Generation and detection of photons in a cavity with a resonantly oscillating boundary

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The problem of photon creation from vacuum in an ideal cavity with vibrating walls is studied in the resonance case, when the frequency of vibrations equals twice the frequency of some unperturbed electromagnetic mode. Analytical solutions are obtained in two cases: for the one-dimensional model (scalar electrodynamics) and for the three-dimensional (3D) cavity. In the first example, we have a strong intermode interaction; nonetheless, an explicit solution in terms of the complete elliptic integrals is found. The rate of photon generation in the principal mode rapidly assumes a constant value proportional to the product of the frequency by the dimensionless amplitude of oscillations. The total amount of photons created in all the modes increases in time as t^2. In the second example, the eigenmode spectrum is nonequidistant and the problem can be reduced to the problem of a single harmonic oscillator with a time-dependent frequency. The number of photons in the resonant mode of a 3D cavity increases exponentially in time and the field appears in a highly squeezed state with a strongly oscillating photon distribution function. The problem of detecting the created photons is analyzed in the framework of a simplified model, when a detector is replaced with a harmonic oscillator. It turns out that the presence of the detector changes the picture drastically: both the detector and the field mode occur in highly mixed (nonthermal) quantum states, with identical nonoscillating photon distribution functions. The detector gains exactly half of the total energy of excitation inside the cavity. The estimations show a possibility of creating up to several hundred or even thousand photons, provided that the cavity's Q factor exceeds 10^{10} and the amplitude of the wall's oscillations is greater than 10^{-10} cm at a frequency of the order of 10 GHz.

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I. INTRODUCTION

Recent progress in experiments with high-Q electromagnetic cavities provides a possibility to verify the most delicate features of quantum physics [1–4]. Among them there is a wide class of phenomena combined under the name *nonstationary Casimir effect* (NSCE) [5], which are caused by the reconstruction of the quantum state of a field due to a time dependence of the geometrical configuration. The manifestations of these phenomena can be divided conventionally in two (closely connected) groups. The first one is related to the modification of the known Casimir force [6], when it acts on a *moving* boundary. This problem was studied, e.g., in Refs. [7–13] (for ideal mirrors) and [14–19] (for partially reflecting and dielectric mirrors). Here we address another manifestation of the NSCE, namely, the effect of *photon creation* in an empty cavity with moving boundaries.

As far as we know, the first exact solution of the wave equation in one space dimension (we assume $c=\hbar=1$)

$$\frac{\partial^2 A}{\partial t^2}-\frac{\partial^2 A}{\partial x^2}=0 \tag{1.1}$$

with the time-dependent boundary conditions

$$A(0,t)=A(L(t),t)=0 \tag{1.2}$$

[$L(t)$ is the given law of motion of the right boundary, whereas the left boundary is assumed to be at rest at $x=0$] was obtained in Ref. [20], where the transverse vibrations of a string with a uniformly varying length $L(t)=L_0(1+\alpha t)$ were studied (see also [21], where not only the string, but also the electromagnetic field in a one-dimensional (1D) cavity with a uniformly increasing length was considered). Forty years later, this solution was rediscovered in [22] [where an exact solution for $L(t)=(t^2+1)^{1/2}$ was also found and some graphical method of finding the solution for an arbitrary law of motion $L(t)$ was proposed]. The evolution of the classical electromagnetic field between uniformly moving (in opposite directions) ideal walls of a one-dimensional resonator was investigated in [23,24]. The three-dimensional rectangular cavities (waveguides) with uniformly moving walls were considered in the framework of the classical electrodynamics in [25]. The case of a spherical ideal cavity whose radius was a linear function of time was studied in [26]. Exact solutions for the transverse vibrations of a two-dimensional rectangular membrane, when the length of one boundary uniformly increased in time, was found in [27].

Moore's paper [28] seems to be the first one devoted to the quantum aspects of the problem, in particular to the prob-

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lem of generating photons from vacuum in the cavities with moving boundaries. Considering a model of the ''scalar electrodynamics'' (when the field depends on a single space coordinate), Moore found a complete set of solutions to the problem (1.1) and (1.2) in the form

$$A_n(x,t)=(4\pi n)^{-1/2}\{\exp[-i\pi nR(t-x)] - \exp[-i\pi nR(t+x)]\}, \tag{1.3}$$

where function $R(\xi)$ must satisfy the functional equation

$$R(t+L(t))-R(t-L(t))=2. \tag{1.4}$$

This solution was discovered independently in [29] [note that for a *linear* function $L(t)$ such an approach was used actually in [20,21]]. However, only a few exact solutions of Eq. (1.4) are known nowadays for the functions $L(t)$ different from the linear one. Most of them were found in the framework of the ''inverse problem,'' where the unknown function $L(t)$ was extracted from Eq. (1.4) with the given dependence $R(\xi)$ [30,31]. Approximate solutions of Eq. (1.4), valid in the case of a nonrelativistic motion of the boundary, were found in [11,28,29,32,33] in the form of the asymptotic series with respect to a small parameter of the order of dL/dt (see also [24]). However, these solutions are not valid when the boundary oscillates at a frequency comparable with the frequencies of the field modes [34]. But just this case seems to be the most interesting from the point of view of a possible experimental verification. Indeed, if the characteristic time of changing the position of the wall is much longer than the period of the field oscillations, then we have an adiabatic situation, where no photons can be produced (see Appendix A).

Since the maximal velocity of the boundary, that could be achieved under the laboratory conditions is very small in comparison with the speed of light, the only hope to observe the effect can be connected with the *resonance* between the mechanical and field oscillations, where a gradual accumulation of the small changes in the quantum state results finally in a significant effect. Thus it is worth considering first an ideal parametric resonance case, where the boundary performs small harmonic oscillations at twice the unperturbed eigenfrequency of the lowest field mode. For this law of motion, we found earlier [12,35] an asymptotical solution of Eq. (1.4), valid in the long-time limit $\varepsilon\omega t \gg 1$, ε being the dimensionless amplitude of the boundary oscillations. Surprisingly, this solution gives the correct result even in the opposite case $\varepsilon\omega t \ll 1$, which was considered earlier in [34]. According to [12,35], with $\varepsilon\omega t \gg 1$ the photon generation rate becomes time independent, being proportional to the product of the amplitude of wall oscillations by their frequency. An exact solution for an *almost harmonic* law of motion was given recently in [13]. The evolution of the *classical* field in a 1D ideal cavity, whose boundaries performed harmonic oscillations at a frequency comparable with the field eigenfrequencies, was investigated in [36], while a similar problem for the spherical cavity was considered in [37].

In the present paper we shall obtain explicit expressions for the rate of photon generation, both in the principal mode and in all modes, which hold for any moment of time and turn into the asymptotic formulas of Refs. [12,34,35] in the appropriate limit cases. For this purpose we use an approach different from Moore's. Namely, we shall expand the field operator in the Heisenberg picture over the ''instantaneous'' eigenfunctions of the Helmholtz equation. Actually, such a method of solving the problems with moving boundaries was known for a long time (see, e.g., [38]). In connection with the problem involved it was used in Refs. [39–43], but only approximate or numerical solutions of the arising infinite set of coupled ordinary differential equations were found. We shall demonstrate that in the resonance case these equations can be simplified significantly so that they can be solved *exactly*.

In general, the three-dimensional situation is more complicated than the simplified 1D model of the scalar electrodynamics. For this reason, the exact solutions of the quantum problem in three space dimensions were found until now only in the case of *uniformly* moving ideal walls [8,44]. A *single* ideal boundary moving along an arbitrary prescribed nonrelativistic trajectory was considered in [45] (see also [46] for the 1D model); a more general case of a moving dielectric medium with a single boundary was studied in [19]. We shall demonstrate that in the parametric resonance case the problem can be also treated analytically. This remarkable possibility is due to the fact that the intervals between the neighboring field eigenfrequencies are not constant in the 3D cavity. As a consequence, a reduction of the field problem to the problem of a parametrically excited one-dimensional oscillator becomes possible. In contradistinction to the 1D model, the number of photons in the resonance mode increases in time exponentially and the field appears in a *highly squeezed* quantum state with a strongly oscillating photon distribution function.

An important point is how to detect the photons created from the vacuum due to the motion of the wall. One of the possibilities is to put some probe object inside the cavity. For the cavities with dimensions of several centimeters (i.e., for the eigenfrequencies about 10 GHz), a realistic detector could be some Rydberg atom. Here we confine ourselves to a simplified model of a probe *oscillator* resonantly interacting with the field mode by means of the standard electromagnetic coupling. We show that the interaction drastically changes the photon statistics, so that both the field and the probe oscillators appear in highly mixed, but moderately squeezed, nonthermal states, with identical photon distribution functions, which do not exhibit oscillations.

The plan of the paper is as follows. Sections II–VI are devoted to the 1D model. In Sec. II we give the scheme of the field quantization in the case of moving boundaries. In Sec. III we take advantage of the method of averaging over fast oscillations to obtain simplified equations for slowly varying amplitudes in the resonance case. The solutions of these equations will be found in Secs. IV and V. In Sec. VI we calculate the rate of photon generation in the principal mode and the total number of photons in all the modes. Then we turn to the 3D case. In Sec. VII we reduce the field problem to the problem of a one-dimensional parametric oscillator. Then, using the theory of a quantum nonstationary oscillator [47], we calculate all the characteristics of the resonance mode. The interaction between the probe oscillator and the resonance mode is the subject of Sec. VIII. Section

IX is devoted to the discussion of the possibility of observing the phenomenon under laboratory conditions. We arrive at the conclusion that the current experimental level is sufficient, in principle, to discover the effect.

The aim of Appendix A is to demonstrate explicitly the impossibility of photon creation in the case of an adiabatic motion of the wall. Appendix B contains the details of calculations of the propagator for the two-dimensional coupled system ‘‘field oscillator plus probe oscillator.’’ Some useful formulas related to the properties of multidimensional Gaussian quantum states are given in Appendix C.

II. FIELD QUANTIZATION WITH MOVING BOUNDARIES

We adopt the scheme of the field quantization in an empty cavity with time-dependent boundary conditions, proposed by Moore [28] [for its generalization to the case when the cavity is filled with a (nondispersive) nonuniform and time-dependent dielectric medium see [48]]. Let us consider a cavity formed by two infinite ideal plates, one being at rest at $x=0$, with the other moving according to a prescribed law $L(t)>0$ when $t>0$. Confining ourselves to the electromagnetic modes whose vector potential is directed along z axis, we can write down the field operator *in the Heisenberg representation* $\hat{A}(x,t)$ at $t\leqslant 0$ (when both the plates were at rest) as

$$\hat{A}_{in}=2\sum_n \frac{1}{\sqrt{n}}\sin\frac{n\pi x}{L_0}\hat{b}_n\exp(-i\omega_n t)+\text{ H.c.},\qquad (2.1)$$

where $\hat{b}_n$ means the usual annihilation photon operator and $\omega_n=\pi n/L_0$. The choice of coefficients in Eq. (2.1) corresponds to the standard form of the field Hamiltonian

$$\hat{H}\equiv\frac{1}{8\pi}\int_0^{L_0}dx\left[\left(\frac{\partial A}{\partial t}\right)^2+\left(\frac{\partial A}{\partial x}\right)^2\right]=\sum_n \omega_n\left(\hat{b}_n^\dagger\hat{b}_n+\frac{1}{2}\right).\qquad (2.2)$$

For an arbitrary moment of time the field operator can be written as

$$\hat{A}(x,t)=2\sum_n \frac{1}{\sqrt{n}}[\hat{b}_n\psi^{(n)}(x,t)+\text{ H.c.}].$$

To find the explicit form of functions $\psi^{(n)}(x,t)$, $n=1,2,\ldots,$ we have to take into account that the field operator must satisfy (i) the wave equation (1.1), (ii) the boundary conditions (1.2), and (iii) the initial condition

$$\psi^{(n)}(x,t<0)=\sin\frac{n\pi x}{L_0}\exp(-i\omega_n t).\qquad (2.3)$$

Following the approach of Refs. [39–43], we expand the function $\psi^{(n)}(x,t)$ in a series with respect to the *instantaneous basis*

$$\psi^{(n)}(x,t>0)=\sum_k Q_k^{(n)}(t)\sqrt{\frac{L_0}{L(t)}}\sin\frac{\pi kx}{L(t)},\qquad (2.4)$$

with the initial conditions

$$Q_k^{(n)}(0)=\delta_{kn},\dot{Q}_k^{(n)}(0)=-i\omega_n\delta_{kn}.$$

This way we satisfy automatically both the boundary conditions (1.2) and the initial condition (2.3). Putting expression (2.4) into the wave equation (1.1), one can, after some algebra, arrive at an infinite set of coupled differential equations [43]

$$\ddot{Q}_k^{(n)}+\omega_k^2(t)Q_k^{(n)}=2\lambda(t)\sum_j g_{kj}\dot{Q}_j^{(n)}+\dot{\lambda}(t)\sum_j g_{kj}Q_j^{(n)}+\lambda^2(t)\sum_{j,l} g_{jk}g_{jl}Q_l^{(n)},\qquad (2.5)$$

where

$$\omega_k(t)=\frac{k\pi}{L(t)},\quad \lambda(t)=\frac{\dot{L}(t)}{L(t)},$$

and the constant antisymmetric coefficients g_{jk} read

$$g_{kj}=-g_{jk}=(-1)^{k-j}\frac{2kj}{j^2-k^2}\quad (j\neq k).\qquad (2.6)$$

Equation (2.5) possesses exact first integrals. Namely, for any two sets of solutions $\{Q_k^{(1)}\}$ and $\{Q_k^{(2)}\}$,

$$\sum_k [Q_k^{(1)}\dot{Q}_k^{(2)}-\dot{Q}_k^{(1)}Q_k^{(2)}]+2\lambda(t)\sum_{k,l} g_{lk}Q_k^{(1)}Q_l^{(2)}=\text{const}$$

for arbitrary functions $\omega(t)$ and $\lambda(t)$. These integrals correspond to the following conserved quantity

$$\int_0^{L(t)}(A_1\dot{A}_2-\dot{A}_1A_2)dx=\text{const}$$

where $A_1(x,t)$ and $A_2(x,t)$ are arbitrary solutions of the wave equation satisfying the boundary conditions.

