$$

where $u=+(\epsilon^2-\gamma^2)^{1/2}$ and we have arbitrarily defined the time origin such that $\phi_0/2=\pi/2$ at $t=0$. The steady-state solution $(t\rightarrow\infty)$ is

$$\tan(\phi_0/2)=[\epsilon-(\epsilon^2-\gamma^2)^{1/2}]/\gamma=(\epsilon-u)/\gamma\equiv\gamma/(\epsilon+u). \tag{11}$$

It is not difficult to complete the solution by solving Eqs. (9a) and (9c) at least numerically if not analytically. Thus we have found the fixed point by letting the dynamic equation, Eq. (9b), evolve from an arbitrary value of ϕ_0. The unstable solution, $\tan(\phi_0/2)=(\epsilon+u)/\gamma$, can be found by letting time go to $-\infty$ in Eq. (10).

As stated earlier, there is another (standard) approach to solve for the possible stationary states (ss) of Eq. (9b); it is to set $d\phi_0/dt=0$. This yields $\sin(\phi_0)_{\mathrm{ss}}=\gamma/\epsilon$ [77]. Figure 2 shows that there are two solutions to the equation, which for γ positive, lie between $\phi_0=0$ and π. Having found two stationary solutions the next step is to determine their stability. From Eq. (9b), the dynamical equation for a small perturbation, $\delta\phi_0$, about the stationary point is

$$\begin{aligned} d(\delta\phi_0)/dt&=-4[\epsilon\cos\phi_0]_{\mathrm{ss}}(\delta\phi_0)\\ &=\pm4\epsilon[1-(\gamma/\epsilon)^2]^{1/2}(\delta\phi_0)\\ &=\pm4u(\delta\phi_0), \end{aligned} \tag{12}$$

which has the obvious solutions $\delta\phi_0(t)=\delta\phi_0(0)e^{\pm4ut}$. The stable solution, the one with the minus sign, corresponds to $(\phi_0)_{\mathrm{ss}}$ between 0 and $\pi/2$. In this second method of solution we obtain information about fluctuations in the azimuth, $\phi_0/2$; they decay to or grow away from the steady-state values without oscillations. Equations (9) govern the dynamics of solutions that are restricted to the equator of the Poincaré sphere, i.e., to linearly polarized light. We have chosen this method of presentation in order to get at the underlying physics. Nevertheless, a complete linear stability analysis, starting from Eqs. (6), leads to the same conclusions provided we are dealing with quasi-isotropic lasers.

In the last paragraph we showed that there is a second fixed-point solution, which is unstable. As γ approaches ϵ, the stable and unstable solutions both approach $\phi_0=\pi/2$

(orientation equal to $\pi/4$). The stable and unstable branches "collide" when $\gamma=\epsilon$. This aspect of the stationary states is contained in Fig. 2. By now, it is clear that we are on the familiar ground of coupled nonlinear equations. There the usual picture of an instability is a "collision" between stable and unstable stationary states. A mathematical collision means there is a degeneracy in the solutions; a collision between a stable and unstable state results in a channel opening up whereby the system can escape from the stable state. In the linear stability analysis of the two fixed-point solutions we found the stability exponents $\pm(\epsilon^2-\gamma^2)^{1/2}$. As the exponent goes from positive to negative through zero at the critical field H_c, we conclude that the discontinuity is a saddle-node bifurcation. Turning Fig. 2 on its side (which then becomes a plot of ϕ_0 versus the control parameter γ/ϵ) shows that the unstable and stable branches collide at a turning point.

Above we used two approaches to find the fixed-point solution for $\gamma<\epsilon$. There are no fixed-point solutions when the Faraday rotation, as measured by γ, is larger than the dichroism, as measured by ϵ. Figure 2 also captures this aspect of the problem. Setting $\nu=(\gamma^2-\epsilon^2)^{1/2}$, with $\gamma>\epsilon$, we find for the time-dependent (limit cycle) solution [75]

$$\tan[\phi_0/2]=[\nu\tan(2\nu t)-\epsilon]/\gamma \tag{13}$$

where in this case we have set $\tan(\phi_0/2)=-\epsilon/\gamma$ at $t=0$. The azimuth is periodic, generally not harmonic, and has a period $\pi(\gamma^2-\epsilon^2)^{-1/2}$. If ϵ is small relative to γ, then $\tan[\phi_0/2]\simeq\tan[2\gamma t]$ and the orientation $(\phi_0/2)$ of the linearly polarized light, now rotates with a constant frequency equal to 2γ rad. per roundtrip time. In this large γ limit, one could have derived the result directly from Eq. (9b).

Substituting the solution for ϕ_0, Eq. (13), into Eq. (9a) allows one to determine the periodic variation in the intensity S_0. It is clear from Eq. (9a) that the modulation of the intensity will be small for quasi-isotropic lasers, i.e., when the dichroism ϵ is small compared to the net gain. Even without integration, Eq. (9a) tells us that the maximum and minimum of the intensity (which occur for $dS_0/dt=0$) coincide with the linearly polarized light parallel to the axis of minimum and maximum loss, respectively. We conclude that the laser suffers weak amplitude modulation, and consequently, during the limit cycle, that the tip of the Stokes vector does not lie on a perfect circle, the equator of the Poincaré sphere.

