Theory of the Lamb shift in muonic hydrogen

Krzysztof Pachucki*
Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straβe 1, 85748 Garching, Germany
(Received 28 August 1995)

Muonic hydrogen (μH) is a unique tool to study the low-energy properties of the proton form factors. The energy levels of μH are very sensitive to QED, recoil, and proton finite-size effects. We calculate the corrections to the Lamb shift and also to the fine and hyperfine structures that contribute at the 0.01-meV precision level. This result may allow for the precise determination of the proton charge radius from measurements of the $2P$-$2S$ transition energy in μH. A more accurate value for the proton radius is necessary for further improvements of QED tests based on the hydrogen atom.

PACS number(s): 31.30.Jv, 12.20.Ds, 36.10.Dr, 06.20.Jr

I. INTRODUCTION

Muonic hydrogen (μH) is a bound system of a muon and a proton. Its finite lifetime is determined by the muon decay $\tau_\mu = 2.197\,03(4)\times 10^{-6}$ s. The energy levels of the μH system are very sensitive to the QED, recoil, and nuclear structure effects. The muon is about 200 times heavier than the electron and therefore its wave function overlaps with the proton $(m_\mu/m_e)^3 \approx 10^7$ stronger than that of the electron in the hydrogen atom. The effective potential that the muon experiences is significantly modified by the proton charge distribution. Therefore, a measurement of the $2P$-$2S$ Lamb shift could give a precise value for the proton charge radius r_p. In fact, the current tests of QED on atomic hydrogen are limited by the experimental uncertainty in the proton form factors. For example, the theoretical predictions for the $2S$ Lamb shift in hydrogen [1]

$$L(2S) = 1\,045\,003(4)(3) \ \ \text{kHz} \tag{1}$$

have an error of 4 kHz coming from the uncertainty in the proton charge radius $r_p = 0.862(12)$ fm (the second error in an estimate for unknown higher-order corrections). The experimental precision for the $2S$ Lamb shift is expected to reach an accuracy of about 1 kHz, thus the precise knowledge of the proton charge radius is very desirable for further improvements of QED tests. Taqqu [2] is currently working on the measurement of the $2P$-$2S$ transition frequency in muonic hydrogen using a phase-space-compressed muon beam technique. Recent investigations indicate that appreciable fractions of the $2S$ state are long lived at low target pressure. The predicted relative accuracy is of order 10^{-4} and corresponds to about 25% of the natural linewidth of the $2P$-$1S$ transition, $\Gamma = 0.08$ meV. Such a measurement would allow one to increase the precision of Lamb shift tests on hydrogen by one order of magnitude.

The Lamb shift in μH was first investigated by Giacomo in [3]. Several significant corrections have been omitted there and in general there has been some progress in the bound-state QED since this paper was written. The Lamb shift in other muonic atoms has been worked out in detail by Borie and Rinker in a series of papers summarized in a comprehensive review [4]. μH was also treated by Borie in [5], where the leading contributions were calculated. A detailed analysis for light muonic ions such as μLi, μBe, and μB was given by Drake and Byer in [6]. We make an attempt here to collect the data for various corrections to the Lamb shift in muonic hydrogen, with particular attention to its correct mass dependence. The precision level for our calculation is determined by the unknown three-loop vacuum polarization, which is estimated to be of the order of 0.01 meV. This correction is α^2 times smaller than the leading contribution, i.e., one-loop vacuum polarization (VP) (the known two-loop VP is approximately α times smaller than the one-loop VP).

In our calculation we use the following values for masses and other physical constants:

$$m_\mu = 105.658\,389(34) \ \ \text{MeV}, \tag{2}$$

$$\frac{m_p}{m_\mu} = 8.880\,244\,4(13), \tag{3}$$

$$\frac{m_e}{m_\mu} = 4.836\,332\,18(71)\times 10^{-3}, \tag{4}$$

$$\alpha^{-1} = 137.035\,989\,5(61), \tag{5}$$

$$\frac{\hbar}{m_e c} = 386.159\,323(25) \ \ \text{fm}. \tag{6}$$

II. CORRECTIONS TO THE LAMB SHIFT

The Lamb shift in muonic hydrogen differs from the usual hydrogen in that the electron vacuum polarization gives the most significant contribution. This can be explained by the fact that the Compton wavelength of the electron (which determines the spatial distribution of the vacuum polarization charge density) is of the order of the Bohr radius of muonic hydrogen

$$\beta = \frac{m_e}{\mu\alpha} = 0.737\,386, \tag{7}$$

where μ is the reduced mass of the proton-muon system.

