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Unpolarized light: Classical and quantum states

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We give a general definition of unpolarized light that rests on invariance properties rather than specific values of second-order field correlations or, equivalently, the Stokes parameters. It turns out that to require invariance with respect to rotation around the propagation axis and symmetry with respect to left- and right-handed circular polarization is not sufficient to fully characterize completely unpolarized light, such as natural light. Actually, a third requirement demanding invariance with respect to phase retardation is needed. Depending on whether it is satisfied or not, we speak of type I or type II unpolarized light, and we specify the general form of both classical distribution functions and density operators consistent with the respective conditions.

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I. INTRODUCTION

Unpolarized light is usually characterized in terms of second-order correlations, with respect to the electric-field strength, or, equivalently, physical quantities such as the Stokes parameters [1]. One speaks of unpolarized light when (i) there are no correlations between the components of the (complex) electric-field strength in two orthogonal directions, $\langle E_x^* E_y\rangle = \langle E_x E_y^*\rangle = 0$, and (ii) the partial intensities are equal, $\langle E_x^* E_x\rangle = \langle E_y^* E_y\rangle$. Here the angular brackets denote a classical average. Of course, the quantum-mechanical description is quite similar: the corresponding expectation values with respect to the field operators $\hat{E}_x, \hat{E}_y$ fulfill the conditions $\langle \hat{E}_x^\dagger \hat{E}_y\rangle = \langle \hat{E}_x \hat{E}_y^\dagger\rangle = 0$ and $\langle \hat{E}_x^\dagger \hat{E}_x\rangle = \langle \hat{E}_y^\dagger \hat{E}_y\rangle$. In other words, the coherence matrix composed of the averages mentioned is a multiplum of the unit matrix. Trivially, the latter is an invariant with respect to a unitary transformation, i.e., either a rotation of the x,y basis around the propagation axis or the transition from linear to circular polarization. We feel that rotational symmetry is a general property, namely, not restricted to second-order correlations, of unpolarized light. More experimentally speaking, we will require that any two observers who use a different (rotated) linear-polarization basis, when dealing with unpolarized light, will arrive at the same results, whatever they measure. For instance, when they split an unpolarized light beam, with the help of a polarizing prism, into two linearly polarized beams, not only the mean photon numbers in those beams but also their mean square fluctuations should coincide, and this should be so for all orientations of the polarizing prism. Actually, our point of view is not shared by Karasev [2], who *defines* unpolarized light by the specific values $s_1 = s_2 = s_3 = 0$ of the Stokes parameters considered as quantum-mechanical expectation values. Not surprisingly, he finds for special examples that the fluctuations in the aforementioned experiment, in fact, depend on the orientation of the polarizing prism, which leads him to the paradoxical statement that "unpolarized light has a polarization structure, which is latent when the mean intensities are measured and detectable when the noise intensities are measured."

However, it is well known that truly unpolarized light in our sense, i.e., light without any polarization structure, actually exists, namely, in the form of natural light emitted by any thermal source, like the Sun. So our concept is physically meaningful, and we will ask what is the most general state, both classical and quantum mechanical, that fulfills the condition of rotational invariance. Actually, this requirement does not exclude (eventually partial) circular polarization. Hence an additional symmetry property, namely, invariance with respect to the interchange of left- and right-handed circular polarization, has to be demanded. In addition, it should be noted that natural light has a further characteristic property: It remains unchanged when passing a phase retarder, e.g., a quarter-wave plate [1]. Hence we will impose, in a second step, also the condition of retardation invariance on unpolarized light. It will turn out that this leads to a drastic reduction of the variety of possible states.

The paper is organized as follows. In Sec. II we present in some detail our general definition of unpolarized light and give a classical description of unpolarized light in terms of distribution functions. In Sec. III we extend this description by interpreting the classical distribution function as Glauber's P function, to quantum mechanics. Moreover, we formulate necessary conditions to be obeyed by density operators describing unpolarized light. In Sec. IV we construct, by forming averages over the Poincaré sphere or suitable parts of it, a very general form of a field density operator that is both rotationally invariant and symmetric with respect to left- and right-handed circular polarization. Requiring, in addition, retardation invariance, we find the most general form of the density operator that describes truly unpolarized light. Further we list general properties of fourth-order field correlations characteristic of unpolarized light. Finally, in Sec. V we discuss some relevant examples of unpolarized-light states, in particular, the analogs of Fock and Glauber states, and give a short summary.

II. BASIC CONSIDERATIONS

A. Definition of unpolarized light

According to what has been said in the Introduction, we require, as the first necessary condition, *all* measurable properties of unpolarized light to remain unchanged when the

x,y basis is rotated by an angle ϑ around the axis of light propagation, i.e., with respect to the transformation

$$\begin{pmatrix} E_{x'} \\ E_{y'} \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{1}$$

(We focus first on a classical description. The extension to quantum mechanics will be given later.)

Since (eventually partially) circularly polarized light is not excluded by the requirement of rotational invariance, we introduce a second (necessary) condition: The field distribution function (or the density operator in the quantum mechanical description) must be symmetric with respect to left- and right-handed circular polarization. We will call light that satisfies the two conditions mentioned type II unpolarized light. We reserve the term type I unpolarized light for such light that, in addition, is invariant with respect to phase retardation. Since only the relative phase is of physical relevance, we will write the corresponding transformation as

$$\begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix} = \begin{pmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{2}$$

We emphasize again that type I unpolarized light really exists in the form of so-called natural light, that is, light emitted by thermal sources. Our goal will be to find out the most general state of unpolarized light, of both type I and type II.