In addition to amplitude modulation, the laser also suffers weak frequency modulation. In the case of the fixed-point solution Eq. (9c) self-consistently determines the stationary frequency of the solution. In the case of a limit cycle solution, since the intensity is periodic, the frequency of the mode is also periodic for a fixed laser length. At line center, α^i,β^i, and θ^i are all equal to zero and the frequency is stationary. Away from line center the modulation is small, of the order of ϵ times the perturbation of the frequency associated with saturation of the gain medium. Since the laser is only very weakly modulated in frequency, it is reasonable to evaluate the gain and saturation parameters (α, β, and θ) at a fixed frequency, as is done in the calculations.

To complete the solution of the modes of the dichroic Faraday laser, we have also carried out a stability analysis of the periodic solution. We find that perturbations of the intensity and ellipticity of the light decay monotonically to zero provided the laser is quasi-isotropic. The Floquet exponent for the relative phase, ϕ_0, is zero indicating neutral stability in this variable. Of course this is expected on physical grounds, as a displacement in position on the equator of the Poincaré "sphere" is simply a displacement of the time origin for the limit cycle we have found. Thus, the limit cycle solutions are physically stable against small perturbations.

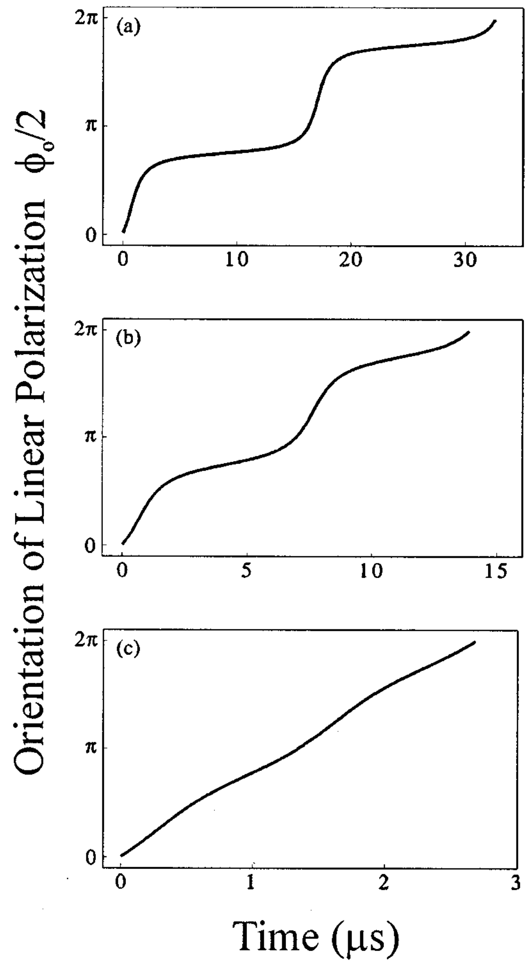

FIG. 3. Calculated azimuth of the linearly polarized field for (a) $\gamma=1.05\gamma_c$, (b) $\gamma=1.25\gamma_c$, and (c) $\gamma=4\gamma_c$, plotted as a function of time. A round-trip time of 3.3×10^{-9} s was assumed.

At this point we have all the basic ingredients to explain the experimental observations, (i) the light is linearly polarized, (ii) the orientation is along the axis of minimum dichroism for zero Faraday rotation, (iii) the orientation changes, and reaches a maximum value of $\phi_0/2=\pi/4$ at a critical field defined by $\gamma_c=\epsilon$, and (iv) the orientation is periodic for $H>H_0$ $(\gamma>\gamma_c=\epsilon)$, and becomes harmonic for larger fields. At all times the theory has been interpreted in terms of a single mode. We now present some calculations, using our single-mode model, which may be compared with experimental results found in the literature.

Figures 3(a)–3(c) show theoretical plots of the orientation of the linearly polarized mode as a function of time for three values of the Faraday rotation, $\gamma=1.05\gamma_c$, $\gamma=1.25\gamma_c$, and $\gamma=4\gamma_c$. The dichroism, or equivalently γ_c, was set at 10^{-3}. All the other parameters were chosen consistent with those of a He-Ne laser operating at 3.39 μm, near line center