*Permanent address: Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warsaw, Poland.

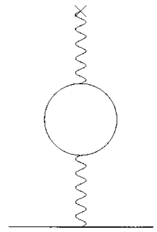

FIG. 1. Leading-order electron vacuum polarization contribution.

In our calculation we employ a perturbation expansion in α and $Z\alpha$, where $Z=1$ in our case, but, in general, it is convenient to distinguish between the charge of the lepton and of the nucleus. It differs from previous treatments [4], which were based on numerical solutions of the Dirac equation in the Coulomb field modified by the presence of vacuum polarization and finite-size effects. Our approach is more suited for light muonic atoms and moreover allows for the easy incorporation of the reduced mass and recoil effects. In many cases we follow the corresponding calculations in electronic hydrogen.

A. Electron vacuum polarization

The electron vacuum polarization modifies the photon propagator

$$-\frac{g_{\mu\nu}}{k^2}\to-\frac{g_{\mu\nu}}{k^2[1+\bar{\omega}(k^2)]}. \tag{8}$$

At the one-loop level $\bar{\omega}$ (in electron mass units) is given by [7]

$$\bar{\omega}(k^2)=\frac{\alpha}{\pi}k^2\int_4^\infty d(q^2)\frac{1}{q^2(m_e^2q^2-k^2)}u(q^2), \tag{9}$$

$$u(q^2)=\frac{1}{3}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right). \tag{10}$$

The vacuum polarization potential as described by Fig. 1 is

$$V_{\rm VP}(r)=-\frac{Z\alpha}{r}\frac{\alpha}{\pi}\int_4^\infty\frac{d(q^2)}{q^2}e^{-m_eqr}u(q^2). \tag{11}$$

The leading-order contribution to the energy shift is

$$E(2P-2S)=\int d^3r V_{\rm VP}(r)\rho(r), \tag{12}$$

$$\rho=\rho_{2P}-\rho_{2S}, \tag{13}$$

where ρ_{2S} and ρ_{2P} are nonrelativistic charge densities of the corresponding $2S$ and $2P$ states with the reduced mass μ. The radial wave functions for these states are

$$R_{20}=\frac{1}{\sqrt{2}}e^{-\mu\alpha r/2}\left(1-\frac{\mu\alpha r}{2}\right), \tag{14}$$

$$R_{21}=\frac{1}{2\sqrt{6}}e^{-\mu\alpha r/2}(\mu\alpha r). \tag{15}$$

We first assume that the nucleus is a pointlike particle. The finite-size effects will be considered separately later. After the integration with respect to r, E is

$$E(2P-2S)=\mu(Z\alpha)^2\frac{\alpha}{\pi}\int_4^\infty\frac{d(q^2)}{q^2}u(q^2)\frac{(\beta q)^2}{2(1+\beta q)^4}$$
$$=\frac{\mu\alpha^3}{\pi}\frac{2}{3}0.026\,178\,9=205.006\text{ meV}. \tag{16}$$

This value will be a reference point for all other corrections. We will neglect all corrections much smaller than α^2 times this energy, since they are smaller than the unknown three-loop vacuum polarization.

The relativistic corrections to the VP are calculated from the two-body Breit Hamiltonian. The external field approximation does not give an accurate result. The Breit Hamiltonian consists of

$$H_B=\frac{p^2}{2m_1}+\frac{p^2}{2m_2}-\frac{\alpha}{r}+\delta H+\delta V, \tag{17}$$

$$\delta H=-\frac{p^4}{8m_1^3}-\frac{p^4}{8m_2^3}, \tag{18}$$

$$\delta V=\frac{\pi\alpha}{2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\delta^3(r)-\frac{\alpha}{2m_1m_2r}\left(p^2+\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p})\mathbf{p}}{r^2}\right)$$
$$+\frac{\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)\mathbf{r}\times\mathbf{p}\cdot\boldsymbol{\sigma}. \tag{19}$$