If the wall returns to its initial position L_0 after some interval of time T, then the right-hand side of Eq. (2.5) disappears, so at $t>T$ one gets

$$Q_k^{(n)}(t)=\xi_k^{(n)}e^{-i\omega_k t}+\eta_k^{(n)}e^{i\omega_k t},\qquad (2.7)$$

$\xi_k^{(n)}$ and $\eta_k^{(n)}$ being some constant coefficients. Consequently, at $t>T$ the initial annihilation operators $\hat{b}_n$ cease to be ‘‘physical’’ due to the contribution of the terms with ‘‘incorrect signs’’ in the exponentials $\exp(i\omega_k t)$. Introducing a new set of physical operators $\hat{a}_m$ and $\hat{a}_m^\dagger$, which result, at $t>T$, in relations such as (2.1) and (2.2), but with $\hat{a}_m$ instead of $\hat{b}_m$, one can easily check that the two sets of operators are related by means of the Bogoliubov transformation

$$\hat{a}_m=\sum_n (\hat{b}_n\alpha_{nm}+\hat{b}_n^\dagger\beta_{nm}^*),\qquad (2.8)$$

with the coefficients

$$\alpha_{nm}=\sqrt{\frac{m}{n}}\xi_m^{(n)},\quad \beta_{nm}=\sqrt{\frac{m}{n}}\eta_m^{(n)}.\qquad (2.9)$$

The amount of photons created in the mth mode equals the average value of the operator $\hat{a}_m^\dagger \hat{a}_m$, since just this operator has a physical meaning at $t>T$. Assuming that initially the field was in the vacuum state $|0_b\rangle$ (defined with respect to the *initial* operators $\hat{b}_n$), one gets

$$\mathscr{P}_m \equiv \langle 0_b | \hat{a}_m^\dagger \hat{a}_m | 0_b \rangle = \sum_n |\beta_{nm}|^2 = m \sum_{n=1}^{\infty} \frac{1}{n} |\eta_m^{(n)}|^2. \tag{2.10}$$

The calculation of the last sum in this equation is our first goal.

III. SIMPLIFIED EQUATIONS IN THE RESONANCE CASE

Equation (2.5) forms an infinite set of coupled equations with time-dependent coefficients, moreover, each equation also contains an infinite number of terms. The simplest attempt to simplify these equations seems to neglect their right-hand sides completely, since the function $\lambda(t)$, which is proportional to the velocity of the wall, is very small in real conditions. Then one would arrive at a system of equations describing *uncoupled* oscillators with time-dependent frequencies. However, this simple idea is *wrong*. Indeed, let us evaluate the relative magnitudes of terms in both sides of Eq. (2.5). If the relative displacement of the wall's position $\delta L/L$ is characterized by some dimensionless parameter ϵ and the characteristic frequency (the inverse characteristic time) of the motion is ω_w, then $\lambda \sim \epsilon\omega_w$, so the three groups of terms on the right-hand side are proportional to $\epsilon\omega_w\omega_j$, $\epsilon\omega_w^2$, and $\epsilon^2\omega_w^2$, respectively. Since ϵ is very small under the laboratory conditions (see the discussion in Sec. IX), all these terms are much smaller than the leading terms on the left-hand side, whose order is ω_k^2. However, one should compare the terms on the right-hand side not with the leading terms on the left-hand side, but with the *corrections to the leading terms* due to the variations of the frequency. These corrections are proportional to $\delta(\omega_k^2) \sim \omega_k \delta\omega_k \sim \omega_k^2 \epsilon$. Consequently, the right-hand side terms can be neglected only under the condition $\omega_w \ll \omega_k$, i.e., in the case of an *adiabatic* motion of the wall. But in the adiabatic situation, evidently, no photons can be generated, as was demonstrated explicitly, e.g., in [32,40,49] (see also Appendix A).

Thus, the photons can be created only provided that $\omega_w \sim \omega_k$. In such a case, the first two groups of terms on the right-hand side of Eq. (2.5) have the same order of magnitude as the corrections on the left-hand side. Besides, *a strong intermode interaction* exists [12,35,41] due to the *equidistant* character of the eigenfrequency spectrum in the 1D cavity. Nonetheless, there exists a special *nonadiabatic* case, when Eq. (2.5) can be simplified significantly. Moreover, just this case seems to be the most similar to the real experimental situation. Indeed, the most realistic law of motion of the wall corresponds to its *harmonic oscillations* with some small amplitude and it is clear that the most significant effect can be achieved under the condition of the *parametric resonance*, when the frequency of the wall's vibrations coincides with twice the frequency of some electromagnetic mode. So let us consider the following law of motion of the wall:

$$L(t) = L_0[1 + \varepsilon \sin(2\omega_1 t)], \quad \omega_1 = \pi/L_0. \tag{3.1}$$

Assuming $\varepsilon \ll 1$, it is natural to look for the solutions of Eq. (2.5) in the form (2.7), but now we allow the coefficients $\xi_m^{(n)}$ and $\eta_m^{(n)}$ to be *slowly varying functions of time*. The further procedure is well known in the theory of parametrically excited systems [50–52]. First we put expression (2.7) into Eq. (2.5) and neglect the terms $\ddot{\xi}, \ddot{\eta}$ (having in mind that $\dot{\xi}, \dot{\eta} \sim \varepsilon$, while $\ddot{\xi}, \ddot{\eta} \sim \varepsilon^2$), as well as the terms proportional to $\lambda^2 \sim \varepsilon^2$. Multiplying the resulting equation for Q_k by the factors $\exp(i\omega_k t)$ and $\exp(-i\omega_k t)$ and performing *averaging over fast oscillations* with frequencies proportional to ω_k (since the functions ξ, η practically do not change their values at the time scale of $2\pi/\omega_k$) we get the equations for ξ_k and η_k, respectively. After some algebra we arrive at the system of equations looking much more simple than the initial one:

$$\frac{d}{d\tau} \xi_1^{(n)} = -\eta_1^{(n)} + 3\xi_3^{(n)}, \tag{3.2}$$

$$\frac{d}{d\tau} \xi_k^{(n)} = (k+2)\xi_{k+2}^{(n)} - (k-2)\xi_{k-2}^{(n)}, \quad k \geqslant 2 \tag{3.3}$$

$$\frac{d}{d\tau} \eta_1^{(n)} = -\xi_1^{(n)} + 3\eta_3^{(n)}, \tag{3.4}$$

$$\frac{d}{d\tau} \eta_k^{(n)} = (k+2)\eta_{k+2}^{(n)} - (k-2)\eta_{k-2}^{(n)}, \quad k \geqslant 2. \tag{3.5}$$

Here we have introduced a new "slow" time scale

$$\tau = \frac{1}{2}\varepsilon\omega_1 t. \tag{3.6}$$

The initial conditions now read

$$\xi_k^{(n)}(0) = \delta_{kn}, \quad \eta_k^{(n)}(0) = 0. \tag{3.7}$$

Although the new system also consists of an infinite number of equations, each equation separately contains now only three terms.

It is worth noting that if we deleted the right-hand side of Eq. (2.5) from the beginning (following an incorrect "naive" approach), then we would obtain a quite different system of equations, without terms ξ_m and η_m with $m>1$ on the right-hand sides. This example demonstrates explicitly why the naive approach is wrong.

In contradistinction to *exact* equations (2.5), Eqs. (3.2)–(3.5) are approximate. However, it should be emphasized that the degree of approximation is very good when $\varepsilon \ll 1$ (this parameter can hardly exceed the value of the order of 10^{-7} in realistic situations; see Sec. IX). In particular, the *unitarity condition* for the Bogoliubov transformation (2.8),

$$\sum_m (|\alpha_{nm}|^2 - |\beta_{nm}|^2) = 1, \tag{3.8}$$

is *an exact consequence* of the new equations. Indeed, multiplying both sides of Eqs. (3.3) and (3.5) by $k\xi_k^{(n)}$ or $k\eta_k^{(n)}$, respectively, and performing the summation over k, one gets the relations (hereafter the over-dot denotes differentiation with respect to the slow time τ)

$$\sum_k k\xi_k^{(n)}\dot{\xi}_k^{(n)} = \sum_k k\eta_k^{(n)}\dot{\eta}_k^{(n)} = -\xi_1^{(n)}\eta_1^{(n)}, \qquad (3.9)$$

resulting in the identity

$$\sum_k k([\xi_k^{(n)}]^2 - [\eta_k^{(n)}]^2) = n, \qquad (3.10)$$

which is equivalent to Eq. (3.8) due to Eq. (2.9).

It is remarkable that even modes are not coupled to the odd ones. Moreover, one can easily verify that due to the initial conditions (3.7) all the functions $\eta_k^{(n)}(t)$ equal zero, provided either of the suffices n or k is an even number:

$$\eta_{2l}^{(n)}(t) = \eta_k^{(2j)}(t) = 0.$$

Consequently, there is no creation of photons in the even modes (see also [12,35,41]). Thus we need the solutions of Eqs. (3.2)–(3.5) only for the odd values of the indices n and k.

IV. REDUCING TO A SINGLE INTEGRAL EQUATION

It is convenient to introduce the variables

$$\mu_k^{(n)} = \xi_k^{(n)} + \eta_k^{(n)}, \quad \nu_k^{(n)} = \xi_k^{(n)} - \eta_k^{(n)}, \qquad (4.1)$$

which satisfy a ‘‘more homogeneous’’ set of equations

$$\frac{d}{d\tau}\mu_1^{(n)} = -\mu_1^{(n)} + 3\mu_3^{(n)}, \qquad (4.2)$$

$$\frac{d}{d\tau}\mu_k^{(n)} = (k+2)\mu_{k+2}^{(n)} - (k-2)\mu_{k-2}^{(n)}, \quad k\geqslant 3 \qquad (4.3)$$

$$\frac{d}{d\tau}\nu_1^{(n)} = \nu_1^{(n)} + 3\nu_3^{(n)}, \qquad (4.4)$$

$$\frac{d}{d\tau}\nu_k^{(n)} = (k+2)\nu_{k+2}^{(n)} - (k-2)\nu_{k-2}^{(n)}, \quad k\geqslant 3 \qquad (4.5)$$

and the initial conditions

$$\mu_k^{(n)}(0) = \nu_k^{(n)}(0) = \delta_{kn}. \qquad (4.6)$$

To get rid of the infinite number of equations we introduce two *generating functions*

$$M^{(n)}(\tau,z) = \sum_{k=1}^{\infty} \mu_k^{(n)}(\tau)z^k, \qquad (4.7)$$

$$N^{(n)}(\tau,z) = \sum_{k=1}^{\infty} \nu_k^{(n)}(\tau)z^k, \qquad (4.8)$$

where z is an auxiliary variable and the summation is performed over odd values of k. Using relations such as $kz^k = z(dz^k/dz)$, it is not difficult to obtain the following first-order partial differential equations:

$$\frac{\partial M^{(n)}}{\partial\tau} = \left(\frac{1}{z} - z^3\right)\frac{\partial M^{(n)}}{\partial z} - \mu_1^{(n)}(\tau)\left(z + \frac{1}{z}\right), \qquad (4.9)$$

$$\frac{\partial N^{(n)}}{\partial\tau} = \left(\frac{1}{z} - z^3\right)\frac{\partial N^{(n)}}{\partial z} + \nu_1^{(n)}(\tau)\left(z - \frac{1}{z}\right). \qquad (4.10)$$

Let us consider Eq. (4.9). For the given function $\mu_1^{(n)}(\tau)$ its solution satisfying the initial condition $M^{(n)}(0,z) = z^n$ reads

$$M^{(n)}(\tau,z) = \left[\frac{1+z^2-e^{-4\tau}(1-z^2)}{1+z^2+e^{-4\tau}(1-z^2)}\right]^{n/2} - 2\int_0^\tau \left[1 - \left(\frac{1-z^2}{1+z^2}\right)^2 e^{-8x}\right]^{-1/2} \mu_1^{(n)}(\tau - x)\,dx. \qquad (4.11)$$

Since the summation in Eq. (4.7) begins at $k=1$, the function $M^{(n)}(\tau,z)$ must satisfy the boundary condition

$$\partial M^{(n)}(\tau,z)/\partial z\big|_{z=0} = \mu_1^{(n)}(\tau). \qquad (4.12)$$

But the derivative of the first term in the right-hand side of Eq. (4.11) with respect to z equals zero at $z=0$. Thus

$$\partial M^{(n)}(\tau,z)/\partial z\big|_{z=0} = \lim_{z\to 0}\int_0^\tau \mu_1^{(n)}(\tau-x) f(x;z)\,dx,$$

where the function

$$f(x;z) = 8ze^{-8x}\left[1 - \left(\frac{1-z^2}{1+z^2}\right)^2 e^{-8x}\right]^{-3/2}$$

possesses the following properties: $\lim_{z\to 0} f(x=0;z) = \lim_{z\to 0}(1+z^2)^3/z^2 = \infty$, $\lim_{z\to 0} f(x\neq 0;z) = 0$, and $\int_0^\infty f(x;z)dx = (1+z^2)^2/(1+z)^2$. Consequently, $\lim_{z\to 0} f(x;z) = \delta(x)$, giving rise to Eq. (4.12).