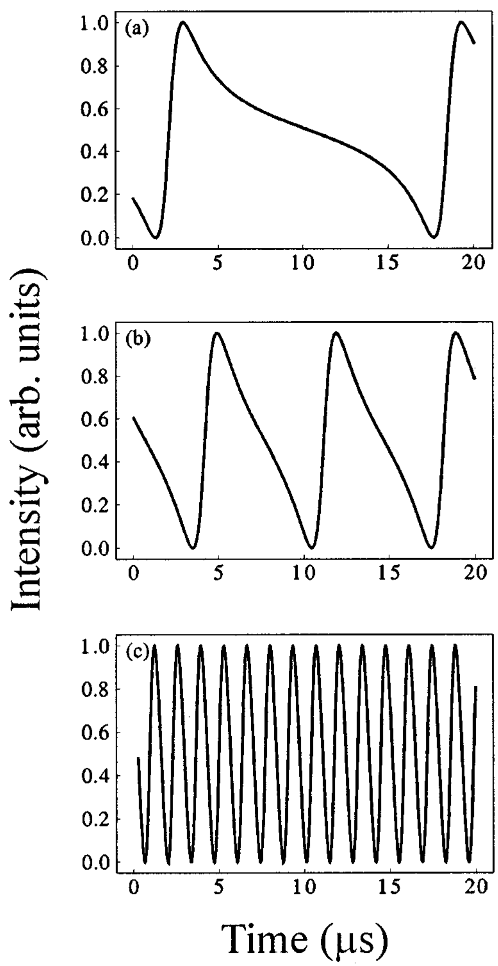

FIG. 4. Calculated intensity observed through a polarizer oriented along the preferred polarization direction in zero magnetic field. Same conditions as in Fig. 3.

[63]. The figure shows the evolution of the orientation from simply periodic to almost purely harmonic with increasing $\gamma/\gamma_c = \gamma/\epsilon$. We are not aware of any direct measurements of $\phi_0/2$ as a function of time. What has been observed is the intensity measured through a linear polarizer. As the intensity, S_0, is nearly constant the intensity transmitted by a polarizer aligned with the x axis ($\phi_0/2=0$) varies directly as $\cos^2(\phi_0/2)$. Figures 4(a)–4(c) show the results, calculated under the same set of conditions as for Fig. 3. While we cannot make a quantitative comparison with the existing data, the computed curves are strikingly similar to those shown in Refs. [3 (Fig. 6), 20 (Fig. 10a), 50 (Fig. 3)]. We have also computed the variation in intensity for a polarizer oriented at 0, 45, 90, and 135° with respect to the orientation of the polarization at zero magnetic field, all for $\gamma = 1.05\gamma_c$. These are shown in Figs. 5(a)–5(d). The computed curves are very similar to the experimental results reported in Ref. [20 (Fig. 10b)] although one must interchange the "0°" and "90°" traces to bring the results, as reported, into agreement with our calculations. We have redone the experiment of Culshaw and Kannelaud. Our results are in agreement with the theory presented here and point to a printing error in Fig. 10b of [20].

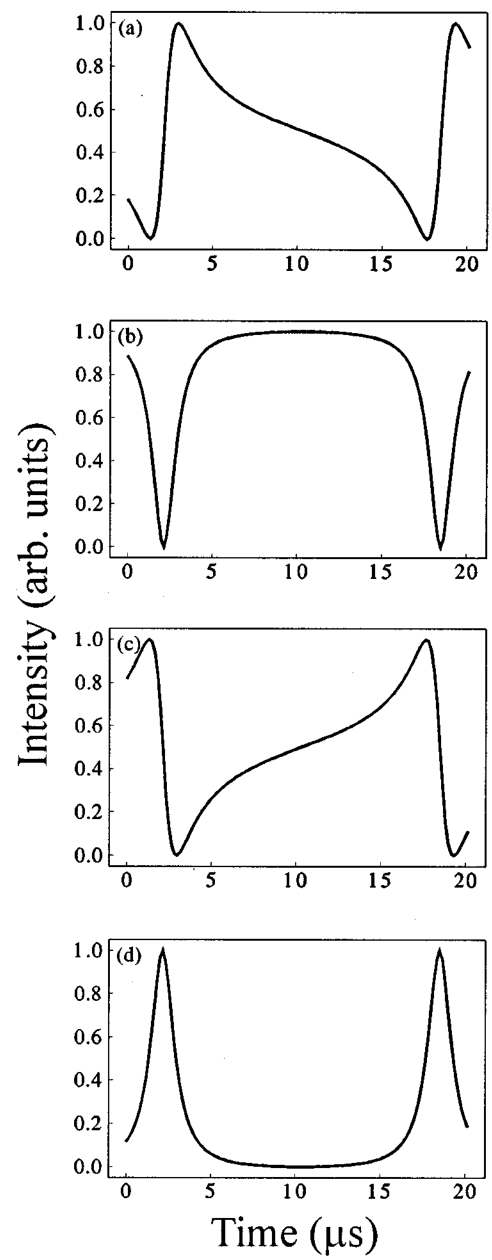

FIG. 5. Calculated intensity for $\gamma = 1.05\gamma_c$, for a polarizer oriented (a) as in Fig. 4, (b) at 45° with respect to orientation (a), (c) at 90° with respect orientation (a), (d) at 135° with respect to orientation (a). See text for discussion.