B. Classical description

Classically, we will describe a light field by a distribution function $f(E_x,E_y)$. Then, the requirements stated in Sec. II A have to be fulfilled by this function. Obviously, this is the case, when f is a function of corresponding invariants only. So our primary goal is to construct the latter. This will be done for the different invariance conditions.

(a) Rotational invariance. Certainly, the appropriate way to deal with this problem is to pass from linear to circular polarization, according to the unitary transformation

$$\begin{pmatrix} E_l \\ E_r \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{3}$$

where the subscripts l and r refer to left- and right-handed circular polarization [3]. In terms of E_l and E_r, the rotation (1) simply reads

$$E_l' = e^{-i\vartheta} E_l, \quad E_r' = e^{i\vartheta} E_r. \tag{4}$$

From Eq. (4) it becomes obvious that the rotation has the following fundamental invariants:

$$J_1 = |E_l|, \quad J_2 = |E_r|, \quad J_3 = E_l E_r. \tag{5}$$

Hence any distribution function that depends only on them, $\tilde{f}(|E_r|,|E_l|,E_r E_l)$, satisfies our first requirement.

(b) Left-right symmetry. This condition evidently imposes on $\tilde{f}$ the condition that it be symmetric in $|E_r|$ and $|E_l|$. So we arrive at the result that type II unpolarized light is generally described by a distribution function that depends symmetrically on $|E_l|$ and $|E_r|$ and, moreover, on $E_l E_r$.

(c) Retardation invariance. To discuss this requirement we have to go back to linear polarization. From Eq. (3) we get

$$|E_l|^2 = \tfrac{1}{2}(|E_x|^2 + iE_x^* E_y - iE_x E_y^* + |E_y|^2),$$

$$|E_r|^2 = \tfrac{1}{2}(|E_x|^2 - iE_x^* E_y + iE_x E_y^* + |E_y|^2),$$

$$E_l E_r = \frac{i}{2}(E_x^2 + E_y^2). \tag{6}$$

Obviously, the terms $E_x^* E_y$, $E_x E_y^*$, and E_x^2, E_y^2 are not invariant under the retardation transformation (2). It is only the sum $|E_l|^2 + |E_r|^2$ that satisfies this requirement. Hence the general state of type I unpolarized light is characterized by a distribution function that depends only on the total intensity $I = |E_l|^2 + |E_r|^2 = |E_x|^2 + |E_y|^2$. In fact, natural light is of this type, the distribution function being given by [4]

$$f(E_x,E_y) = \text{const} \times \exp\{-2I/\langle I \rangle\}. \tag{7}$$

III. QUANTUM-MECHANICAL DESCRIPTION

A. Glauber's *P* representation

It is well known that there exists a precise correspondence between the quantum-mechanical and the classical description of light in the sense that any classical distribution function $f(\alpha_x,\alpha_y)$ can be interpreted as Glauber's P function [5], where α_x and α_y are the complex field amplitudes E_x and E_y normalized such that $|\alpha_x|^2$ gives us the mean photon number in the x-polarized mode, etc. This means that the quantum-mechanical analog of a classical field characterized by a distribution $f(\alpha_x,\alpha_y)$ is a quantum field described by the density operator

$$\hat{\rho} = \int f(\alpha_x,\alpha_y) |\alpha_x,\alpha_y\rangle\langle\alpha_y,\alpha_x| d^2\alpha_x d^2\alpha_y, \tag{8}$$

where $|\alpha_x,\alpha_y\rangle$ stands for the direct product of two Glauber states $|\alpha_x\rangle_x$ and $|\alpha_y\rangle_y$ referring to two orthogonal linear polarization modes and the integration extends over the whole (complex) α_x plane as well as the α_y plane. With the help of Eq. (8) or, equivalently, the representation

$$\hat{\rho} = \int \tilde{f}(\alpha_l,\alpha_r) |\alpha_l,\alpha_r\rangle\langle\alpha_r,\alpha_l| d^2\alpha_l d^2\alpha_r \tag{9}$$

with respect to circular-polarization modes, the results obtained in Sec. II for classical fields are readily transferred to quantum mechanics. However, in this way one will not find the most general density operator for unpolarized light, since the manifold of quantum states, compared to that of classical distribution functions, is in fact greater: Glauber's P function may take negative values (which is not possible for a classical distribution function) or it may even not exist as a nonpathological function. Actually, quantum states that allow no positive definite P representation have attracted special interest, since they represent nonclassical light.

We would like to add that the arguments presented above apply not only to Glauber's P function but also to the Wigner function (in fact, to any s-parametrized quasiprobability distribution [6]). Compared to the P function, the Wigner function has the advantage that it is free from singularities, in any case. So it provides a general description of quantum states, and in order to represent unpolarized light it has to have the same mathematical structure as a classical distribution function (see Sec. II B). However, this is a necessary condition only, and it should be noticed that a Wigner function cannot be chosen as an arbitrary function, it is rather subjected to subtle mathematical restrictions. So one cannot construct, in an easy way, a general *explicit* form of Wigner functions representing unpolarized light.