Due to the definition (4.7), $M^{(n)}(\tau,0) = 0$ for any value of τ. This observation immediately results in the closed integral equations for the functions $\mu_1^{(n)}$ with $n = 1,3, \ldots$.

$$\int_0^\tau \frac{\mu_1^{(n)}(\tau - x)}{\sqrt{1-e^{-8x}}}\,dx = \frac{1}{2}\left(\frac{1-e^{-4\tau}}{1+e^{-4\tau}}\right)^{n/2}. \qquad (4.13)$$

In a similar way one can obtain an integral equation for the function $\nu_1^{(n)}(\tau)$. However, it is convenient to introduce a function

$$\lambda_1^{(n)}(\tau) = e^{4\tau}\nu_1^{(n)}(\tau) \qquad (4.14)$$

obeying the equation (again for $n = 1,3, \ldots$)

$$\int_0^\tau \frac{\lambda_1^{(n)}(\tau - x)}{\sqrt{1-e^{-8x}}}\,dx = \frac{1}{2}e^{4\tau}\left(\frac{1-e^{-4\tau}}{1+e^{-4\tau}}\right)^{n/2}. \qquad (4.15)$$

Let us consider first the case of $n=1$. Differentiating both sides of Eqs. (4.13) and (4.15) with respect to τ and taking into account the initial conditions $\mu_1^{(1)}(0) = \lambda_1^{(1)}(0) = 1$ one can verify the relations

$$\mu_1^{(3)} = -\mu_1^{(1)} - \dot{\mu}_1^{(1)}, \qquad (4.16)$$

$$\lambda_1^{(3)}=\lambda_1^{(1)}+\dot{\mu}_1^{(1)}. \tag{4.17}$$

Besides, there exists an additional identity

$$\dot{\mu}_1^{(1)}+\dot{\lambda}_1^{(1)}=4\lambda_1^{(1)},$$

which is equivalent to

$$\dot{\mu}_1^{(1)}+e^{4\tau}\dot{\nu}_1^{(1)}=0. \tag{4.18}$$

From Eqs. (4.16)–(4.18) one gets

$$\eta_1^{(3)}=-\xi_1^{(1)}-\dot{\eta}_1^{(1)}, \tag{4.19}$$

$$\xi_1^{(3)}=-\eta_1^{(1)}-\dot{\xi}_1^{(1)}. \tag{4.20}$$

At $n\geqslant 3$ the same procedure [but with different initial conditions $\mu_1^{(n)}(0)=\lambda_1^{(n)}(0)=0$] leads after simple algebraic manipulations to the recurrence relations

$$2n\mu_1^{(n+2)}=-2n\mu_1^{(n)}-4\lambda_1^{(n)}-\dot{\mu}_1^{(n)}+\dot{\lambda}_1^{(n)}, \tag{4.21}$$

$$2n\lambda_1^{(n+2)}=(2n+4)\lambda_1^{(n)}+\dot{\mu}_1^{(n)}-\dot{\lambda}_1^{(n)}. \tag{4.22}$$

Their consequence is the relation

$$\mu_1^{(n+2)}+\lambda_1^{(n+2)}=\lambda_1^{(n)}-\mu_1^{(n)}, \tag{4.23}$$

which is valid actually for $n\geqslant 1$. Subtracting from Eq. (4.21) a similar equation with n replaced by $n-2$, and replacing the difference $\lambda_1^{(n)}-\lambda_1^{(n-2)}$ by the sum $\mu_1^{(n)}+\mu_1^{(n-2)}$, one can, after simple algebra, arrive at the three-term recurrence relation

$$n\mu_1^{(n+2)}=n\mu_1^{(n-2)}-\dot{\mu}_1^{(n)}.$$

Similarly, adding to Eq. (4.22) a similar equation with $n-2$ instead of n and using again Eq. (4.23), one can arrive at a relation for the functions $\lambda_1^{(n)}$, which results in exactly the same recurrence relation for the functions $\nu_1^{(n)}$:

$$n\nu_1^{(n+2)}=n\nu_1^{(n-2)}-\dot{\nu}_1^{(n)}.$$

Consequently, we have the same relations for the functions $\xi_1^{(n)}$ and $\eta_1^{(n)}$:

$$n\xi_1^{(n+2)}=n\xi_1^{(n-2)}-\dot{\xi}_1^{(n)}, \quad n\geqslant 3 \tag{4.24}$$

$$n\eta_1^{(n+2)}=n\eta_1^{(n-2)}-\dot{\eta}_1^{(n)}, \quad n\geqslant 3. \tag{4.25}$$

An immediate consequence of Eqs. (4.19) and (4.25) is a closed expression for the photon generation rate in the principal cavity mode [see Eq. (2.10) with $m=1$; remember that the summation is performed over *odd* numbers]

$$\frac{d\mathscr{P}_1}{d\tau}=\sum_{n=1}^{\infty}\frac{2}{n}\eta_1^{(n)}\dot{\eta}_1^{(n)}=-2\eta_1^{(1)}(\tau)\xi_1^{(1)}(\tau). \tag{4.26}$$

Moreover, Eqs. (4.20) and (4.24) result in the relation

$$\sum_{n=1}^{\infty}\frac{2}{n}\xi_1^{(n)}\dot{\xi}_1^{(n)}=-2\eta_1^{(1)}(\tau)\xi_1^{(1)}(\tau). \tag{4.27}$$

Comparing it with the second equality in (4.26), we arrive at the identity

$$\sum_n\frac{1}{n}([\xi_1^{(n)}]^2-[\eta_1^{(n)}]^2)=1, \tag{4.28}$$

which is *exactly* equivalent to the unitarity condition of the Bogoliubov transformation (2.8)

$$\sum_n(|\alpha_{n1}|^2-|\beta_{n1}|^2)=1, \tag{4.29}$$

due to Eq. (2.9).

Comparing Eqs. (3.2)–(3.5) with superscript $n=1$ and Eqs. (4.19), (4.20), (4.24), and (4.25), it is not difficult to check the relations

$$\xi_1^{(2j+1)}=(-1)^j(2j+1)\xi_{2j+1}^{(1)},$$

$$\eta_1^{(2j+1)}=(-1)^j(2j+1)\eta_{2j+1}^{(1)}.$$

For the *total* rate of photon generation in *all the modes* we get, using Eqs. (2.10) and (3.9), the expression

$$\frac{d\mathscr{P}}{d\tau}=\sum_k\sum_n\frac{2k}{n}\eta_k^{(n)}\dot{\eta}_k^{(n)}=-\sum_n\frac{2}{n}\eta_1^{(n)}(\tau)\xi_1^{(n)}(\tau). \tag{4.30}$$

Differentiating it once more, we get

$$\frac{d^2\mathscr{P}}{d\tau^2}=-\sum_n\frac{2}{n}[\dot{\eta}_1^{(n)}\xi_1^{(n)}+\eta_1^{(n)}\dot{\xi}_1^{(n)}].$$

Then Eqs. (4.19), (4.20), (4.24) and (4.25) lead to the simple formula

$$\frac{d^2\mathscr{P}}{d\tau^2}=2([\xi_1^{(1)}]^2+[\eta_1^{(1)}]^2). \tag{4.31}$$

Consequently, we need only two functions, $\eta_1^{(1)}(\tau)$ and $\xi_1^{(1)}(\tau)$ to calculate all physically interesting characteristics of the system under study. To find these functions we must solve Eqs. (4.13) and (4.15).

V. SOLVING INTEGRAL EQUATIONS

Since we have *a convolution* of two functions on the left-hand sides of Eqs. (4.13) and (4.15), it is natural to apply the *Laplace transformation* to these equations. Using the known integrals [see, e.g., [53], Eqs. (3.191.3) and (3.197.3)] one can find the following expressions for the Laplace transforms $\overline{\mu}_1^{(n)}(s)$ and $\overline{\nu}_1^{(n)}(s)$ of the functions $\mu_1^{(n)}(\tau)$ and $\nu_1^{(n)}(\tau)$, respectively:

$$\bar{\mu}_1^{(n)}(s)=\frac{\Gamma\left(\frac{s}{8}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{8}\right)\Gamma\left(\frac{1}{2}\right)}I(s,n),$$

$$\bar{\nu}_1^{(n)}(s)=\frac{\Gamma\left(\frac{s}{8}+1\right)}{\Gamma\left(\frac{s}{8}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}I(s,n),$$

where

$$I(s,n)=\int_0^1 x^{s/4-1}(1-x)^{n/2}(1+x)^{-n/2}\,dx$$
$$=\frac{\Gamma\left(\frac{s}{4}\right)\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(\frac{s}{4}+1+\frac{n}{2}\right)}F\left(\frac{s}{4},\frac{n}{2};\frac{s}{4}+1+\frac{n}{2};-1\right)$$

and $F(a,b;c;z)$ is the Gauss hypergeometric function. For $n=1$ we have $c=a-b+2$ for which an explicit expression for the hypergeometric function is known [54]:

$$F(a,b;a-b+2;-1)=\frac{2^{-a}\sqrt{\pi}}{b-1}\left\{\frac{\Gamma(a-b+2)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{3}{2}+\frac{a}{2}-b\right)}-\frac{\Gamma(a-b+2)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(1+\frac{a}{2}-b\right)}\right\}.$$

This formula leads to the relations

$$\bar{\mu}_1^{(1)}(s)=\frac{1}{2}\left\{1-\frac{s}{8}\left[\frac{\Gamma\left(\frac{s}{8}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{8}+1\right)}\right]^2\right\}, \tag{5.1}$$

$$\bar{\nu}_1^{(1)}(s)=-\frac{1}{2}\left\{1-\frac{s}{8}\left[\frac{\Gamma\left(\frac{s}{8}\right)}{\Gamma\left(\frac{s}{8}+\frac{1}{2}\right)}\right]^2\right\}. \tag{5.2}$$

So we have to find the explicit expression for the function $R(\tau)$, whose Laplace transform reads

$$\bar{R}(s)=\left[\frac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^2.$$

The known integral representation for Euler's β-function [53,54]

$$\frac{\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-sy}(1-e^{-y})^{-1/2}\,dy$$

yields

$$\bar{R}(s)=\frac{1}{\pi}\int_0^\infty\int_0^\infty\frac{dy\;dz\;e^{-s(y+z)}}{\sqrt{(1-e^{-y})(1-e^{-z})}}.$$

Since $\exp(-sa)$ is the Laplace transform of the function $\delta(\tau-a)$, we get

$$R(\tau)=\frac{1}{\pi}\int_0^\infty\int_0^\infty\frac{\delta(\tau-y-z)dy\;dz}{\sqrt{(1-e^{-y})(1-e^{-z})}}$$
$$=\frac{1}{\pi}\int_{e^{-\tau}}^1\frac{dx}{\sqrt{x(1-x)(x-e^{-\tau})}}.$$