Figures 6(a)–6(c) show the computed homodyne spectra, or Fourier analysis of the intensity variation, $\cos^2(\phi_0/2)$, for the same three cases as for Fig. 3. Again we cannot make a quantitative comparison with existing experimental results. Nevertheless Fig. 6 bares a strong resemblance to experimental results given in Refs. [1 (Fig. 1), 20 (Fig. 11), 36 (Fig. 6)]. Note, in the last reference, many frequency components were observed without a polarizer in front of the detector. The experiment involved a laser with Brewster angle windows and consequently there existed considerable amplitude modulation (AM) as the linearly polarized light rotated. As mentioned above one must solve Eqs. (9a) and (9c) as the laser is both AM and FM modulated in this case.

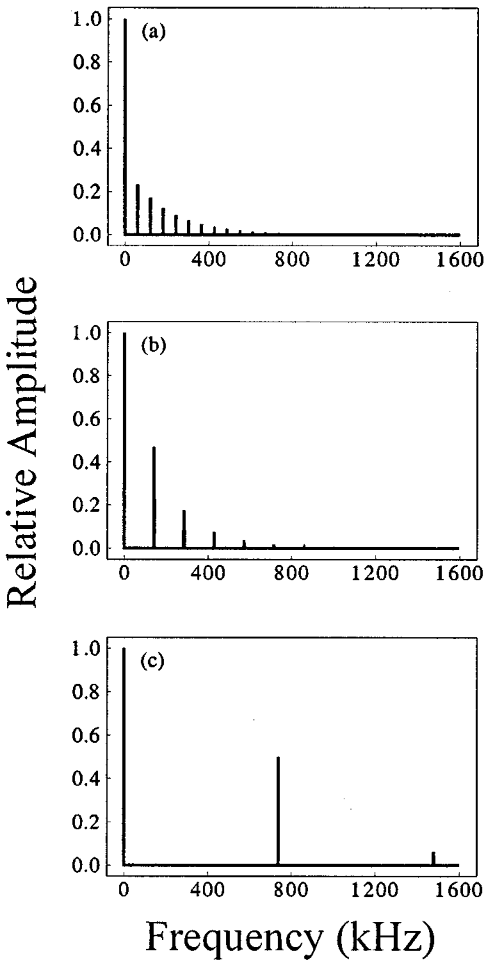

FIG. 6. Amplitude of the Fourier components of light transmitted by a polarizer, as calculated from Fig. 4.

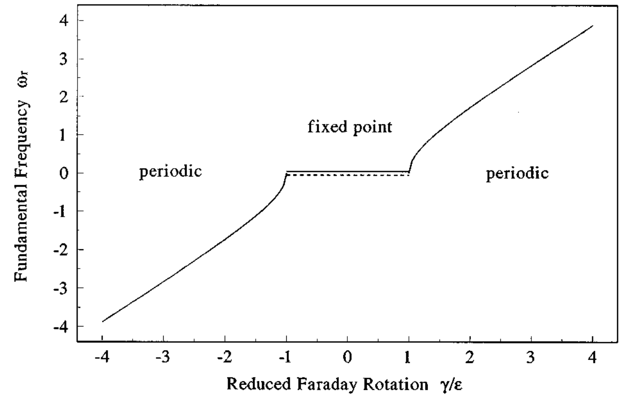

FIG. 7. Normalized fundamental frequency of rotation, ω_r, as a function of the reduced Faraday rotation, γ/ϵ. The stable periodic and fixed-point curves are shown as solid lines. The dashed line shows the fixed-point unstable solution for the dichroic Faraday laser. The two fixed-point solutions are shown as slightly separated. While really overlapping in this figure, they are separated by their values of ϕ_0.

In general, the homodyne spectrum shows that there are not just two frequencies present. In the literature the multiplicity of frequencies has been dismissed as indicating that "other weak (unidentified) modes" were running and not relevant to the general behavior of the Zeeman laser. Here they are relevant and are a direct consequence of the periodic but not harmonic motion. They provide strong support for our treatment of the Faraday-Zeeman laser.

The fundamental frequency of rotation ω_f, or inverse of the period has been measured experimentally. Let us define the reduced frequency as $\omega_r = \omega_f/2\epsilon$ a quantity which is given by $\omega_r = [(\gamma/\epsilon)^2 - 1]^{1/2}$ for $\gamma > \epsilon$. It is zero for $\gamma < \epsilon$. Figure 6 is a plot of ω_r versus γ/ϵ for γ (or the magnetic field) both positive and negative. The overlap between the stable (solid line) and unstable (dashed line) fixed point solutions is fictitious. The solutions have different values of ϕ_0, as Fig. 2 has stressed. The solid curves are the ones expected in any measurement. Experimental evidence confirming the predicted dependence of the frequency of rotation on the strength of the Faraday effect can be found in Refs. [3 (Fig. 7), 6 (Fig. 1), 11 (Fig. 2), 36 (Fig. 5), 51 (Fig. 1), 55 (Fig. 2)].