B. Restrictions on the density operator

The preceding discussion motivates us to analyze the problem of appropriately describing unpolarized light in the framework of quantum-mechanical density operators. As a basis we will use Fock states with respect either to linearly polarized modes, $|n_x,n_y\rangle=|n_x\rangle_x|n_y\rangle_y$, or circularly polarized modes, $|n_l,n_r\rangle=|n_l\rangle_l|n_r\rangle_r$. To study the effect of a rotation on the density operator, we will use a Fock basis corresponding to circular-polarization modes, i.e., we expand the density operator in the form

$$\hat{\rho}=\sum_{\substack{n_l,n_r\\m_l,m_r}}\rho(n_l,n_r;m_l,m_r)|n_l,n_r\rangle\langle m_r,m_l|. \tag{10}$$

How does a field state $|n_l,n_r\rangle$ transform under a rotation? To answer this question, we use the familiar representation of Fock states in terms of photon creation operators, $\hat{a}_l^\dagger$ and $\hat{a}_r^\dagger$, respectively,

$$|n_l,n_r\rangle=\frac{(\hat{a}_l^\dagger)^{n_l}(\hat{a}_r^\dagger)^{n_r}}{\sqrt{n_l!n_r!}}|\text{vac}\rangle. \tag{11}$$

Rotation of an orthogonal basis is described by the quantum-mechanical analog of Eq. (4),

$$\hat{a}_l'=e^{-i\vartheta}\hat{a}_l,\quad \hat{a}_r'=e^{i\vartheta}\hat{a}_r. \tag{12}$$

Hence the Fock state $|n_l,n_r\rangle$ transforms as

$$|n_l,n_r\rangle\to\exp\{i(n_l-n_r)\vartheta\}|n_l,n_r\rangle. \tag{13}$$

Requiring the density operator (10) to be invariant under this transformation leads us to the first necessary condition:

$$\rho(n_l,n_r;m_l,m_r)=0\quad\text{unless}\quad n_l-n_r=m_l-m_r. \tag{14}$$

The left-right symmetry, on the other hand, requires the second necessary condition,

$$\rho(n_l,n_r;m_l,m_r)=\rho(n_r,n_l;m_r,m_l), \tag{15}$$

to be satisfied.

The conditions (14) and (15) can be used to check whether a *given* density operator describes unpolarized light that is (at least) of type II. However, they are not suitable to *construct* density operators with the desired properties, since, in addition, the requirement that the density operator is a (normalized) positive definite Hermitian operator has to be fulfilled, which is no simple task. Hence we will proceed in Sec. IV in a different way, namely, we will construct states of unpolarized light by suitably averaging over the Poincaré sphere, that is, over different polarization states.

First, let us formulate the condition of retardation invariance. To do this in a transparent manner, one has to rewrite the density operator as an expansion in terms of Fock states $|n_x,n_y\rangle$ referring to linear-polarization modes. Using the quantum-mechanical analog of the transformation (3)

$$\begin{pmatrix}\hat{a}_l\\ \hat{a}_r\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}\hat{a}_x\\ \hat{a}_y\end{pmatrix}, \tag{16}$$

one can readily express the states $|n_x,n_y\rangle$ in terms of states $|n_l,n_r\rangle$ and vice versa. Then the density operator takes the form

$$\hat{\rho}=\sum_{\substack{n_x,n_y\\m_x,m_y}}\rho(n_x,n_y;m_x,m_y)|n_x,n_y\rangle\langle m_y,m_x|, \tag{17}$$

and phase retardation is described by the transformation

$$\begin{pmatrix}\tilde{\hat{a}}_x\\ \tilde{\hat{a}}_y\end{pmatrix}=\begin{pmatrix}\exp(i\varphi/2) & 0\\ 0 & \exp(-i\varphi/2)\end{pmatrix}\begin{pmatrix}\hat{a}_x\\ \hat{a}_y\end{pmatrix}, \tag{18}$$

in accordance with Eq. (2). It follows from Eq. (18), together with the equivalent of Eq. (11) for linear-polarization Fock states, that the latter transform as

$$|n_x,n_y\rangle\to\exp\{i(n_y-n_x)\varphi/2\}|n_x,n_y\rangle. \tag{19}$$

Hence retardation invariance is ensured when

$$\rho(n_x,n_y;m_x,m_y)=0\quad\text{unless}\quad n_x-n_y=m_x-m_y. \tag{20}$$

This equation represents the third necessary condition to be fulfilled by type I unpolarized light. In the Appendix we will show that this requirement, together with the condition of rotational invariance, leads to a diagonal form of the density operator.

IV. CONSTRUCTION OF DENSITY OPERATORS

A. Type II unpolarized light

From a classical point of view, one will think of unpolarized light as a light beam that, nevertheless, has a definite, in general elliptic, polarization in any sufficiently short time interval. Of course, observed over a long time interval, the elliptic polarization will change randomly, and the same will hold true also for the phase. (To give the phase of elliptically polarized light a unique meaning, we will identify it with the phase of the x component of the electric-field vector.) This picture suggests modeling unpolarized light by an ensemble of fields that are completely, however differently, polarized and, moreover, have random phases. Formally, this means we construct the density operator by forming a suitable mixture of states corresponding to different, in general elliptic, polarizations. Since the best visualization of the variety of polarizations is provided by the Poincaré sphere, our first step will be to integrate over the latter, or eventually suitable parts of it.