The last integral can be transformed to the standard form of the complete elliptic integral of the first kind [53]

$$K(\kappa)=\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}},$$

resulting in the expression

$$R(\tau)=\frac{2}{\pi}K(\sqrt{1-e^{-\tau}})\,\theta(\tau), \tag{5.3}$$

$\theta(\tau)$ being Heaviside's step function. Then Eqs. (5.1)–(5.3), together with the initial conditions (4.6), lead to the relations

$$\mu_1^{(1)}(\tau)=-\frac{1}{\pi}\frac{\partial}{\partial\tau}[e^{-4\tau}K(\sqrt{1-e^{-8\tau}})],$$

$$\nu_1^{(1)}(\tau)=\frac{1}{\pi}\frac{\partial}{\partial\tau}[K(\sqrt{1-e^{-8\tau}})].$$

It is convenient to introduce the notation

$$\kappa=\sqrt{1-e^{-8\tau}},\qquad \tilde{\kappa}=\sqrt{1-\kappa^2}=e^{-4\tau} \tag{5.4}$$

and to use the differentiation rules [53]

$$\frac{dK(\kappa)}{d\kappa}=\frac{E(\kappa)}{\kappa\tilde{\kappa}^2}-\frac{K(\kappa)}{\kappa},\quad \frac{dE(\kappa)}{d\kappa}=\frac{E(\kappa)-K(\kappa)}{\kappa}, \tag{5.5}$$

where

$$E(\kappa)=\int_0^{\pi/2}d\alpha\sqrt{1-\kappa^2\sin^2\alpha}$$

is the complete elliptic integral of the second kind [53]. Finally we get

$$\xi_1^{(1)}=\frac{2}{\pi}\frac{E(\kappa)+\tilde{\kappa}K(\kappa)}{1+\tilde{\kappa}},\qquad \eta_1^{(1)}=-\frac{2}{\pi}\frac{E(\kappa)-\tilde{\kappa}K(\kappa)}{1-\tilde{\kappa}}. \tag{5.6}$$

The general structure of solutions to Eqs. (4.24) and (4.25) is

$$\eta_1^{(n)} = \frac{2}{\pi} y\{[\tilde{\kappa} g_-^{(n)}(y) + \tilde{\kappa}^2 g_+^{(n)}(y)]K(\kappa) - [f_+^{(n)}(y) + \tilde{\kappa} f_-^{(n)}(y)]E(\kappa)\},$$

$$\xi_1^{(n)} = \frac{2}{\pi} y\{[\tilde{\kappa} g_-^{(n)}(y) - \tilde{\kappa}^2 g_+^{(n)}(y)]K(\kappa) + [f_+^{(n)}(y) - \tilde{\kappa} f_-^{(n)}(y)]E(\kappa)\},$$

where $y \equiv \kappa^{-2}$ and functions $f_\pm^{(2j+1)}(y), g_\pm^{(2j+1)}(y)$ are polynomials of the degree j. In particular,

$$f_+^{(1)} = f_-^{(1)} = g_+^{(1)} = g_-^{(1)} = 1,$$

$$f_+^{(3)} = 8y - 7, \quad f_-^{(3)} = 8y - 1,$$

$$g_+^{(3)} = 8y - 3, \quad g_-^{(3)} = 8y - 5,$$

while the polynomials with $n \geqslant 5$ have more complicated explicit expressions resulting from the recurrence relations (4.24) and (4.25). However, we do not need these expressions to calculate the total numbers of photons produced in the principal mode and in all the modes.

VI. RATE OF PHOTON GENERATION

Putting expressions (5.6) into Eq. (4.26), we obtain the following explicit formula for the photon generation rate in the first cavity mode:

$$\frac{d\mathscr{P}_1}{dt} = \frac{4\varepsilon\omega_1}{\pi^2} \frac{E^2(\kappa) - \tilde{\kappa}^2 K^2(\kappa)}{\kappa^2}. \quad (6.1)$$

If $\tau \ll 1$, then $\kappa = \sqrt{8\tau} \ll 1$, and the power series expansions of the elliptic integrals [53]

$$K(\kappa) = \frac{\pi}{2}\left\{1 + \frac{1}{4}\kappa^2 + \frac{9}{64}\kappa^4 + \cdots\right\},$$

$$E(\kappa) = \frac{\pi}{2}\left\{1 - \frac{1}{4}\kappa^2 - \frac{3}{64}\kappa^4 - \cdots\right\}$$

lead to the linear law

$$\frac{d\mathscr{P}_1}{dt} \approx \frac{1}{2}\varepsilon^2\omega_1^2 t, \quad \varepsilon\omega_1 t \ll 1. \quad (6.2)$$

This expression coincides exactly with the result of Ref. [34], which was obtained in the framework of a quite different approach.

In the long-time limit $\tau \gg 1$ we have $\tilde{\kappa} \to 0$. Then the leading terms of the asymptotic expansions [53]

$$K(\kappa) \approx \ln\frac{4}{\tilde{\kappa}} + \frac{1}{4}\left(\ln\frac{4}{\tilde{\kappa}} - 1\right)\tilde{\kappa}^2 + \cdots,$$

$$E(\kappa) \approx 1 + \frac{1}{2}\left(\ln\frac{4}{\tilde{\kappa}} - \frac{1}{2}\right)\tilde{\kappa}^2 + \cdots$$

result in the constant rate of photon creation

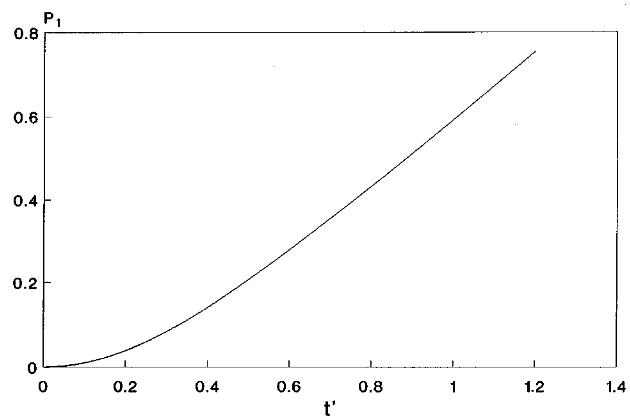

FIG. 1. Number of photons created in the first mode versus the dimensionless time $t' \equiv \tau$.

$$\frac{d\mathscr{P}_1}{dt} = \frac{4\varepsilon\omega_1}{\pi^2}, \quad \varepsilon\omega_1 t \gg 1. \quad (6.3)$$

This dependence was found earlier in [12,35], where an asymptotical solution of the Moore equation (1.4) was obtained. However, in that papers a factor 2 was missed and the answer was half that of Eq. (6.3).

The total number of photons in the first mode can be obtained by integrating Eq. (6.1). Taking into account the relation

$$d\tau = \frac{\kappa d\kappa}{4\tilde{\kappa}^2}, \quad (6.4)$$

one can check that the expression on the right-hand side of Eq. (6.1) is a complete differential due to Eq. (5.5). So the final result appears surprisingly simple:

$$\mathscr{P}_1(\kappa) = \frac{2}{\pi^2}E(\kappa)K(\kappa) - \frac{1}{2}. \quad (6.5)$$

The plot of this dependence, given in Fig. 1, shows how the initial parabola

$$\mathscr{P}_1(t) = \frac{1}{4}(\varepsilon\omega_1 t)^2, \quad \varepsilon\omega_1 t \ll 1 \quad (6.6)$$

is smoothly transformed into the linear dependence

$$\mathscr{P}_1(t) = \frac{4}{\pi^2}\varepsilon\omega_1 t + \frac{2}{\pi^2}\ln 4 - \frac{1}{2} + \mathscr{O}(\tau e^{-8\tau}) \quad (6.7)$$

for $\varepsilon\omega_1 t \gg 1$. Actually the asymptotical formula (6.7) works quite well beginning at $\tau \approx 0.5$.

Using Eqs. (4.31) and (5.6) we find the second time derivative of the total amount of photons created in all the modes:

$$\frac{d^2\mathscr{P}}{d\tau^2} = \frac{16}{\pi^2\kappa^4}\{[E(\kappa) - \tilde{\kappa}^2 K(\kappa)]^2 + \tilde{\kappa}^2[E(\kappa) - K(\kappa)]^2\}. \quad (6.8)$$

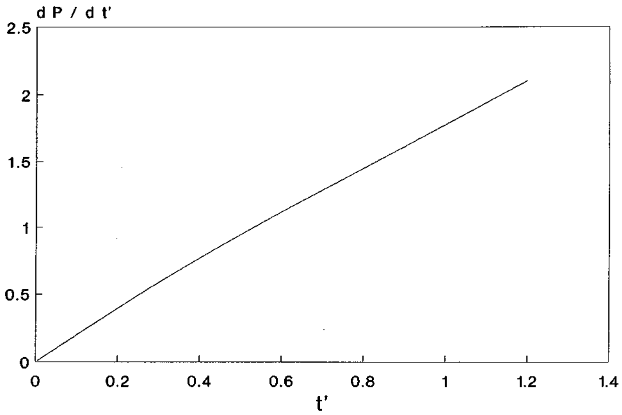

FIG. 2. Total rate of photon generation versus the dimensionless time $t' \equiv \tau$.

It decreases from the value $\frac{1}{2}(\varepsilon\omega_1)^2$ at $t=0$ [it is seen immediately from Eq. (4.31)] to the value $(2\varepsilon\omega_1/\pi)^2$ at $t\to\infty$ (i.e., its variation is only 20%). To integrate Eq. (6.8) we use, besides Eq. (6.4), the relations [see, e.g., Eqs. (5.113.4) and (5.113.5) from [53]]

$$\int [K(\kappa)-E(\kappa)]\frac{d\kappa}{\kappa^2}=\frac{1}{\kappa}[E(\kappa)-\tilde{\kappa}^2K(\kappa)],$$

$$\int [E(\kappa)-\tilde{\kappa}^2K(\kappa)]\frac{d\kappa}{\kappa^2\tilde{\kappa}^2}=\frac{1}{\kappa}[K(\kappa)-E(\kappa)].$$

The result is

$$\frac{d\mathscr{P}}{d\tau}=\frac{4}{\pi^2\kappa^2}[E(\kappa)-\tilde{\kappa}^2K(\kappa)][K(\kappa)-E(\kappa)]. \quad (6.9)$$

The dependence on τ appears to be practically linear (see Fig. 2). For $\tau\ll 1$ we have $d\mathscr{P}/d\tau=2\tau=d\mathscr{P}_1/d\tau$, i.e., the photons are created only in the first mode. But if $\tau\gg 1$, then $d\mathscr{P}/d\tau=16\tau/\pi^2\gg d\mathscr{P}_1/d\tau$.

Equation (6.9) can be also integrated analytically. The following formula holds due to Eqs. (5.5) and (6.4):

$$\mathscr{P}=\frac{1}{\pi^2}\left[\left(1-\frac{1}{2}\kappa^2\right)K^2(\kappa)-E(\kappa)K(\kappa)\right]. \quad (6.10)$$

Thus we have quadratic dependences of the total amount of photons on time in both the short-time and long-time limits: $\mathscr{P}=\tau^2$ at $\tau\ll 1$ and

$$\mathscr{P}=\frac{8}{\pi^2}\tau^2+\frac{4\tau}{\pi^2}(\ln 4-1)-\frac{\ln 4}{\pi^2}(1-\ln 2)+O(\tau e^{-8\tau})$$

at $\tau\gg 1$. Figure 3 shows this parabolic law explicitly. Moreover, the plot of $\mathscr{P}$ versus τ^2 is visually undistinguishable from the straight line. However, while at $t\to 0$ the total number almost coincides with that in the first mode, $\mathscr{P}\approx\mathscr{P}_1$, at $t\to\infty$ we have $\mathscr{P}_1\ll\mathscr{P}$.