We now turn to a general discussion. Equation (9b) has played a pivotal role in developing our single-mode description of the Faraday laser. But Eq. (9b) has appeared at several places in the literature where it was used to explain the "locking" and "unlocking" in the usual two-mode approach. Once again, this raises the central question, if this paper is only concerned with a question of semantics, i.e., is a two-mode description also tenable. Essentially we have shown that a single-mode interpretation of the model equations is possible. The strongest argument against a two-mode description applies above the critical field. Since either approach starts with the same basic equations, the question is "Can the two circular components be interpreted as two modes?" The answer is no. A single circularly polarized field does not satisfy the equations. The reader may easily verify this claim by examining Eqs. (2a) and (2b) with say, $E_+ \neq 0$, $E_- = 0$, and $c \neq 0$. One quickly concludes that both components must be present to satisfy the basic equations. They produce a periodic solution above the critical field and a fixed-point solution below the critical field. If both components of the field must be present, then it is a single mode, a single *vector* mode.

In place of the traditional locking of two oscillators, the view developed here is one of "clamping." When the saturated gain medium shows a preference for linear polarization and there is no Faraday rotation it is the linear dichroism that fixes or clamps the orientation of the polarization. If only a Faraday rotation were present, the laser would still operate linearly polarized but it would rotate 2γ per round trip. Consequently, in a laser with a linear dichroism, as the magnetic field is increased from zero, the Faraday rotation will eventually break the clamping of the light field to the dichroism and enforce a clamping to the gyro-optic properties of the cavity. This view is supported by the polarization propagation picture given in Appendix A.

It is probably a coincidence that the clamping of the polarization in the Zeeman or Faraday laser and the traditional description of the locking of two coupled oscillators are described by the same mathematical equation, here Eq. (9b). The consequence of this is that the behavior of the fundamental frequency, as depicted by the solid curves in Fig. 7, is the same in both cases. That one mathematical description can describe many different physical problems is the strength of theoretical physics. However, to reverse the thought process and conclude that the equivalence of the mathematics means the physical situation is the same, is false.

The single-mode interpretation that we have presented contains none of the problems we have identified with the usual two-mode view. The understanding of the complicated behavior of the Zeeman or Faraday laser is "demystified" when presented in terms of the general properties of nonlinear systems. Physics has changed in the 30 years since the first experiments on Zeeman lasers. Fixed points, limit cycles, bifurcations, etc., simply were not even part of the vocabulary at that time. That is not the case now. We would argue that not only have we simplified both the mathematical treatment and the interpretation of the experimental results, we have also deepened our understanding of the Zeeman or Faraday laser by relating their behavior to general properties of nonlinear multidimensional systems.

There remains the question of how closely the Zeeman laser approximates the dichroic Faraday model discussed. Near line center the Faraday rotation, due to a field applied parallel to the gain tube, is nearly frequency independent as we have taken in our model. If the experimental apparatus includes a tilted plate inside the cavity, then the dominant linear dichroism will also be frequency independent. Consequently the dichroic Faraday laser and dichroic Zeeman laser should show similar behavior.

This completes the presentation of our single-mode view of the Faraday laser. In Appendix A we present a polarization propagation picture that sheds more light on the physics of the problem. In Appendixes B and C we discuss what are essentially two non-mean-field approaches to the Zeeman or Faraday laser. They are not central to the question of a single-mode description and are included more with an eye to completing an overview of the mono-mode dichroic Zeeman or Faraday laser. We note in passing that the true two-mode case has recently been addressed by Svirina [78].

SUMMARY

In this paper we have given an overview of the properties of a dichroic Faraday or Zeeman laser than can support only a single spatial mode. We have shown that it is possible to use a single-mode, theoretical model to interpret, at least qualitatively, all of the experimental results. The theory provides a continuous description of such lasers for zero magnetic field and fields below and above the critical field. The transition is identified as a saddle-point instability. There remains the problem of looking for all of the unstable solutions [79], solutions that may be involved in new instabilities when other anisotropies, such as birefringence, are included in the calculations. There also remains the problem of looking for differences rather than similarities between the Zeeman and Faraday laser, or more generally, differences that arise from allowing for frequency dependence of the anisotropies of the gain medium or cavity elements. However, what is required first, before addressing such interesting questions, is a quantitative experimental test of the theory at the present level. Finally, we have shown that a two-mode interpretation of a dichroic Faraday laser is not tenable.