The Poincaré sphere is parametrized by spherical angles Θ and ϕ limited by $0\leqslant\Theta\leqslant\pi$ and $-\pi\leqslant\phi<\pi$ [7]. In particular, the poles represent circular-, and the equator linear-polarization states. Following Glauber [7], we introduce a creation operator for photons possessing arbitrary polarization

$$\hat{a}^{\dagger}_{\Theta\phi}=e^{i\phi/2}\cos\frac{\Theta}{2}\,\hat{a}^{\dagger}_{l}+e^{-i\phi/2}\sin\frac{\Theta}{2}\,\hat{a}^{\dagger}_{r}. \tag{21}$$

The corresponding n-photon (Fock) state can then be written in the familiar form [see Eq. (11)]

$$|n\rangle_{\Theta\phi}=\frac{(\hat{a}^{\dagger}_{\Theta\phi})^{n}}{\sqrt{n!}}|\text{vac}\rangle \tag{22}$$

or, utilizing Eq. (21),

$$|n\rangle_{\Theta\phi}=\sum_{j=0}^{n}\sqrt{\binom{n}{j}}\left(\cos\frac{\Theta}{2}\right)^{j}\left(\sin\frac{\Theta}{2}\right)^{n-j}\times e^{\,i(2j-n)\phi/2}|j\rangle_{l}|n-j\rangle_{r}. \tag{23}$$

Then an arbitrary density operator for the polarization state characterized by Θ and ϕ, can be represented in the form

$$\hat{\rho}_{\Theta\phi}=\sum_{n,m}\rho_{nm}|n\rangle_{\Theta\phi}\,{}_{\Theta\phi}\langle m|=\sum_{n,m}\rho_{nm}\sum_{j=0}^{n}\sum_{k=0}^{m}\sqrt{\binom{n}{j}\binom{m}{k}}e^{i(2j-n-2k+m)\phi/2}\times\left(\cos\frac{\Theta}{2}\right)^{j+k}\left(\sin\frac{\Theta}{2}\right)^{n+m-j-k}|j\rangle_{l}|n-j\rangle_{r}\,{}_{r}\langle m-k|_{l}\langle k|. \tag{24}$$

The density matrix ρ_{nm} has to satisfy the condition that it is a (normalized) positive definite Hermitian matrix, the simplest case being a pure state with expansion coefficients c_n for which $\rho(n,m)=c_nc_m^{*}$. It should be noted that *formally* the operator (21) is not periodic with 2π, but with 4π. In fact, it follows from Eq. (21) that $\hat{a}^{\dagger}_{\Theta,\phi+2\pi}=-\hat{a}^{\dagger}_{\Theta\phi}$. *Physically*, however, this does not matter because the resulting change of the sign of $|n\rangle_{\Theta\phi}$ for odd n has no physical effect, since quite generally a state vector is defined only up to an arbitrary phase factor. To exclude mathematical artifacts we will integrate over the azimuthal angle ϕ from $-\pi/2$ to $5\pi/2$, or, equivalently, from 0 to 4π. This gives us the averaged density operator

$$\hat{\rho}_{\Theta}\equiv\frac{1}{4\pi}\int_{0}^{4\pi}\hat{\rho}_{\Theta\phi}d\phi=\sum_{n,m}\rho_{nm}\sum_{j=0}^{n}\sum_{k=0}^{m}\sqrt{\binom{n}{j}\binom{m}{k}}\delta(2j-n,2k-m)\left(\cos\frac{\Theta}{2}\right)^{j+k}\left(\sin\frac{\Theta}{2}\right)^{n+m-j-k}|j\rangle_{l}|n-j\rangle_{r}\,{}_{r}\langle m-k|_{l}\langle k|. \tag{25}$$

Due to the Kronecker symbol in Eq. (25) $\hat{\rho}_{\Theta}$ is a sum of terms that are separately invariant under the transformation (13); i.e., it is rotationally invariant, for any value of Θ.

The density operator (25) is, however, in general not symmetric with respect to left- and right-handed circular polarization. In fact, the relevant prefactors of the term $|j\rangle_{l}|n-j\rangle_{r}\,{}_{r}\langle m-k|_{l}\langle k|$ and its "image" $|j\rangle_{r}|n-j\rangle_{l}\,{}_{l}\langle m-k|_{r}\langle k|$ are $[\cos(\Theta/2)]^{j+k}[\sin(\Theta/2)]^{n+m-j-k}$ and $[\cos(\Theta/2)]^{n+m-j-k}[\sin(\Theta/2)]^{j+k}$, respectively. This means that the symmetry operation in question amounts to interchanging $\cos(\Theta/2)$ and $\sin(\Theta/2)$, for fixed Θ. It is only for the special case $\Theta=\pi/2$ that $\hat{\rho}_{\Theta}$ is symmetric. Noticing, however, that $\sin(\pi/4+\alpha)=\cos(\pi/4-\alpha)$ and $\cos(\pi/4+\alpha)=\sin(\pi/4-\alpha)$, we can readily symmetrize (25) by adding $(1/2)\hat{\rho}_{\Theta_1}$ and $(1/2)\hat{\rho}_{\Theta_2}$, where Θ_1 and Θ_2 lie symmetrically with respect to the equator, $\Theta=\pi/2$. More generally, we can construct from (25) a density operator that has the desired symmetry property, in addition to being rotationally invariant, by averaging over Θ, thereby using a (positive definite) weight function $w(\Theta)$ that is symmetric with respect to the equator, $\Theta=\pi/2$, but otherwise *arbitrary*. We thus arrive at a very general representation of type II unpolarized light by a density operator of the form

$$\hat{\rho}^{(\mathrm{II})}=\int_{0}^{\pi}d\Theta\;w(\Theta)\;\hat{\rho}_{\Theta}=\int_{0}^{\pi}d\Theta\;w(\Theta)\sum_{n,m}\rho_{nm}\sum_{j=0}^{n}\sum_{k=0}^{m}\sqrt{\binom{n}{j}\binom{m}{k}}\;\delta\,(2j-n,\;2k-m)\times\left(\cos\frac{\Theta}{2}\right)^{j+k}\left(\sin\frac{\Theta}{2}\right)^{n+m-j-k}|j\rangle_{l}\,|n-j\rangle_{r}\,{}_{r}\langle m-k|_{l}\langle k|\;. \tag{26}$$