The energy of the principal mode is simply $\omega_1\mathscr{P}_1$. For the total energy in all the modes we have the formula [see Eq. (2.10)]

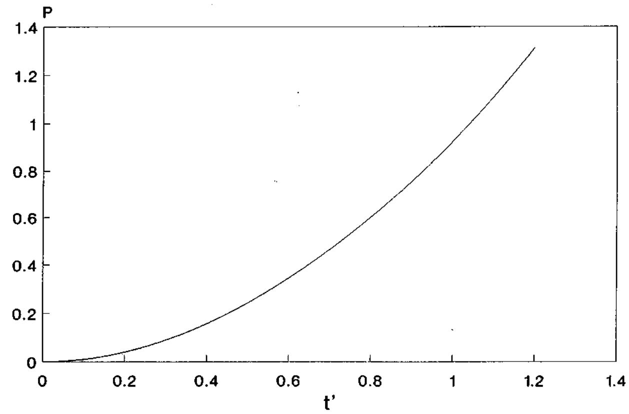

FIG. 3. Total number of photons versus the dimensionless time $t' \equiv \tau$.

$$\mathscr{E}=\omega_1\sum_{m=1}^{\infty} m\mathscr{P}_m=\omega_1\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2}{n}|\eta_m^{(n)}|^2. \quad (6.11)$$

It is remarkable that this double sum can be calculated exactly *without a knowledge of the explicit expressions for coefficients* $\eta_m^{(n)}$. Let us introduce the function

$$S^{(n)}=\sum_{m=1}^{\infty} m^2|\eta_m^{(n)}|^2.$$

Differentiating it with respect to the slow time τ and using Eqs. (3.4) and (3.5) we get (remember that all the functions $\eta_m^{(n)}$ and $\xi_m^{(n)}$ are real)

$$\dot{S}^{(n)}=-2\xi_1^{(n)}\eta_1^{(n)}-4\sum_{m=1}^{\infty} m(m+2)\eta_m^{(n)}\eta_{m+2}^{(n)}.$$

Differentiating this equation once more, we get

$$\ddot{S}^{(n)}=2\xi_1^{(n)}\dot{\eta}_1^{(n)}-2\dot{\xi}_1^{(n)}\eta_1^{(n)}+4[\xi_1^{(n)}]^2-4[\eta_1^{(n)}]^2+16S^{(n)}.$$

The total energy equals $\mathscr{E}=\omega_1\Sigma_{n=1}^{\infty}S^{(n)}/n$. But one can check, using relations (4.19)–(4.20), (4.24) and (4.25), the identity

$$\sum_{n=1}^{\infty}\frac{1}{n}(\dot{\xi}_1^{(n)}\eta_1^{(n)}-\xi_1^{(n)}\dot{\eta}_1^{(n)})=1.$$

Combining it with identity (4.28), we arrive at a simple equation

$$\ddot{\mathscr{E}}-16\mathscr{E}=2\omega_1.$$

Its solution, satisfying the evident initial conditions $\mathscr{E}(0)=\dot{\mathscr{E}}(0)=0$, reads

$$\mathscr{E}(\tau)=\frac{1}{4}\omega_1\sinh^2(2\tau). \quad (6.12)$$

The total energy grows much faster than the total number of photons. Therefore we have a rapid pumping in the high-frequency modes at the expense of the energy of the low-frequency field oscillations. A similar phenomenon in the case of a classical field in a 1D cavity was discovered in [36] in the framework of a different approach, namely, with the aid of an explicit account of the repeated wave reflections from an oscillating boundary. The physical mechanism of such a pumping is the Doppler up-shift of the field reflected from the mirror oscillating at the frequency coinciding with the interval between the equidistant eigenfrequencies.

VII. A 3D RESONANCE CAVITY: REDUCING TO A SINGLE PARAMETRIC OSCILLATOR

Now let us proceed to the three-dimensional case. For definiteness we choose a rectangular cavity with dimensions L_x, L_y, L_z (briefly designated by the symbol $\{L\}$). If these dimensions do not depend on time, each field mode is determined by three integers m,n,l, responsible for the eigenfrequency

$$\omega_{mnl}=\pi[(m/L_x)^2+(n/L_y)^2+(l/L_z)^2]^{1/2}, \tag{7.1}$$

and by two orthogonal directions of polarization. In order to simplify the exposition and to get rid of extra unessential indices, let us consider the case when $L_z \ll L_x \sim L_y$. Then the frequencies with $l \neq 0$ are much greater than those with $l=0$. It is clear that the interaction between low- and high-frequency modes in the nonstationary case is weak. Consequently, studying the excitation of the *lowest modes* we may confine ourselves to the case of $l=0$. Then the only possible polarization of the vector potential is along the z axis, so the low-frequency part of the *Heisenberg field operator* at $t<0$ reads

$$\hat{A}_z(x,y,t<0)=\sum_{\mathbf{n}} (2\pi/\omega_{\mathbf{n}})^{1/2}\psi_{\mathbf{n}}(x,y|\{L\}) \times[\hat{b}_{\mathbf{n}}\exp(-i\omega_{\mathbf{n}}t)+\hat{b}_{\mathbf{n}}^{\dagger}\exp(i\omega_{\mathbf{n}}t)]. \tag{7.2}$$

The difference from the similar expression (2.1) is that now the suffix n is replaced by its "vector" counterpart $\mathbf{n}=(m,n)$ and the function $\psi_{\mathbf{n}}(x,y|\{L\})$ depends on two space coordinates

$$\psi_{\mathbf{n}}(x,y|\{L\})=2(L_xL_yL_z)^{-1/2}\sin\frac{m\pi x}{L_x}\sin\frac{n\pi y}{L_y}.$$

The coefficients in Eq. (7.2) are chosen again in correspondence with the standard form of the field Hamiltonian (2.2).

Now let the dimension L_x depend on time according to the given law $L(t)$. To satisfy the boundary conditions

$$A_z|_{x=0}=A_z|_{x=L(t)}=A_z|_{y=0}=A_z|_{y=Ly}=0$$

we write the field operator at $t>0$ in the same functional form (7.2), but with the time-dependent parameter $L(t)$:

$$\hat{A}_z(x,y,t)=2\sqrt{\pi}\sum_{\mathbf{n}} \psi_{\mathbf{n}}(x,y|L(t),L_y)\hat{Q}_{\mathbf{n}}(t). \tag{7.3}$$

In the stationary case the operators $\hat{Q}_{\mathbf{n}}(t)$ coincide with the (coordinate) quadrature components of the field mode operators. Putting (7.3) into the wave equation

$$\partial^2 A_z/\partial t^2-\Delta A_z=0,$$

we arrive at an equation similar to Eq. (2.5). The difference is that now all the indices are "two-vectors" and the frequencies are given by Eq. (7.1) with $l=0$ and $L(t)$ instead of L_x. The *constant* numerical coefficients $g_{\mathbf{kj}}$ are given by a formula

$$g_{\mathbf{kj}}=L\int_0^L dx\int_0^{L_y}dy\int_0^{L_z}dz\ \psi_{\mathbf{j}}(\mathbf{r}|L)\frac{\partial\psi_{\mathbf{k}}(\mathbf{r}|L)}{\partial L}.$$

Their explicit form is more complicated now than the simple formula (2.6). However, these coefficients remain antisymmetrical $g_{\mathbf{kj}}=-g_{\mathbf{jk}}$, due to the normalization of functions $\psi_{\mathbf{k}}$,

$$\int_0^{\{L\}}d\mathbf{r}\ \psi_{\mathbf{m}}\psi_{\mathbf{n}}=\delta_{\mathbf{mn}},$$

and due to zero boundary conditions at $x=L$. (Moreover, they do not depend on the cavity dimensions.)

Although we use the same notation as in the 1D case, the operators $\hat{Q}_{\mathbf{n}}(t)$ in Eq. (7.3) differ from their analogs in a similar decomposition (2.4). In Sec. II we dealt with complex functions $Q_k^{(n)}(t)$, whereas now $\hat{Q}_{\mathbf{n}}(t)$ means the Hermitian operator coinciding with the ("coordinate") *quadrature component* of the field mode operator.

Supposing again that the wall oscillates at twice the eigenfrequency of some unperturbed mode

$$L(t)=L_0[1-\varepsilon\cos(2\omega_{\mathbf{m}}t)], \quad |\varepsilon|\ll 1$$

[we have changed the phase of vibrations in comparison with Eq. (3.1) because of technical reasons], we look for the solution in the form (carets over operators are omitted)

$$Q_{\mathbf{k}}(t)=\xi_{\mathbf{k}}(\varepsilon t)\exp(-i\omega_{\mathbf{k}}t)+\eta_{\mathbf{k}}(\varepsilon t)\exp(i\omega_{\mathbf{k}}t). \tag{7.4}$$

But now *all the terms* on the right-hand side of Eq. (2.5) disappear after averaging over fast oscillations, since the spectrum $\omega_{\mathbf{j}}$ *is not equidistant*. Indeed, the first and second sums on the right-hand side do not contain functions $Q_{\mathbf{k}}$ due to the antisymmetricity of coefficients $g_{\mathbf{kj}}$, whereas the last sum is proportional to $\lambda^2 \sim \varepsilon^2$. Consequently, after the multiplication by the proper exponential functions, the right-hand side will consist of the terms containing factors such as $\exp(i[\pm\omega_{\mathbf{j}}\pm\omega_{\mathbf{k}}\pm 2\omega_{\mathbf{m}}]t)$ with $\mathbf{j}\neq\mathbf{k}$. After averaging, all these terms turn into zero.

Thus, in this special (*nonadiabatic*) case the problem is reduced to that of a one-dimensional parametric oscillator with the time dependence of the eigenfrequency in the form

$$\omega(t)=\omega_0[1+2\gamma\cos(2\omega_0 t)], \tag{7.5}$$

$\omega_0 \equiv \omega_{mn}$ being the unperturbed eigenfrequency of the resonance mode. The frequency modulation depth γ is related to the cavity length modulation depth ε as

$$\gamma=\frac{1}{2}\varepsilon[1+(nL_0/mL_y)^2]^{-1/2}.$$

At this point it is convenient to abandon the Heisenberg picture and to proceed to the Schrödinger representation. Of course, both representations are equivalent, as soon as the field problem has been reduced to studying a finite-dimensional quantum system. However, the most of numerous investigations of the time-dependent quantum oscillator, since Husimi's paper [55], were performed in the Schrödinger picture. So it is natural to use the known results. According to [47,55,56], all the characteristics of the *quantum* oscillator are determined completely by the *complex* solution of the *classical* oscillator equation of motion

$$\ddot{u}+\omega^2(t)u=0, \tag{7.6}$$

satisfying the normalization condition

$$\dot{u}u^*-\dot{u}^*u=2i. \tag{7.7}$$

Let us assume that the function $\omega(t)$ takes the constant value ω_0 at $t\leqslant 0$ and $t>t_f>0$. Moreover, it is convenient to choose the initial conditions for the u-function as

$$u(0)=1/\sqrt{\omega_0}, \quad \dot{u}(0)=i\sqrt{\omega_0}. \tag{7.8}$$

Then the quantum-mechanical average number of photons created from the ground state due to the time dependence of the frequency in the interval of time $0<t<t_f$ is given by the formula

$$\langle n\rangle=\frac{1}{4\omega_0}(|\dot{u}|^2+\omega_0^2|u|^2)-\frac{1}{2}. \tag{7.9}$$

Looking for the solution of Eq. (7.6) in the parametric resonance case (7.5) in the form

$$u(t)=\frac{1}{\sqrt{\omega_0}}[\xi(t)e^{i\omega_0 t}+\eta(t)e^{-i\omega_0 t}] \tag{7.10}$$

[the opposite signs in the arguments of the exponential functions in Eqs. (7.4) and (7.10) are due to the different representations: the former equation is written in the Heisenberg picture, while the latter in the Schrödinger one] and using the method of averaging over fast oscillations, one can easily obtain the first-order differential equations for the amplitudes (provided that $|\gamma|\ll 1$)

$$\dot{\xi}=i\omega_0\gamma\eta, \quad \dot{\eta}=-i\omega_0\gamma\xi. \tag{7.11}$$

Their solutions satisfying the initial conditions (7.8) (up to the terms of the order of γ) read [57]

$$\xi(t)=\cosh(\omega_0\gamma t), \quad \eta(t)=-i\sinh(\omega_0\gamma t). \tag{7.12}$$

Due to Eqs. (7.9), (7.10), and (7.12), the average number of photons (and the total energy in the cavity) grows exponentially in time:

$$\langle n\rangle=|\eta|^2=\sinh^2(\omega_0\gamma t). \tag{7.13}$$

It is well known that the initial vacuum state of the oscillator is transformed into the *squeezed vacuum state* if the frequency depends on time (see, e.g., reviews [47,58] and numerous references therein). Moreover, looking at Eq. (7.13) one can immediately recognize the combination $\omega_0\gamma t$ as the so-called *squeezing parameter*. Therefore the probability to registrate n photons exhibits typical oscillations

$$\mathscr{P}_{2m}=\frac{[\tanh(\omega_0\gamma t)]^{2m}}{\cosh(\omega_0\gamma t)}\frac{(2m)!}{(2^m m!)^2}, \quad \mathscr{P}_{2m+1}=0. \tag{7.14}$$

This distribution possesses the photon-number variance $\sigma_n=\frac{1}{2}\sinh^2(2\omega_0\gamma t)$. Similar formulas for the amount of photons created in a cavity filled with a medium with a time-dependent dielectric permeability (and stationary boundaries) were found in [48].

VIII. INTERACTION WITH A PROBE OSCILLATOR

To detect the photons created inside the cavity due to the motion of the wall, at least two approaches are conceivable. The first one is to make small holes in the walls, to accumulate a sufficient amount of photons inside the cavity during some time interval (about 1 s, as will be shown in Sec. IX), then to pass a beam of Rydberg atoms through the cavity (eigenfrequencies of a cavity with dimensions of several centimeters have the order of 10 GHz and correspond to the transitions between the excited levels of the Rydberg atoms with the principal quantum number $n\sim 100$), and to use the elaborated methods [1–4] of extracting the information on the field inside the cavity, comparing the states of the atoms in the input and output beams.