ACKNOWLEDGMENTS

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APPENDIX A: POLARIZATION PROPAGATION PICTURE

In this appendix we use a polarization propagation approach, first given by D'Yakonov and Fridrikhov [27] and subsequently exploited by Le Floch and Stéphan [39]. It is valid for lasers that show a preference for linear polarization [57,80]. The basis of the picture is best illustrated graphically, as in Fig. 8. We take the order of the cavity elements, as seen by the propagating field, as mirror one, the dichroic plate, Faraday rotator, mirror two, etc. As will become apparent, the position of the gain medium, is not important in this picture. Amplification by the gain medium, as well as the isotropic loss by the mirrors, has been omitted in the diagram. In the figure, E_1 is the linearly polarized field leaving the first mirror, E_d is the field after the dichroic plate, E_{f1} the field after the first pass through the Faraday rotator, E_{f2} the field after the second pass through the rotator, and E_1' the field returning to the first mirror after passing again through the dichroic plate. For simplicity we have assumed ideal mirrors and that the transmission of the dichroic plate is 1 and 0.5 in the x and y direction, respectively. The physical ideas built into the picture are (i) the x axis is the "low loss" axis, (ii) the Faraday element rotates the field away from the low loss axis in a sense determined by the sign of the Faraday coefficient, (iii) the dichroism reduces the y component more than the x component and thus rotates the field towards the low loss axis, (iv) the second pass through the Faraday rotator returns the final vector $E_{1'}$ to a direction that it is parallel to the original vector E_1, but shorter. Having made the ansatz that the field is linearly polarized, the only action of the gain medium, wherever it is located, is to multiply the size of one or more of the intermediate fields such as to make $E_{1'}$ the same size as E_1. This remarkably simple description greatly demystifies the behavior of the single-mode Zeeman or Faraday laser. For example,

(a) We now see why the azimuth departs from zero as the Faraday rotation is "turned on" and we understand the direction of the reorientation of the azimuth.

(b) We appreciate the competition between the rotation by the Faraday effect and the rotation created by the dichroic plate.

(c) With a second sketch one can see that there are two solutions to the problem posed by Fig. 6, one with the azimuth closer to the low loss axis and another closer to the high loss axis. The first requiring less gain to restore $E_{1'}$ to the size of E_1.

(d) We see that the high loss field E_1 and low loss field E_1 are the same for $\phi_0/2=\pi/4$, thus setting an upper limit to the reorientation of the linearly polarized light.

(e) Above the critical field we can see that there will be a

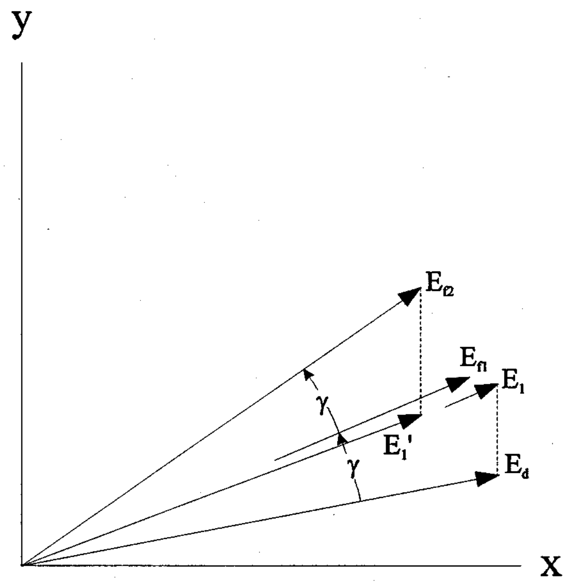

FIG. 8. Polarization diagram showing the round trip variation of the orientation and amplitude of linearly polarized light in a cavity with linear dichroism and Faraday rotation.

net rotation of the linearly polarized light per round trip and that the rotation will be nonuniform, the dichroism and Faraday rotation acting in concert in the second and fourth quadrant but in opposition in the first and third. One can derive this picture directly from Eq. (9b). Tomlinson and Fork [29] developed the same picture based directly upon an equation similar to ours. They arrived at their equation by assuming that the intensity was constant, an approximation that was justified by numerical integration of the full equations for small anisotropies.

(f) Finally we can appreciate that the rotation will become uniform at high magnetic-field strength, since the rotation per pass will be dominated by the Faraday rotation.

Le Floch and co-workers have used this polarization propagation picture and a pseudopotential approach [39,50,81] to construct two semiempirical equations for the polarization dynamics of a He-Ne laser (see page 231 in [51]). Their equations are not identical to our equations (9a) and (9b). However, for small anisotropies and low saturation they are numerically very similar. The drawback to their construct and to the polarization propagation picture given here is that one does not yet know how to generalize either to include birefringence or to treat lasers other than those that prefer linearly polarized fields. Nevertheless, the polarization propagation picture does give significant insight into the physics of a Zeeman or Faraday laser in a dichroic cavity.

APPENDIX B: EXTENSIONS TO THE QUASI-ISOTROPIC MODEL

In the main body of the text, we have used a single-mode, mean-field, vector extension of Lamb's single-mode scalar theory. It is recognized that such a mean-field theory is only valid for low gain, quasi-isotropic lasers. For large anisotropies one can always resort to a numerical evaluation of the beam propagation problem, as we have performed in a recent publication [60]. However, such a calculation is numerically intensive, gives little further physical insight, and so far has been applied only to finding the fixed-point solutions of several anisotropic He-Ne lasers. In this section we consider cases where the present mean-field theory, with or without a simple extension, may be applied to cavities containing strongly anisotropic elements.