In particular, for a state with a fixed total photon number n, Eq. (26) reduces to

$$\hat{\rho}^{(\mathrm{II})}(n)=\int_{0}^{\pi}d\Theta\;\;w(\Theta)\sum_{j=0}^{n}\binom{n}{j}\left(\cos\frac{\Theta}{2}\right)^{2j}\left(\sin\frac{\Theta}{2}\right)^{2n-2j}|j\rangle_{l}\,|n-j\rangle_{r}\,{}_{r}\langle n-j|_{l}\langle j|. \tag{27}$$

We have already indicated by the superscript II that what we describe by (26) and (27) is unpolarized light that lacks retardation invariance. This is readily checked. We would like to mention that there are additional quantum states of type II unpolarized light not covered by Eq. (26). In fact, it is obvious that the Fock states $|k\rangle_l|k\rangle_r$ ($k=1,2,\ldots$), and hence also arbitrary superpositions of them, are both rotationally invariant and symmetric with respect to left- and right-handed circular polarization, the simplest case being a pair of a left- and a right-handed circularly polarized photon. (This has been observed already in Ref. [2].) So we arrive at the interesting result that quantum optics, in contrast to classical theory, allows type II unpolarized light to be also in certain *pure states*. This is, nevertheless, not so surprising in view of the fact that even a Fock state with its intrinsic phase uncertainty can be modeled classically only by a statistical mixture. The pure states in question are readily generalized to a mixture of the form

$$\hat{\rho}^{(\mathrm{II})}=\sum_{n,m}\rho(n,m)|n\rangle_r|n\rangle_{l\,l}\langle m|_r\langle m|. \tag{28}$$

To find a general representation of type I unpolarized light, we will follow more closely the intuitive classical concept of unpolarized light mentioned above.

B. Type I unpolarized light

It appears natural that in completely unpolarized light *all* possible polarizations should be present with their correct weight. Formally, this means that the integration should be done over the *full* Poincaré sphere, the weight thus being given by the surface element $\sin\Theta d\Theta d\phi$. (Actually we will integrate *twice* over the Poincaré sphere, as has been motivated in Sec. IV A.) Since the integration over ϕ has actually been performed in the foregoing, we have only to properly integrate the previous result (25) over Θ. This amounts to evaluating integrals of the form

$$I_{st}\equiv\frac{1}{2}\int_0^\pi \sin\Theta\ d\Theta\ \cos^s\frac{\Theta}{2}\sin^t\frac{\Theta}{2}$$
$$=2\int_0^{\pi/2}\cos^{s+1}\Theta\ \sin^{t+1}\Theta\ d\Theta. \tag{29}$$

The result is well known [8]:

$$I_{st}=\frac{\Gamma(s/2+1)\Gamma(t/2+1)}{\Gamma([s+t]/2+2)}. \tag{30}$$

Hence we obtain from Eq. (25)

$$\bar{\hat{\rho}}=\sum_{n,m}\rho_{nm}\sum_{j=0}^{n}\sum_{k=0}^{m}\sqrt{\binom{n}{j}\binom{m}{k}}\,\delta(2j-n,2k-m)$$
$$\times\frac{\Gamma([j+k]/2+1)\Gamma([n+m-j-k]/2+1)}{\Gamma([n+m]/2+2)}$$
$$\times|j\rangle_l|n-j\rangle_{r\,r}\langle m-k|_l\langle k|, \tag{31}$$

where the overbar denotes (twice) the average over the whole Poincaré sphere. A special case of Eq. (31) is the unpolarized n-photon state

$$\bar{\hat{\rho}}(n)=\frac{1}{n+1}\sum_{j=0}^{n}|j\rangle_l|n-j\rangle_{r\,r}\langle n-j|_l\langle j|. \tag{32}$$

Since the corresponding matrix is a multiplum of the unit matrix, it is trivially invariant with respect to any unitary transformation, in particular to rotation and phase retardation, and, moreover, to passing from the circular- to the linear-polarization basis [see Eq. (16)]. This means that $\bar{\hat{\rho}}(n)$ can also be written as

$$\bar{\hat{\rho}}(n)=\frac{1}{n+1}\sum_{j=0}^{n}|j\rangle_x|n-j\rangle_{y\,y}\langle n-j|_x\langle j|. \tag{33}$$

As becomes obvious from Eq. (32), the density operator $\bar{\hat{\rho}}(n)$ is symmetric with respect to left- and right-handed circular polarization too. Hence it describes, in fact, a special state of type I unpolarized light. Clearly, a straightforward generalization of this result is a statistical mixture of n-photon states (32) with arbitrary weights ρ_{nn},

$$\hat{\rho}^{(\mathrm{I})}=\sum_n \rho_{nn}\,\bar{\hat{\rho}}(n). \tag{34}$$

This corresponds either to the assumption that our averaging procedure starts already from a *diagonal*, and hence *stationary*, density matrix $\rho_{nm}=\rho_{nn}\delta_{mn}$ describing the excitation of the polarization modes, or that an additional averaging procedure has been carried out that makes the off-diagonal matrix elements $\rho_{nm}(n\neq m)$ vanish, thus producing a *stationary* density operator.