Another way is to use some trap when a detector occurs inside the cavity all the time. Then the interaction between the field and the detector can change significantly the dynamics of the field excitation by the vibrating wall. Here we demonstrate this effect in the framework of a simplified model, when a *harmonic oscillator* tuned to the frequency of the resonant mode is placed at the point of maximum of the amplitude mode function $\psi_{mn}(x,y|\{L\})$ in the 3D rectangular cavity discussed Sec. VII.

Assuming the interaction between the oscillator and the field to be described by means of the standard minimal coupling term $-(e/mc)\mathbf{p}\cdot\mathbf{A}$, we arrive at the following two-dimensional Hamiltonian governing the evolution of the coupled "field oscillator plus detector" system:

$$H=\frac{1}{2}[P^2+\omega^2(t)Q^2+p^2+\omega_0^2q^2-4\omega_0\kappa pQ]. \tag{8.1}$$

Here P,Q are the quadrature components of the field oscillator and p,q are those of the probe oscillator. We neglect the interaction with nonresonant modes since it is reasonable to suppose that under the resonance conditions their contribution is not essential at $\varepsilon\ll 1$.

In general, the dimensionless coupling coefficient κ must depend on time due to the decomposition (7.3). However, since this coefficient is small, its variations of the order of $\varepsilon\kappa$ can be neglected in comparison with the relative variation of the eigenfrequency $\delta\omega/\omega\sim\varepsilon$. So κ is assumed to be constant.

Suppose that the lowest cavity mode is resonant. Then one can evaluate the dimensionless coupling constant as $\kappa \sim (e^2/2\pi mc^2 L)^{1/2}$ (here we return to the dimensional variables). The maximum value of the parameter ε is (see Sec. IX) $\varepsilon_{\max} \sim \delta_{\max} v_s/2\pi c$, where $\delta_{\max} \sim 0.01$ is the maximal possible relative deformation in the material of the wall and $v_s \sim 5\times 10^3$ m/s is the sound velocity inside the wall. Then the ratio γ/κ cannot exceed the value $\delta_{\max}(mv_s^2 L/8\pi e^2)^{1/2} \sim 0.05$ for $L \sim 1$ cm and $m \approx$ the mass of electron (for these parameters $\kappa \sim 2\times 10^{-7}$). Consequently, one may believe that in the real conditions $\gamma/\kappa \ll 1$.

In the time-independent case $\omega(t) = \text{const} = \omega_0$, we have two eigenfrequencies

$$\omega_{\pm} = \omega_0(1 \pm \kappa) \tag{8.2}$$

(provided that $|\kappa| \ll 1$). Let us assume that the wall vibrates exactly at twice the lower frequency ω_-:

$$\omega(t) = \omega_0[1 + 2\gamma \cos(2\omega_- t)]. \tag{8.3}$$

Then the lower and upper modes practically do not interact in the limit of $\gamma \ll \kappa$.

The propagator of the Schrödinger equation with the Hamiltonian (8.1) can be calculated in the framework of the general theory of multidimensional quantum systems with arbitrary quadratic Hamiltonians, first proposed in [59] and exposed in detail, e.g., in [47]. It reads (see Appendix B for the details of calculations)

$$G(\mathbf{x}_2, \mathbf{x}_1, t) = \frac{1}{\pi i \sqrt{2\Delta}} \exp\left\{ -\frac{i}{2\Delta}[\mathbf{x}_2\mathbf{A}\mathbf{x}_2 - 2\mathbf{x}_2\mathbf{B}\mathbf{x}_1 + \mathbf{x}_1\mathbf{C}\mathbf{x}_1]\right\}. \tag{8.4}$$

Hereafter it is assumed, for the sake of simplicity, that the quadrature component variables are chosen in such dimensionless forms that one may write formally $\hbar = \omega_0 = 1$. The vector $\mathbf{x}_1 = (Q_1, q_1)$ corresponds to the initial point in the two-dimensional coordinate space and $\mathbf{x}_2 = (Q_2, q_2)$ to the final one. $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are 2×2 matrices

$$\mathbf{A} = \left\| \begin{matrix} -\sinh\mu\,\cos\varphi - \cosh\mu\,\sin\Phi & \sinh\mu\,\sin\varphi - \kappa\Delta \\ \sinh\mu\,\sin\varphi - \kappa\Delta & \sinh\mu\,\cos\varphi - \cosh\mu\,\sin\Phi \end{matrix} \right\|,$$

$$\mathbf{B} = \left\| \begin{matrix} -\sinh\mu\,\cos\varphi_- - \cosh\mu\,\sin\varphi_- - \sin\varphi_+ & \cos\varphi_+ - \cosh\mu\,\cos\varphi_- - \sinh\mu\,\sin\varphi_- \\ -\cos\varphi_+ + \cosh\mu\,\cos\varphi_- - \sinh\mu\,\sin\varphi_- & \sinh\mu\,\cos\varphi_- - \cosh\mu\,\sin\varphi_- - \sin\varphi_+ \end{matrix} \right\|,$$

$$\mathbf{C} = \left\| \begin{matrix} -\sinh\mu\,\cos\Phi - \cosh\mu\,\sin\Phi & \kappa\Delta - \sinh\mu\,\sin\Phi \\ \kappa\Delta - \sinh\mu\,\sin\Phi & \sinh\mu\,\cos\Phi - \cosh\mu\,\sin\Phi \end{matrix} \right\|.$$

The new scalar functions are defined as (here it is convenient to retain the unperturbed frequency ω_0)

$$\Phi = (\omega_+ + \omega_-)t = 2\omega_0 t,$$

$$\varphi = (\omega_+ - \omega_-)t = 2\omega_0\kappa t = \kappa\Phi,$$

$$\mu = \gamma\omega_0 t, \quad \Delta = 1 - \cosh\mu\,\cos\Phi.$$

Evidently, $\Phi \gg \varphi \gg \mu$, so the higher-order terms were neglected in the expressions for matrices **A,B,C** in accordance with these relations.

Let the initial state be the ground one, both for the field and the probe oscillators,

$$\psi(Q,q,0) = \sqrt{\frac{1}{\pi}} \exp\left[-\frac{1}{2}(Q^2 + q^2)\right]. \tag{8.5}$$

Then, after some tedious algebra, related to the calculation of a Gaussian integral for the convolution of the propagator with the initial wave function, one can arrive at the expression for the wave function of the coupled "field plus probe oscillator" system at $t > 0$ in the form of a Gaussian wave packet,

$$\psi(Q,q,t) = \sqrt{\frac{1}{\pi\cosh\mu}} \exp\left(-it - \frac{1}{2}[a(t)Q^2 + b(t)q^2 - 2c(t)qQ]\right), \tag{8.6}$$

with the coefficients

$$a(t) = 1 + i\,\tanh(\mu)e^{-2i\varphi_-} - i\kappa e^{i\Phi}\{\tanh(\mu)e^{-i\Phi} \times [1 + \tanh(\mu)\,\sin(\Phi)e^{i\varphi}] - \sin(\varphi)\},$$

$$b(t) = 1 - i\,\tanh(\mu)e^{-2i\varphi_-} - i\kappa e^{i\Phi}\{\tanh(\mu)e^{-i\Phi} \times [1 - \tanh(\mu)\,\sin(\Phi)e^{i\varphi}] + \sin(\varphi)\},$$

$$c(t) = \tanh(\mu)e^{-2i\varphi_-} + i\kappa[1 - \cos(\varphi)e^{-i\Phi} + i\,\tanh^2(\mu)\,\sin(\Phi)e^{i(\varphi - 2\Phi)}].$$

In all the formulas above, the terms of the order of κ^2 were neglected, as well as the terms proportional to γ (except, of course, the arguments of the hyperbolic functions).

Equation (8.6) shows that the coupled system results in a *two-mode squeezed state* at $t > 0$. In the short-time limit

$\mu \ll 1$ the squeezing effect is very small, since coefficients a and b are close to unity, while c is of the order of κ:

$$a(t)=1+i\kappa\,\sin(\varphi)e^{i\Phi},$$

$$b(t)=1-i\kappa\,\sin(\varphi)e^{i\Phi},$$

$$c(t)=i\kappa[1-\cos(\varphi)e^{-i\Phi}].$$

On the contrary, if $\mu \gtrsim 1$, then all the terms proportional to κ can be neglected, so that

$$a(t)=1+i\chi, \quad b(t)=1-i\chi, \quad c(t)=\chi,$$

$$\chi=\tanh(\mu)e^{-2i\varphi_-}.$$

The properties of any Gaussian state are determined completely by its *covariance matrix*

$$\mathbf{M}=|\mathscr{M}_{\alpha\beta}|=\begin{Vmatrix} \mathbf{M}_{\pi\pi} & \mathbf{M}_{\pi x} \\ \mathbf{M}_{x\pi} & \mathbf{M}_{xx} \end{Vmatrix},$$

$$\mathscr{M}_{\alpha\beta}=\frac{1}{2}\langle \hat{z}_\alpha \hat{z}_\beta+\hat{z}_\beta \hat{z}_\alpha\rangle, \quad \alpha,\beta=1,2,3,4,$$

where the four-dimensional (in the present case) vector $\mathbf{z}$ is defined as $\mathbf{z}=(\pi,\mathbf{x})=(P,p,Q,q)$ (evidently, $\langle \mathbf{z}\rangle=\mathbf{0}$ in the case under study). Using the general formulas for multidimensional Gaussian states given in [47] (see Appendix C), one can obtain the following explicit expressions for the two-dimensional blocks of matrix $\mathbf{M}$ in the most interesting *long-time* limit ($\mu \gtrsim 1$):

$$\mathbf{M}_{\pi\pi}=\frac{1}{2}\cosh^2\mu \begin{Vmatrix} 1+\tanh\mu\,\sin\phi & -\tanh\mu\,\cos\phi \\ -\tanh\mu\,\cos\phi & 1-\tanh\mu\,\sin\phi \end{Vmatrix},$$

$$\mathbf{M}_{xx}=\frac{1}{2}\cosh^2\mu \begin{Vmatrix} 1-\tanh\mu\,\sin\phi & \tanh\mu\,\cos\phi \\ \tanh\mu\,\cos\phi & 1+\tanh\mu\,\sin\phi \end{Vmatrix},$$

$$\mathbf{M}_{\pi x}=\frac{1}{4}\sinh(2\mu) \begin{Vmatrix} -\cos\phi & -\tanh\mu-\sin\phi \\ \tanh\mu-\sin\phi & \cos\phi \end{Vmatrix},$$

where $\phi=2\varphi_-$. Consequently, there exists a strong correlation between the field and probe oscillators in the long-time limit. For instance, the correlation coefficient between the quadrature components reads

$$r_{qQ}\equiv\frac{\langle qQ\rangle}{\sqrt{\langle q^2\rangle\langle Q^2\rangle}}=\frac{\sinh\mu\,\cos\phi}{\sqrt{1+(\sinh\mu\,\cos\phi)^2}}.$$

(If $\phi \approx \pi/2$, this coefficient, as well as other analogous elements of the covariance matrix, does not turn exactly into zero; in such a special case $r_{qQ}\sim\kappa$, due to neglected terms of the order of κ.)