If the gain medium is next to an isotropic or weakly anisotropic mirror, then the fields entering and leaving the other extremity of the gain medium have nearly the same polarization, provided of course that the polarization is compatible with the polarization preference of the saturated gain. Consequently, to be on resonance, the field entering and returning from the rest of the cavity must also be of nearly the same polarization, independent of the size of the anisotropies in the rest of the cavity. We can then replace the rest of the cavity by an effective quasi-isotropic mirror [58] and the entire laser must behave as quasi-isotropic as far as the polarization is concerned. Of course the fields thus determined are the fields in the vicinity of the gain medium. At the far end of the laser the fields may be completely different. The system behaves somewhat as a quasi-isotropic laser with some additional optics external to the cavity to change the polarization state of the light. This pseudo quasi-isotropic laser is the model used by many of the Russian investigators [45]. Understanding the physical basis of the pseudo quasi-isotropic model makes it easy to understand why, in the specific cavity considered, Le Floch and Stéphan find their "Lamb's vector" [37,39] located at the output mirror next to the gain medium in the cavity. The pseudo-quasi-isotropic property concerns the polarization, a property related to the relative phase of the two components of the field. The same cavity may be highly anisotropic with respect to the mean phase, i.e., the modes may have significantly different frequencies. This leads us to consider another aspect of the problem that can have a profound influence on the behavior of any dual polarization laser.

We saw above that one of the critical parameters is the ratio of β to θ. While the ratio depends upon the angular-momentum quantum numbers of the states and actually upon the relative values of certain relaxation rates, it is possible to alter the ratio in a Zeeman laser. The saturation parameters β and θ both contain resonant denominators involving an atomic resonance. In a Zeeman laser, the atomic resonance for a σ_+ transition is displaced differently from the operating frequency than is the σ_- atomic resonance. If this difference is comparable or larger than the relaxation rates of the gain medium, then the values of the β's and θ's will be altered, possibly leading to a change in preference of the gain medium from linear to circular polarization. At the same time a difference between α_+ and α_- will appear and will grow with both increasing applied longitudinal magnetic field and increasing displacement from line center. Under these conditions the Zeeman laser and Faraday laser can be expected to show different behavior. For instance, it is possible to force the He-Ne laser to operate on a pure circularly polarized mode at very high fields [82]. This case could still be handled by the general theory for quasi-isotropic lasers.

The presence of birefringence in the cavity optics can alter the behavior of both Zeeman and Faraday lasers. We see below, in Appendix C, that the bifurcation occurs at the point where the cavity is isotropic, i.e., the cavity modes are degenerate in loss and frequency. If there is a birefringent element then the frequency degeneracy will be lifted. Consequently the nature of the bifurcation may change.

As a specific example, one relevant to single-mode and a true two-mode operation, consider a large birefringence parallel to the axis of the linear dichroism. Much below the critical field we saw that the Faraday laser operated with its linear polarization almost aligned with the low loss axis. The other possible mode was unstable. With birefringence the round trip phase is now different for the two polarizations and in effect one gain curve is displaced in frequency space with respect to the other. If the birefringence is large enough then on one side of the line the high loss mode will come above threshold before the low loss mode. The high loss mode will then oscillate which is opposite to that predicted above. In addition, the cross-saturation between the two possibly stable polarization modes will be weakened, both from the change in the spatial overlap of the modes and because the difference in frequency may exceed some of the relaxation rates of the gain medium. Presumably these are the reasons that several authors [83,84] have observed two polarization modes to oscillate simultaneously even in the less complicated case of zero magnetic field. The message here is clear. When performing a stability analysis of the single-mode operation, one must consider fluctuations in the polarization and intensity, not only at the same frequency of the stable mode but also at other frequencies. For quasi-isotropic lasers, away from frequency-sensitive points like line edge, this problem does not arise, since the modes have frequency modulations within a band that is narrow compared with the inverse of the relaxation times of the gain medium. In lasers with polarized feedback, the output mirror appears to have a frequency-dependent birefringence and dichroism. In competition with the gain medium this leads to catastrophes in the absence of a magnetic field [63]. Such catastrophes are very sensitive to frequency, and thus we can anticipate more complicated behavior in the presence of a magnetic field, particularly above the critical value.

APPENDIX C: BARE CAVITY LASER MODELS

In the literature one encounters laser theories based on cavity decay modes. Bare cavity laser models assume that the role of the gain medium is simply to stop a mode from decaying. Consequently all of the polarization behavior of the laser is contained in the polarization properties of the bare cavity. Such a ‘‘cavity mode’’ laser model has been used by several authors [16,30,39] for the Zeeman or Faraday laser. While there are a few minor variations of the model, all are intrinsically, non-mean-field models of a laser. The cavity modes can be found by diagonalizing the round trip cavity matrix ($M_{\mathrm{rt}}E=\lambda_{\mathrm{rt}}E$), or they may be determined by looking for basis vectors such that Eq. (1) can be written in the form