A closer inspection of Eq. (31) shows that off-diagonal matrix elements ρ_{nm} are not consistent with the requirement of retardation invariance. So we arrive at the result that a general expression for the quantum state of type I unpolarized light is given by Eq. (34). We will show in the Appendix that it is actually the most general one. Moreover, it will turn out from the analysis given there that the requirements of rotational and retardation invariance actually imply the symmetry with respect to left- and right-handed circular polarization. It follows directly from the diagonality of the density operator $\hat{\rho}^{(\mathrm{I})}$, Eq. (34), that the field phase is randomly distributed. In fact, forming in Eq. (34) the trace over the subspace corresponding to y polarization yields, of course, a diagonal density matrix for the x-polarization mode. According to the well-known formula for the phase distribution of a single-mode light field,

$$w(\varphi)=\frac{1}{2\pi}\left(1+\sum_{m\neq n}\rho_{mn}\exp[i(n-m)\varphi]\right), \tag{35}$$

which is based on the phase-state concept originally introduced by London [9], this means that the phase distribution is uniform in this case. Interestingly, this is also so for type II unpolarized light, since in the above-mentioned trace operation applied to Eq. (26), the nondiagonal matrix elements are lost too, as a result of the appearance of the Kronecker symbol in Eq. (26). Hence we can state that the field phase is randomly distributed in both type I and type II unpolarized light. In fact, preference of certain phases would indicate some regularity to be present in unpolarized light, which contradicts the intuitive classical concept of such light mentioned above.

C. Field correlations

We mentioned already in Sec. I that a basic characteristic of unpolarized light is the absence of second-order field correlations related to two orthogonal linear polarization directions (or, equivalently, to left- and right-handed circular polarization), together with the equality of the intensities. However, current technology enables us to measure also higher-order correlations. Of special interest are fourth-order correlations for which different detection schemes have actually been devised [2,10]. Of course, it is a straightforward matter to evaluate any correlation function from a given density operator, Eq. (26) or (34).

In the following, we will specify the nonvanishing fourth-order correlations that occur in stationary unpolarized light fields. We will focus on classical correlations. In fact, the latter are readily translated into quantum mechanics, care has to be taken only to write the field operators in *normal order* in the corresponding expectation values.

Requiring rotational invariance, we find that, in fourth order, only the following correlation functions can be different from zero,

$$B \equiv \langle E_x^{*2} E_x^2 \rangle = \langle E_y^{*2} E_y^2 \rangle,$$

$$C \equiv \langle E_x^* E_y^* E_x E_y \rangle,$$

$$D \equiv \langle E_x^{*2} E_y^2 \rangle = \langle E_y^{*2} E_x^2 \rangle,$$

$$E \equiv \langle E_x^{*2} E_x E_y \rangle = \langle E_x^* E_y^* E_y^2 \rangle,$$

$$F \equiv \langle E_x^* E_y^* E_x^2 \rangle = \langle E_y^{*2} E_y E_x \rangle, \tag{36}$$

and, moreover, the restraints

$$B = 2C + D, \quad E = -F \tag{37}$$

must be obeyed. Exclusion of (eventually partially) circularly polarized light leads to the condition $E = F = 0$. Requiring, in addition, retardation invariance, it is immediately seen that also D must vanish. This is the criterion that distinguishes type II and type I unpolarized light. Higher-order correlation functions for both types of unpolarized light were studied in some detail in Ref. [11].

V. EXAMPLES AND SUMMARY

In this section we will give some examples of unpolarized light.

A. Type I unpolarized light

(a) Natural or thermal unpolarized light. This is of course the most prominent representative of unpolarized light, normally produced by conventional sources. It is distinguished from other forms of unpolarized light by the property that the distribution function factorizes, $f(|E_x|^2 + |E_y|^2) = g(|E_x|^2) g(|E_y|^2)$, or equivalently, $f(|E_l|^2 + |E_r|^2) = g(|E_l|^2) g(|E_r|^2)$ [see Eq. (7)], which implies a similar factorization of the correlation functions, e.g., $\langle E_x^* E_y^* E_x E_y \rangle = \langle |E_x|^2 \rangle \langle |E_y|^2 \rangle$. Moreover, it is an important feature of natural light that any *multitime* correlation function can be expressed through the basic second-order correlation $\langle E_x^*(t_1) E_x(t_2) \rangle = \langle E_y^*(t_1) E_y(t_2) \rangle$ depending only on the time difference $t_2 - t_1$ [4]. In terms of unpolarized n-photon states $\hat{\hat{\rho}}(n)$, Eqs. (32) and (33), the density operator for natural light can be written as

$$\hat{\rho}_{\text{nat}} = (1-p)^2 \sum_{n=0}^{\infty} p^n (n+1) \hat{\hat{\rho}}(n), \tag{38}$$

where $p = \exp\{-\hbar\omega / k_B T\}$ is the Boltzmann factor that can be expressed through the mean photon number as $p = \langle n \rangle / (1 + \langle n \rangle)$.

(b) Amplitude-stabilized unpolarized light. It is described by the distribution function

$$f(E_x, E_y) = \text{const} \times \delta(|E_x|^2 + |E_y|^2 - I_0), \tag{39}$$

where I_0 is the fixed total intensity.

(c) Unpolarized Glauber light. Due to the use of resonator mirrors with polarization-dependent reflectivities laser light is polarized. To a good approximation, it is represented by a Glauber state $|\alpha\rangle$ referring to a definite polarization mode. However, when a laser will be built without any polarization-dependent element, one would expect that the polarization changes in a random manner, similar to the phase in single-mode operation. This means that the polarization, characterized by a point on the Poincaré sphere, will diffuse over the latter [12]. Moreover, the phase will undergo a diffusion process, too. The result is light described by the density operator

$$\hat{\hat{\rho}}_G = \sum_n \exp(-|\alpha|^2) |\alpha|^{2n} (n!)^{-1} \hat{\hat{\rho}}(n), \tag{40}$$

where $\hat{\hat{\rho}}(n)$ is the density operator for the unpolarized n-photon state, Eq. (32) or (33). Actually, the density operator (40) is identical to that following from Eq. (39) when the distribution function is taken as Glauber's P function.