It is clear that the *density matrix* of the probe oscillator [which is obtained from the density matrix of the total system $\rho(Q,q;Q',q')=\psi(Q,q)\psi^*(Q',q')$ by setting $Q=Q'$ and integrating over Q] also has the Gaussian form. Its properties are determined completely by the reduced covariance matrix (accidentally, it coincides with $\mathbf{M}_{xx}$ when $\omega_0=1$)

$$\mathbf{M}_{pr}=\frac{1}{2}\cosh^2\mu \begin{Vmatrix} 1-\tanh\mu\,\sin\phi & \tanh\mu\,\cos\phi \\ \tanh\mu\,\cos\phi & 1+\tanh\mu\,\sin\phi \end{Vmatrix}. \tag{8.7}$$

A similar matrix for the field oscillator can be obtained from Eq. (8.7) by means of changing the sign of parameter μ. As shown in [60], the photon statistics in Gaussian one-mode states is determined completely by two invariants of the covariance matrix

$$d=\det\mathbf{M}, \quad T=\mathrm{Tr}\mathbf{M}.$$

Evidently, the parameter T is twice the energy of quantum fluctuations. The parameter d characterizes the *degree of purity* of the quantum state due to the relation [61]

$$\mathrm{Tr}\hat{\rho}^2=\frac{1}{2\sqrt{d}},$$

where $\hat{\rho}$ is the statistical operator of the system. The *degree of squeezing*, i.e., the minimal possible value of the variance of some quadrature component, normalized by its vacuum value [$\hbar/(2m\omega)$ for an oscillator with mass m and frequency ω], is determined jointly by both parameters T and d according to the relation [62]

$$s\equiv 2\langle q^2\rangle=T-\sqrt{T^2-4d}.$$

Both subsystems have identical invariants

$$T=4d=\cosh^2\mu,$$

so for $\mu \gtrsim 1$ they appear in highly mixed quantum states. As to the degree of squeezing, it turns out rather moderate

$$s=e^{-\mu}\cosh\mu=\frac{1}{2}(1-e^{-2\mu}).$$

The average number of quanta in each subsystem equals

$$\langle n\rangle=\frac{1}{2}(T-1)=\frac{1}{2}\sinh^2\mu,$$

i.e., half that in the free mode case. The variance of the number of quanta (photons) reads [60]

$$\sigma_n\equiv\langle n^2\rangle-\langle n\rangle^2=\frac{1}{4}(2T^2-4d-1)=\frac{1}{4}\sinh^2\mu\,\cosh(2\mu).$$

The so-called Mandel parameter

$$\mathscr{Q}\equiv\frac{\sigma_n}{\langle n\rangle}=\frac{1}{2}\cosh(2\mu)$$

turns out to be much greater than unity for $\mu \gg 1$, indicating that the photon statistics is highly super-Poissonian. The photon distribution function can be easily expressed in terms of the Legendre polynomials, on the basis of the general formula (C2) of Appendix C:

$$\mathscr{P}_n=\frac{2(iz)^n}{\sqrt{1+3\cosh^2\mu}}P_n(-iz), \tag{8.8}$$

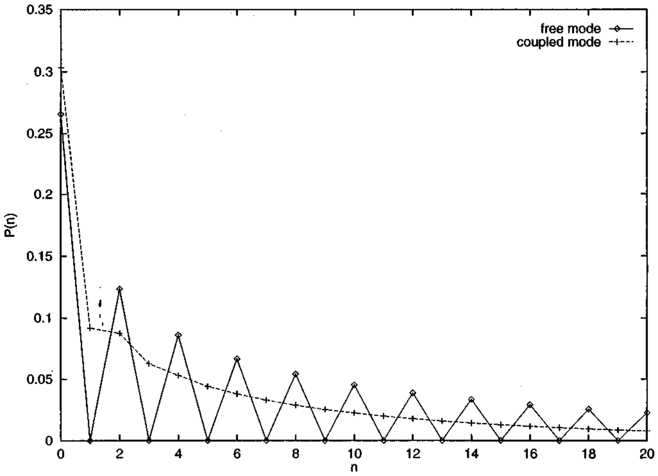

FIG. 4. Plots of the photon distribution functions in the free-mode and coupled-mode cases for $\mu=2$.

where

$$z=\frac{\sinh\mu}{\sqrt{1+3\cosh^2\mu}}.$$

Actually the right-hand side of Eq. (8.8) is a polynomial of degree n with respect to the variable z^2, due to the recurrence relation

$$n\mathscr{P}_n=z^2[(2n-1)\mathscr{P}_{n-1}+(n-1)\mathscr{P}_{n-2}].$$

If $\mu\gg1$, then $z^2\approx1/3$. Figure 4 demonstrates the difference between the free-mode distribution (7.14) and the coupled-mode one (8.8) for $\mu=2$, when $\langle n\rangle\approx13$ in the free-mode case.

Since the expressions for the variances given above were obtained in the framework of the *approximate* method of slowly varying amplitudes, it is desirable to check their accuracy. For this purpose it is natural to check the conservation of the *quantum universal invariants*, first introduced in Ref. [63] and discussed in detail in [47,61] (later similar constructions in classical Hamiltonian dynamics, with applications to the particle beam propagation problems, were considered in [64–66]). As proven in those studies, for any Hamiltonian that is *quadratic* with respect to the coordinates and momenta operators (and more generally, for any Hamiltonian, which can be written as a linear combination of generators of a semisimple Lie algebra [47,61,63]), there exist certain combinations of the second- and higher-order moments, which are conserved in time *independently on concrete time dependences of the coefficients* in the corresponding quadratic or linear forms. For instance, in the two-dimensional case (considered in this section) there are two independent second-order invariants. The first one reads (provided that the Hamiltonian does not contain linear terms, as in the case under study)

$$\mathscr{D}_2=\Delta_{11}+\Delta_{22}+2\Delta_{12},$$

where

$$\Delta_{ij}=\langle\pi_i\pi_j\rangle\langle x_ix_j\rangle-\frac{1}{4}\langle\pi_ix_j+x_j\pi_i\rangle\langle\pi_jx_i+x_i\pi_j\rangle.$$

This special form of the invariant was found also in [67]. Taking the values of the second-order moments from the expressions for matrices $\mathbf{M}_{\pi\pi}$, $\mathbf{M}_{xx}$, and $\mathbf{M}_{\pi x}$, one gets $\Delta_{11}=\Delta_{22}=\frac{1}{4}\cosh^2\mu$ and $\Delta_{12}=-\frac{1}{4}\sinh^2\mu$, so the invariant $\mathscr{D}_2$ *is conserved exactly*: $\mathscr{D}_2\equiv\frac{1}{2}$. The same is true for the second universal invariant $\mathscr{D}_4\equiv\det\mathbf{M}$. Consequently, the method of slowly varying amplitudes yields quite good and reliable results.

IX. DISCUSSION

Let us discuss now the possibility of observing the effect of photon creation due to the motion of a boundary in a laboratory. Obviously, it depends crucially on the achievable values of the wall displacement amplitude. For the cavity dimensions of the order of 1–100 cm, the resonance frequency ω_0/π varies from 30 GHz to 300 MHz. It is difficult to imagine that the wall could be forced to oscillate as a whole at such a high frequency. Rather, one could think of the oscillations of the *surface* of the cavity wall. In such a case one has to find a way of exciting a sufficiently strong standing acoustic wave at frequency $\omega_w=2\omega_0$ inside the wall. The amplitude a of this wave (coinciding with the amplitude of oscillations of the free surface) is connected to the relative deformation amplitude δ inside the wall by the formula $\delta=\omega_w a/v_s$, where v_s is the sound velocity. Since the usual materials cannot bear the deformations exceeding the value $\delta_{\max}\sim10^{-2}$ [68], the maximal possible velocity of the boundary appears to be $v_{\max}\sim\delta_{\max}v_s\sim50$ m/s (independent of the frequency). Thus the maximal dimensionless displacement $\varepsilon=a/L_0$ is $\varepsilon_{\max}\sim(v_s/2\pi c)\delta_{\max}\sim3\times10^{-8}$ for the lowest mode with the frequency $\omega_0\sim c\pi/L_0$. It also does not depend on the frequency. Consequently, the maximal rate of photon generation in the principal mode of a 1D cavity equals [see Eq. (6.3)]

$$\left(\frac{d\mathscr{P}_1}{dt}\right)_{\max}=\frac{4}{\pi^2}\frac{v_s}{c}\delta_{\max}\frac{\omega_1}{2\pi}\sim6\times10^{-8}\omega_1/2\pi. \quad (9.1)$$

It is proportional to the frequency. For $\omega_1/2\pi=10$ GHz (corresponding to a distance between the plates of the order of several centimeters) we get 600 photons/s.

This number can be significantly increased in a 3D cavity, due to the exponential law (7.13). For the same frequency $\omega_0/2\pi=10$ GHz, the maximal value of parameter $\mu=\gamma\omega_0 t$ equals $\mu_{\max}\sim600t$, time t being expressed in seconds. Even if the amplitude of the vibrations were 100 times less than the maximal possible value, in $t=1$ s one could get about $\sinh^2(6)\approx4\times10^4$ photons in an empty cavity. The available level of experiments in cavity QED is quite sufficient to detect such an amount of photons [3]. Note that the concrete shape of a 3D cavity considered in Sec. VII is not important. The significant requirements are (i) the nondegenerate character of the eigenfrequency spectrum and (ii) the condition of the parametric resonance between the oscillating wall and some electromagnetic mode. It is interesting that the *total* energy of photons created in the 1D cavity is approximately the same as in the 3D case (see Sec. VI). But in the 1D situation this energy is spread over many interacting modes, resulting in moderate numbers of quanta in each mode. According to the asymptotical solution found in [12], the rate of photon generation in the mth (odd) mode of the 1D cavity is approximately m times less than in the principal mode with $m=1$ (provided that $\varepsilon\omega t \gg 1$).

To create the above-mentioned 600 or 4×10^4 photons, one should vibrate the wall for not less than 1 s. The necessary Q factor of the cavity must be $Q\sim 3\times 10^{10}$. This value was already achieved in experiments several years ago [3].

An unsolved problem is how to excite the high-frequency surface vibrations with a sufficiently large amplitude. One could consider, for instance, using some kind of piezoeffect. This method was successfully applied in early experiments devoted to solving the mode-locking and pulse production problems in lasers with the aid of vibrating mirrors. The displacements of the mirror from 0.1 to 0.7 μm at the frequency 500 kHz were achieved in [69]. In [70] the resonance vibrations of the mirror in a laser with a length of 250 cm (i.e., at a frequency of about 100 MHz) were excited with the aid of a quarz transducer. However, for our purposes the frequency 100 MHz is too small, since the parameter μ becomes 100 times smaller compared to the estimations given above (remember that $\varepsilon_{\max}$ does not depend on the frequency). Besides, in such a case one must cool a 3-m cavity below 4 mK to eliminate the thermal noise. Diminishing the dimensions of the cavity to 1 cm, we simultaneously raise the temperature limit to 0.4 K and increase the parameter μ. Thus we arrive at the value of the frequency 10 GHz and the minimal necessary amplitude of vibrations about 10^{-10} cm.

Of course, many other difficult experimental problems must be solved, for instance, how to avoid heating the wall in a strong acoustic wave in order to prevent the destruction of the superconductivity. Besides, in the case of piezoexcitation, the applied electric voltage and the oscillations of electrons and ions inside the piezoelectric could create their own fields inside the cavity. However, these parasitic fields would be polarized in the direction *perpendicular* to the surface (since the region inside the cavity is the near field zone for the electric dipole arising due to a possible uncompensated charge density in the material). Therefore they could be distinguished from the field created due to the nonstationary vacuum fluctuations (physically, it is created by fluctuating currents at the ideal conducting surface of the wall), which is polarized parallel to the surface.

In any case, it is well worth endeavoring to perform the proposed experiment, since it would be a clear demonstration of the reality of vacuum fluctuations. Indeed, an impressive interpretation of the result could be as follows: one takes an *empty* box, ‘‘knocks’’ on its wall, and then discovers that the box is ‘‘filled with photons.’’

Recently, a possibility of generating a fantastic amount (10^{11} cm^{-2}) of *visible* photons between two ideal plates due to the nonstationary Casimir effect was predicted in the case when one plate performed periodic instantaneous jumps between two stationary positions [71]. However, this result seems dubious for two reasons. First, it was obtained in the framework of the adiabatic approach [i.e., removing all the terms on the right-hand side of Eq. (2.5)], which is obviously incompatible with the ‘‘jump’’ approximation. Second, imagining that one can periodically and almost instantaneously (for a period of time much less than the period of vibrations in the light wave, i.e., $\sim 10^{-14}$ s) change the distance between plates from 10^{-4} to 2×10^{-5} cm is even more fantastic than discussing the possibility of exciting the surface vibrations with the amplitude of the order of 10^{-10} cm. A correct evaluation of the amount of photons created between two plates, one of which exhibits a fast (nonperiodic) effective displacement due to the creation of a ‘‘plasma mirror’’ inside a semiconductor by a powerful femtosecond laser pulse, was given in [72]. It yields not more than 1–100 photons/cm^2 created from vacuum.

In conclusion, let us formulate briefly the results and the unsolved problems that are worth studying. The main result of the paper is the explicit demonstration of the possibility of generating real photons from vacuum in a cavity with resonantly vibrating walls and finding analytical solutions in one and three space dimensions. In addition, we have shown that the interaction between the field mode and the detector can change the picture essentially, due to the back influence of the detector on the mode. Thus the analysis of more realistic models of the detectors seems important. (A two-level detector was considered in the framework of the generalized Jaynes-Cummings model in [73]; its response turned out quite different from the behavior of the harmonic oscillator.) Other theoretical problems include, for instance, the evaluation of the contribution of the nonresonant modes, the quantitative account of different losses, and the analysis of various parasite effects. The existence of the *exact* solution in the 1D resonance case gives rise to the conjecture on the possible internal symmetry of the simplified equations (3.2)–(3.5). Indeed, these equations appear to be closely connected with the algebra $su(1,1)$. Besides, certain relations between our solutions and the solutions of Moore’s equation (1.4) found in Refs. [12,13,35] must also exist. The approach used in our paper can be applied for solving some problems in other fields of physics, where equations such as (1.1) and (1.2) arise. A recent example of such a problem, related to the plasma oscillations in a slab with a corrugated surface, was presented in [74]. We hope that the method of averaging over fast oscillations can result in new analytical solutions for other relations between the wall frequency and the field mode frequencies (in the 1D case), namely, when $\omega_w = k\omega_1$ with an integer $k\neq 2$. For odd values of k the variables ξ and η will not be coupled, so the photon production from vacuum will be impossible. But some other interesting phenomena can be discovered (see [41]). These questions are planned to be considered in future works.