$$\frac{dE}{dt}=(M_{\mathrm{ca}}-\mathscr{L}U)E=\lambda E. \tag{C1}$$

Equation (C1) has the obvious solutions $E(t)=E(0)e^{\lambda t}$, where E is the field, written as a Jones matrix. We can anticipate that the real part of the eigenvalues, λ^r, will be negative since all cavity modes decay with time. The bare cavity laser model then equates $-\lambda^r$ (smallest absolute value) to the gain [16,30,36,39,84]. If λ has a complex part then it is interpreted as a frequency shift. The eigenvectors are the polarization modes of the cavity. Garrett [30] has given a very clear treatment of a cavity with a Faraday rotator and a linear dichroism, in his case a Brewster window. While such a cavity is quite anisotropic, the same eigenvalue equation is found for small anisotropies. Thus his discussion may be applied equally to quasi-isotropic cavities. Garrett uses a Cartesian coordinate system with the axes aligned parallel (π) and perpendicular (σ) to the plane of incidence of the window. We can then identify his low loss axis, π, with our x axis and his σ axis with our y axis. For zero Faraday rotation, he finds, (i) that the modes are x or y polarized, (ii) that the x mode has the lowest decay rate, i.e., that it has the lowest loss, and (iii) that both x and y modes have the same frequency. For $0<\theta<\theta_c$ he finds (i) the modes remain linearly polarized but not orthogonal, (ii) the azimuth of each mode rotates towards an inclination of $\pi/4$ with respect to the plane of incidence, and (iii) the two values of λ^r (decay rates) approach each other, becoming equal at the critical field. If one adds to these properties of the bare cavity the statement that it is the mode with the smaller loss that oscillates one has partially ‘‘explained’’ the properties of the Faraday laser below the critical field. [Our Eq. (9c) predicts that the frequency of the two modes will be different below the critical field.]

In spite of the success of the bare cavity model of the dichroic Faraday laser, the argument is seriously flawed. It works for the He-Ne laser for which the direct saturation parameter β is greater than the cross saturation parameter, θ. The He-Ne laser, in saturation, prefers linearly polarized fields [57] and consequently there is no competition between the gain medium and the bare cavity. If we were to consider a gain medium such as the He-Ne laser operating on a transition for which $\beta<\theta$, then, as we saw above, a linearly polarized laser field is unstable and the laser will not operate on either cavity mode [41]. There are also complications above the critical field. Here the two cavity modes have the same loss but different frequencies. They are equally elliptically polarized, one being right handed, the other left handed. No property of the bare cavity, or of the low signal gain, can be used to determine whether one cavity mode or two cavity modes operate, nor can any relationship be determined between the two, in the two-mode case. To repeat an argument used above, the most serious flaw is the fact that a single bare cavity mode, which is elliptically polarized above H_c, is unstable according to the laser equations. Since both bare cavity modes must be present to satisfy the laser equations, we are required to consider the laser above the critical magnetic field as two cavity modes locked. This is the complete reverse of the original two-mode interpretation of the Zeeman laser. In this way we see again that a single-mode picture is the only basis independent, consistent way to describe the physics involved in the Zeeman or Faraday laser.

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$$I=(b^2-a^2)^{-1/2}\ln\left\{\frac{[a\tan(x/2)+b-(b^2-a^2)^{1/2}]}{[a\tan(x/2)+b+(b^2-a^2)^{1/2}]}\right\}$$

and

$$I=2(a^2-b^2)^{-1/2}\tan^{-1}\left\{\frac{[a\tan(x/2)+b]}{[(a^2-b^2)^{1/2}].}\right\}.$$

If we take our variables in Eq. (9b) as $4t$ and ϕ_0 then we can set $a=\gamma$, $b=-\epsilon$, and $x=\phi_0$. For $\gamma<\epsilon$ it is convenient to use the first form of I and define u as $u=(b^2-a^2)^{1/2}=(\epsilon^2-\gamma^2)^{1/2}$. For $\gamma>\epsilon$ it is convenient to use the second form of I and define ν as $\nu=(a^2-b^2)^{1/2}=(\gamma^2-\epsilon^2)^{1/2}$.

[76] Equation (9a) predicts that laser action can be suppressed if the dichroism is large and the rotation arising from the Faraday action is large enough. This has been observed experimentally. However for quasi-isotropic lasers Eq. (9a) says that there will be little modulation of the intensity of the laser. When the fact that the ellipticity is constant is added to this observation one concludes that the fastest change in the field is in the azimuth. However, Eq. (9b) sets the maximum rate of change of ϕ_0 as γ divided by the round trip time in the cavity. This is slow compared to the relaxation rates for the He-Ne system. Thus we were justified in making the adiabatic approximation in deriving Eqs. (2).

[77] For small ϵ the rotation of the azimuth can be much larger than the Faraday rotation γ. This ''amplification'' effect was exploited by Le Floch and Stéphan [39] although their equation is very slightly different from our Eq. (12), when rewritten in terms of $\sin\phi_0$.

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[80] Most authors use the terminology strong or weak coupling when discussing the relative values of β and θ. We prefer to express the physical process as a preference for circular or linearly polarized fields. The drawbacks to the terminology ''strong or weak coupling'' are (i) it is appropriate to a two-mode rather than a single-mode situation and (ii) strong coupling in a circular basis becomes weak coupling in a Cartesian basis. The expressions, linear or circular preference are not ambiguous.

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