(d) Single unpolarized photons. They can be produced, to some approximation, by drastically attenuating pulses of unpolarized light emitted from a conventional source, e.g., a light-emitting diode. It is well known that an unpolarized photon is in the mixed state,

$$\hat{\hat{\rho}}(1) = \tfrac{1}{2} (|1\rangle_x |0\rangle_y \, {}_y\langle 0| {}_x\langle 1| + |0\rangle_x |1\rangle_y \, {}_y\langle 1| {}_x\langle 0|)$$

$$= \tfrac{1}{2} (|1\rangle_l |0\rangle_r \, {}_r\langle 0| {}_l\langle 1| + |0\rangle_l |1\rangle_r \, {}_r\langle 1| {}_l\langle 0|), \tag{41}$$

which is the simplest unpolarized n-photon state, Eq. (32) or (33).

B. Type II unpolarized light

A possible way to produce light of this type is to start from linearly polarized light and change the polarization direction randomly, by means of a suitable optically active element driven by a randomly varying external force, utilizing, e.g., the Faraday effect. Different forms of linearly polarized light thus give rise to different states of unpolarized light. Formally, we get the latter by integrating along the equator of the Poincaré sphere only, i.e., by specializing $w(\Theta)=\sin\Theta\,\delta(\Theta-\pi/2)$ in Eq. (26)

$$\hat{\rho}_{\rm eq}^{\rm (II)}=\sum_{n,m} 2^{-(n+m)/2}\rho_{nm}\sum_{j=0}^{n}\sum_{k=0}^{m}\sqrt{\binom{n}{j}\binom{m}{k}}\times\delta(2j-n,2k-m)|j\rangle_l|n-j\rangle_r\,{}_r\langle m-k|_l\langle k|. \quad (42)$$

In the following we list some relevant special cases.

(a) Type II n-photon state. The density operator reads [cf. also Eq. (27)]

$$\hat{\rho}_{\rm eq}^{\rm (II)}(n)=2^{-n}\sum_{j=0}^{n}\binom{n}{j}|j\rangle_l|n-j\rangle_r\,{}_r\langle n-j|_l\langle j|. \quad (43)$$

(b) Type II Glauber light. Here we have to insert $\rho_{nm}=\exp(-|\alpha|^2)\alpha^n\alpha^{*m}(n!m!)^{-1/2}$ in Eq. (42). Classically, one can describe this light by an ensemble of linearly polarized fields with fixed amplitude E_0 and random polarization direction. This means that we can write

$$E_x=E_0\cos\varphi, \quad E_y=E_0\sin\varphi, \quad (44)$$

where φ is distributed over the range $0,\ldots,2\pi$ with equal weight. From this definition field correlations are readily evaluated.

(c) Two-mode squeezed vacuum state. While in a two–mode squeezed vacuum state the two modes are normally linearly polarized, one can readily depolarize such a field by converting the linear polarization states into left- and right-handed circular ones, respectively. The result is a special example of the above-mentioned type II unpolarized *pure* states. It is given by [13]

$$|\psi\rangle=\sum_n \frac{1}{\cosh r}\exp(2i\chi n)(-\tanh r)^n|n\rangle_r|n\rangle_l, \quad (45)$$

where r is the squeezing parameter and χ is a phase.

In summary, we have given a general definition of unpolarized light that does not rest on specific properties of second-order field correlations but is based on three general principles: (i) invariance with respect to rotation of an orthogonal basis around the propagation axis, (ii) symmetry with respect to an interchange of left- and right-handed circular polarization, and (iii) invariance with respect to phase retardation, i.e., change of the relative phase between the field oscillations in two orthogonal directions. Fulfilling the requirements (i) and (ii) only, we arrive at a form of light we call type II unpolarized light. When condition (iii) is satisfied, in addition, we speak of type I unpolarized light of which natural (thermal) light is the most prominent and widespread representative. We have given both a classical and a quantum-mechanical description of those light forms, specifying the general form of both classical distribution functions and density operators consistent with the respective conditions. In constructing those quantum states, averaging over the Poincaré sphere proved to be a valuable tool. Our results show that there exists a variety of unpolarized light states. In the second part of this paper we will deal with realistic schemes for actually generating some of those new forms of unpolarized light.

Note added in proof. Recently, we became aware of earlier papers by H. Prakash and N. Chandra [Phys. Rev. A **4**, 796 (1971)] and G. S. Agarwal [Nuovo Cimento Lett. **1**, 53 (1971)] in which similar problems were studied. Those authors, however, restricted themselves to type I unpolarized light and used techniques different from ours.

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APPENDIX: DENSITY OPERATOR FOR TYPE I UNPOLARIZED LIGHT

In this appendix we will show that the form (34) of the density operator for type I unpolarized light follows uniquely from the requirements of both retardation and rotational invariance, and hence is the most general one.