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APPENDIX A: IMPOSSIBILITY OF PHOTON CREATION IN THE ADIABATIC CASE

If the wall vibrates with a small amplitude at frequency $\omega_w \ll \omega_{\mathbf{n}}$, then one may neglect the right-hand side of Eq. (2.5), arriving at the problem of an oscillator with a time-dependent frequency $\omega(t)=\omega_0[1-\epsilon \sin(\omega_w t)]$, where ω_0 is the unperturbed frequency of the given mode and $\epsilon \ll 1$. Then the average number of photons produced in each field mode can be expressed through the energy reflection coefficient R from an effective potential barrier corresponding to the time dependence $\omega^2(t)$ as follows [48,58]: $\langle n\rangle = R/(1-R)$. The general formula for the adiabatic reflection coefficient can be found, e.g., in [75]. In the case under study it can be simplified as

$$R=\left|\frac{1}{4\omega_0^2}\int_0^T \ddot{\omega}(\tau)e^{-2i\omega_0\tau}\,d\tau\right|^2, \tag{A1}$$

where T is the duration of the process. Calculating the integral, it is not difficult to obtain the inequality $R \leqslant (\epsilon\omega_w^2/4\omega_0^2)^2$. It clearly shows that no photons can be produced under the adiabatic conditions, so the only hope to observe the effect may be associated with the *resonance* between the electromagnetic and mechanical modes, when one of the walls of a cavity vibrates with the frequency $\omega_w = 2\omega_0$, and Eq. (1.1) indicates a possibility of a large reflection coefficient. However, in such a case the adiabatic approach is inapplicable.

APPENDIX B: CALCULATION OF THE PROPAGATOR

Suppose that a quantum system is described by a quadratic Hamiltonian

$$H=\frac{1}{2}(\pi,\mathbf{x})\mathcal{B}\begin{pmatrix}\pi\\ \mathbf{x}\end{pmatrix}, \tag{B1}$$

where $\mathbf{x}$ is an N-dimensional coordinate vector, π is the canonically conjugated momentum vector, $[x_j,\pi_k]=i\delta_{jk}$, $j,k=1,\ldots,N$ ($\hbar=1$), and $\mathcal{B}(t)$ is quite arbitrary *symmetric* (but not necessarily real) $2N\times 2N$ matrix consisting of four $N\times N$ blocks

$$\mathcal{B}=\begin{Vmatrix}\mathbf{b}_1 & \mathbf{b}_2\\ \mathbf{b}_3 & \mathbf{b}_4\end{Vmatrix}, \quad \mathbf{b}_1=\tilde{\mathbf{b}}_1, \quad \mathbf{b}_4=\tilde{\mathbf{b}}_4, \quad \mathbf{b}_2=\tilde{\mathbf{b}}_3.$$

The propagator of the Schrödinger equation in the coordinate representation can be expressed as [47,59]

$$G(\mathbf{x}_2,\mathbf{x}_1,t)=[\det(-2\pi i\lambda_3)]^{-1/2}\exp\left\{-\frac{i}{2}[\mathbf{x}_2\lambda_3^{-1}\lambda_4\mathbf{x}_2 - 2\mathbf{x}_2\lambda_3^{-1}\mathbf{x}_1+\mathbf{x}_1\lambda_1\lambda_3^{-1}\mathbf{x}_1]\right\},$$

provided that $N\times N$ matrices $\lambda_j(t)$ satisfy the system of linear differential equations (here $\mathbf{1}$ means an $N\times N$ unit matrix)

$$\dot{\lambda}_1=\lambda_1\mathbf{b}_3-\lambda_2\mathbf{b}_1,\quad \lambda_1(0)=\mathbf{1},\quad \dot{\lambda}_2=\lambda_1\mathbf{b}_4-\lambda_2\mathbf{b}_2,$$

$$\lambda_2(0)=\mathbf{0},\quad \dot{\lambda}_3=\lambda_3\mathbf{b}_3-\lambda_4\mathbf{b}_1,\quad \lambda_3(0)=\mathbf{0},$$

$$\dot{\lambda}_4=\lambda_3\mathbf{b}_4-\lambda_4\mathbf{b}_2,\quad \lambda_4(0)=\mathbf{1}.$$

In the case of the Hamiltonian (8.1) with $\omega_0=1$ the block matrices $\mathbf{b}_j$ read (the terms of the order of κ^2 are neglected)

$$\mathbf{b}_1=\mathbf{1},\quad \mathbf{b}_4=\begin{Vmatrix}1+\Gamma & 0\\ 0 & 1\end{Vmatrix},\quad \mathbf{b}_2=\begin{Vmatrix}0 & 0\\ -2\kappa & 0\end{Vmatrix},$$

where $\Gamma(t)=4\gamma\cos(2\omega_- t)$. The matrices λ_2 and λ_4 can be excluded with the aid of the relations

$$\lambda_{2,4}=\lambda_{1,3}\tilde{\mathbf{b}}_2-\dot{\lambda}_{1,3}. \tag{B2}$$

Two other matrices λ_1 and λ_3 satisfy the same second-order equation

$$\ddot{\lambda}+\dot{\lambda}(\mathbf{b}_2-\tilde{\mathbf{b}}_2)+\lambda\mathbf{b}_4=\mathbf{0}, \tag{B3}$$

but different initial conditions

$$\lambda_1(0)=\mathbf{1},\quad \dot{\lambda}_1(0)=\tilde{\mathbf{b}}_2, \tag{B4}$$

$$\lambda_3(0)=\mathbf{0},\quad \dot{\lambda}_3(0)=-\mathbf{1}. \tag{B5}$$

Let us write the 2×2 matrix λ as

$$\lambda=\begin{Vmatrix}x & y\\ z & u\end{Vmatrix}.$$

Then the elements of both rows, (x,y) and (z,u), must satisfy the identical equations, which in the (x,y) case read

$$\ddot{x}-2\kappa\dot{y}+[1+\Gamma(t)]x=0,$$

$$\ddot{y}+2\kappa\dot{x}+y=0.$$

Differentiating the first equation and comparing the result with the second one, it is easy to get the relation

$$y=-\frac{1}{2\kappa}\{\dddot{x}+(1+\Gamma)\dot{x}+\dot{\Gamma}x\}. \tag{B6}$$

Therefore function $x(t)$ must satisfy the fourth-order differential equation

$$d^4x/dt^4+(2+\Gamma)\ddot{x}+2\dot{\Gamma}\dot{x}+(1+\Gamma+\ddot{\Gamma})x=0. \tag{B7}$$

According to the method of slowly varying amplitudes, one may look for the solution to Eq. (B.7) in the form

$$x(t)=\xi_+e^{i\omega_+t}+\eta_+e^{-i\omega_+t}+\xi_-e^{i\omega_-t}+\eta_-e^{-i\omega_-t}, \tag{B8}$$

$\xi_\pm$ and $\eta_\pm$ being slowly varying functions of time. Due to the condition $\gamma \ll \kappa$, the upper and lower splitted modes do not interact with each other under the condition of the strict parametric resonance at the lower frequency ω_-. Thus one may assume $\xi_+=$const and $\eta_+=$const. Averaging over fast oscillations with the frequency $2\omega_-$ gives rise to the equations for ξ_- and η_- (remember that all the higher-order terms such as κ^2, γ^2, $\kappa\gamma$, etc., are neglected)

$$\dot{\xi}_- = i\gamma\eta_-, \quad \dot{\eta}_- = -i\gamma\xi_-,$$

whose solutions read

$$\xi_- = ae^{\gamma t} + be^{-\gamma t}, \quad \eta_- = -i(ae^{\gamma t} - be^{-\gamma t}).$$

Taking into account the relation $2\kappa = \omega_+ - \omega_-$ one can transform Eq. (B6) into the form

$$y(t) = i(\xi_+ e^{i\omega_+ t} - \eta_+ e^{-i\omega_+ t} - \xi_- e^{i\omega_- t} + \eta_- e^{-i\omega_- t}),$$

up to the terms of the order of κ^2 and γ. Then the constant coefficients ξ_+, η_+, a, and b can be determined from the initial conditions (B.4) and (B.5). After rather tedious algebra one can arrive finally at the following expressions for the elements of matrices λ_1 and λ_3:

$$\lambda_1^{11} = \frac{1}{2}[\omega_-(\cosh\mu\ \cos\varphi_- - \sinh\mu\ \sin\varphi_-) + \omega_+ \cos\varphi_+],$$

$$\lambda_1^{12} = \frac{1}{2}[\omega_-(\sinh\mu\ \cos\varphi_- + \cosh\mu\ \sin\varphi_-) - \omega_+ \sin\varphi_+],$$

$$\lambda_1^{21} = \frac{1}{2}[\omega_+(\sinh\mu\ \cos\varphi_- - \cosh\mu\ \sin\varphi_-) + \omega_- \sin\varphi_+],$$

$$\lambda_1^{22} = \frac{1}{2}[\omega_+(\cosh\mu\ \cos\varphi_- + \sinh\mu\ \sin\varphi_-) + \omega_- \cos\varphi_+];$$

$$\lambda_3^{11} = \frac{1}{2}[\sinh\mu\ \cos\varphi_- - \cosh\mu\ \sin\varphi_- - \sin\varphi_+],$$

$$\lambda_3^{12} = \frac{1}{2}[\cosh\mu\ \cos\varphi_- + \sinh\mu\ \sin\varphi_- - \cos\varphi_+],$$

$$\lambda_3^{21} = \frac{1}{2}[\sinh\mu\ \sin\varphi_- - \cosh\mu\ \cos\varphi_- + \cos\varphi_+],$$

$$\lambda_3^{22} = \frac{1}{2}[-\sinh\mu\ \cos\varphi_- - \cosh\mu\ \sin\varphi_- - \sin\varphi_+].$$

From these expressions one can obtain matrices $\mathbf{B} = \lambda_3^{-1}\Delta$ and $\mathbf{C} = \lambda_1\lambda_3^{-1}\Delta$ (for the notation see Sec. VIII). The matrix $\mathbf{A} = \lambda_3^{-1}\lambda_4\Delta$ was calculated with the aid of the relation [see Eq. (B2)] $\lambda_3^{-1}\lambda_4 = \mathbf{b}_3 - \lambda_3^{-1}\dot{\lambda}_3$.

APPENDIX C: SOME RELATIONS FOR GAUSSIAN STATES

If an N-dimensional quantum system is described by a Gaussian wave function (in the coordinate representation) $\psi(\mathbf{x}) = \mathcal{N}\exp(-\frac{1}{2}\mathbf{xRx})$, where $\mathcal{N}$ is a normalization constant and $\mathbf{R}$ is a symmetric complex $N\times N$ matrix, then the variance matrices, defined in Sec. VIII, can be expressed through the matrix $\mathbf{R}$ as [47] ($\hbar = 1$)

$$\mathbf{M}_{xx} = (\mathbf{R} + \mathbf{R}^*)^{-1},$$

$$\mathbf{M}_{\pi x} = \frac{i}{2}(\mathbf{R} - \mathbf{R}^*)\mathbf{M}_{xx},$$

$$\mathbf{M}_{\pi\pi} = \frac{1}{4}(\mathbf{1} + 4\mathbf{M}_{\pi x}^2)\mathbf{M}_{xx}^{-1}.$$

The last formula is equivalent to the identity

$$\mathbf{M}_{\pi\pi}\mathbf{M}_{xx} - \mathbf{M}_{\pi x}^2 = \frac{1}{4}\mathbf{1}. \tag{C1}$$

The left-hand side of this identity is one of the universal invariants, because any quadratic Hamiltonian transforms an arbitrary Gaussian state into another Gaussian state. Moreover, taking the trace of both sides of Eq. (C1), one arrives at a special form of the *generalized uncertainty relation* [63,76], which holds for Gaussian states (usually the variance uncertainty relations for coordinates and momenta are minimized at Gaussian states).

The photon distribution function in any *one-mode* Gaussian state *with zero mean values* of $\langle q\rangle$ and $\langle p\rangle$ can be expressed in terms of the invariants of the variance matrix T and d (see Sec. VIII) as [60,77]

$$\mathcal{P}_n = \frac{2}{\sqrt{1+2T+4d}}\left(\frac{1+4d-2T}{1+4d+2T}\right)^{n/2} \times P_n\left(\frac{4d-1}{\sqrt{(4d+1)^2-4T^2}}\right), \tag{C2}$$

where $P_n(x)$ is the Legendre polynomial. More general expressions in terms of the multidimensional Hermite polynomials, which hold for multimode systems with nonzero first-order average values, can be found in [78].

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