We start from an expansion of the density operator with respect to a linear-polarization Fock basis

$$\hat{\rho}=\sum_{\substack{n_x,n_y\\m_x,m_y}}\rho(n_x,n_y;m_x,m_y)|n_x,n_y\rangle\langle m_y,m_x|. \quad (A1)$$

First we require invariance under a phase-retardation transformation, Eq. (18). The transformed density operator is given by

$$\hat{\hat{\rho}}=\sum_{\substack{n_x,n_y\\m_x,m_y}}\rho(n_x,n_y;m_x,m_y)\exp[i\varphi(-n_x+n_y+m_x-m_y)/2]\times|n_x,n_y\rangle\langle m_y,m_x|. \quad (A2)$$

The invariance condition, to be fulfilled for arbitrary values of φ, then restricts the density operator to the form

$$\hat{\rho}=\sum_{\substack{n_x,n_y\\m_x,m_y}}\rho(n_x,n_y;m_x,m_y)\,\delta(n_x-n_y,m_x-m_y)\times|n_x,n_y\rangle\langle m_y,m_x|. \quad (A3)$$

In order to impose the condition of rotational invariance on this density operator, we make use of the well-known fact that rotation of the orthogonal basis, Eq. (12), is generally described by the unitary transformation

$$\hat{P}(\vartheta)=\exp(-i\vartheta\hat{L}_2), \tag{A4}$$

where $\hat{L}_2$ is the angular-momentum operator, which can be expressed as [14]

$$\hat{L}_2=\frac{1}{2i}(\hat{a}_x^\dagger\hat{a}_y-\hat{a}_y^\dagger\hat{a}_x). \tag{A5}$$

Invariance with respect to rotation is then equivalent to commutability of $\hat{\rho}$ and $\hat{L}_2$,

$$[\hat{\rho},\hat{L}_2]=0\,. \tag{A6}$$

Let us now calculate this commutator. Applying the operator (A5) to Eq. (A3) we obtain

$$2i\hat{L}_2\hat{\rho}=\sum_{\substack{n_x,n_y\\m_x,m_y}}\rho(n_x,n_y;m_x,m_y)\,\delta(n_x-n_y,m_x-m_y)\{\sqrt{(n_x+1)n_y}|n_x+1,n_y-1\rangle\langle m_x,m_y| - \sqrt{n_x(n_y+1)}|n_x-1,n_y+1\rangle\langle m_y,m_x|\}, \tag{A7}$$

$$2i\hat{\rho}\hat{L}2=\sum_{\substack{n_x,n_y\\m_x,m_y}}\rho(n_x,n_y;m_x,m_y)\,\delta(n_x-n_y,m_x-m_y)\{\sqrt{m_x(m_y+1)}|n_x,n_y\rangle\langle m_x-1,m_y+1| - \sqrt{(m_x+1)m_y}|n_x,n_y\rangle\langle m_y-1,m_x+1|\}. \tag{A8}$$

Then the requirement (A6) implies the equation

$$\begin{aligned}&\rho(n_x-1,n_y+1;m_x,m_y)\sqrt{n_x(n_y+1)}\,\delta(n_x-n_y-2,m_x-m_y)-\rho(n_x+1,n_y-1;m_x,m_y)\sqrt{(n_x+1)n_y}\,\delta(n_x-n_y+2,m_x-m_y)\\&-\rho(n_x,n_y;m_x+1;m_y-1)\sqrt{(m_x+1)m_y}\,\delta(n_x-n_y-2,m_x-m_y)\\&+\rho(n_x,n_y;m_x-1,m_y+1)\sqrt{m_x(m_y+1)}\,\delta(n_x-n_y+2,m_x-m_y)=0\end{aligned} \tag{A9}$$

from which it follows that the relations

$$\rho(n_x,n_y;m_x-1,m_x-n_x+n_y-1)\sqrt{(n_x+1)n_y}=\rho(n_x+1,n_y-1;m_x,m_x-n_x+n_y-2)\sqrt{m_x(m_x-n_x+n_y-1)} \tag{A10}$$

and

$$\rho(n_x,n_y;m_x-1,m_x-n_x+n_y-1)\sqrt{m_x(m_x-n_x+n_y-1)}=\rho(n_x+1,n_y-1;m_x,m_x-n_x+n_y-2)\sqrt{(n_x+1)n_y} \tag{A11}$$

must hold. Obviously, they have a nonzero solution only if

$$(n_x+1)n_y=m_x(m_x-n_x+n_y-1). \tag{A12}$$

Putting here

$$n_x+1=m_x-p \quad (p \text{ an integer}), \tag{A13}$$

we arrive at the condition

$$-pn_y=pm_x\,, \tag{A14}$$

which implies $p=0$. [The alternative solution $n_y=m_x=0$ makes no sense due to the appearance of m_x-1 and n_y-1 in Eq. (A11).] Hence the density matrix $\rho(n_x,n_y;m_x,m_y)$ must be diagonal, and, moreover, Eq. (A11) gives us the relationship

$$\rho(n_x,n_y;n_x,n_y)=\rho(n_x+1,n_y-1;n_x+1,n_y-1). \tag{A15}$$

Substituting here $n_x=n-n_y$ (n fixed) and using the abbreviation $\rho(n_x,n_y;n_x,n_y)=\rho(n_x,n_y)$, we can rewrite Eq. (A.15) as

$$\rho(n-n_y,n_y)=\rho(n-[n_y-1],n_y-1), \tag{A16}$$

which means that $\rho(n-n_y,n_y)$ depends only on n, $\rho(n-n_y,n_y)=\rho_{nn}$. This is in perfect agreement with Eq. (34), which, therefore, has been shown to be a necessary implication of the required invariance with respect to both phase retardation and rotation. Moreover, one learns from the above analysis that invariance with respect to left- and right-handed circular polarization, in fact, need not be demanded, as an additional condition, for type I unpolarized light.

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