Here we neglect the nuclear spin (here the nucleus is a particle with spin 2) since we are not considering the hyperfine structure. To include the vacuum polarization, we assume that a photon has a mass ρ and follow the standard derivation of the Breit interaction

$$V_{\rm VP}=-\frac{\alpha}{r}e^{-\rho r}, \tag{20}$$

$$\delta V_{\rm VP}=\frac{\alpha}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\left(4\pi\delta^3(r)-\frac{\rho^2}{r}e^{-\rho r}\right)$$
$$-\frac{\alpha\rho^2}{4m_1m_2}\frac{e^{-\rho r}}{r}\left(1-\frac{\rho r}{2}\right)$$
$$-\frac{\alpha}{2m_1m_2}p^i\frac{e^{-\rho r}}{r}\left(\delta_{ij}+\frac{r_ir_j}{r^2}(1+\rho r)\right)p^j$$
$$+\frac{\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)e^{-\rho r}(1+\rho r)\mathbf{r}\times\mathbf{p}\cdot\boldsymbol{\sigma}. \tag{21}$$

The relativistic correction to the energy due to the exchange of the massive photon is given by

$$E(\rho)=\langle\phi|\delta V_{\rm VP}|\phi\rangle$$
$$+2\left\langle\phi\left|(\delta H+\delta V)\frac{1}{(E-H)'}\left(-\frac{\alpha}{r}e^{-\rho r}\right)\right|\phi\right\rangle, \tag{22}$$

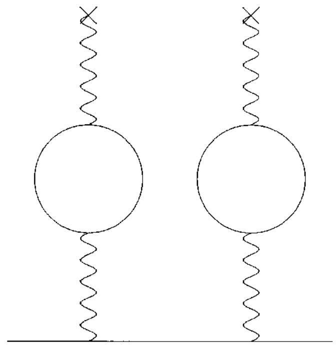

FIG. 2. Double vacuum polarization correction.

where $H=(p^2/2\ \mu)-(\alpha/r)$. In the following

$$G=\left\langle r_1\left|\frac{1}{(H-E)'}\right|r_2\right\rangle$$

denotes the reduced Coulomb Green function. Since have not found its explicit form for $2S$ and $2P$ states in the literature, we have calculated it. The results are ($x=\mu\alpha r$)

$$\begin{aligned}G(2S)=&\frac{\alpha\mu^2e^{-(x_1+x_2)/2}}{4x_1x_2}\{8\,x_<-4\,x_<^2+8\,x_>+12\,x_<x_>\\&-26\,x_<^2x_>+2\,x_<^3x_>-4\,x_>^2-26\,x_<x_>^2\\&+23\,x_<^2\ x_>^2-x_<^3x_>^2+2\,x_<x_>^3-x_<^2x_>^3\\&+4\,e^{x_<}(1-x_<)(x_>-2)x_>+4\,(x_<-2)\\&\times x_<(x_>-2)x_>[-2\,C+\mathrm{Ei}(x_<)-\ln(x_<)\\&-\ln(x_>)]\}\frac{1}{4\,\pi},\end{aligned} \tag{23}$$

$$\begin{aligned}G(2P)=&\frac{\alpha\mu^2e^{-(x_1+x_2)/2}}{36x_1^2x_2^2}\{24\,x_<^3+36\,x_<^3x_>+36\,x_<^3x_>^2\\&+24\,x_>^3+36\,x_<x_>^3+36\,x_<^2x_>^3+49\,x_<^3x_>^3\\&-3\,x_<^4x_>^3-12\,e^{x_<}(2+x_<+x_<^2)x_>^3-3\,x_<^3x_>^4\\&+12\,x_<^3x_>^3[-2\,C+\mathrm{Ei}(x_<)-\ln(x_<)-\ln(x_>)]\}\\&\times\left(\frac{3}{4\,\pi}\,\frac{\mathrm{x}_1\cdot\mathrm{x}_2}{x_1x_2}\right),\end{aligned} \tag{24}$$

where $x_<=\min(x_1,x_2)$, $x_>=\max(x_1,x_2)$, and C is the Euler constant $C=0.577\,216$.

The relevant energy shift to the $2P$-$2S$ transition using (11) is

$$\begin{aligned}E(2P_{1/2}\text{-}2S_{1/2})=&\frac{\alpha}{\pi}\int_4^\infty\frac{d(\rho^2)}{\rho^2}u\left(\frac{\rho^2}{m_e^2}\right)[E_{2P_{1/2}}(\rho)-E_{2S_{1/2}}(\rho)]\\=&0.059\text{ meV}.\end{aligned} \tag{25}$$

If one only takes relativistic corrections to the Dirac wave function with the reduced mass μ, the result would be $E=\int d^3r V_{\mathrm{VP}}(r)\,\delta(\rho_{2P_{1/2}}-\rho_{2S_{1/2}})(r)=0.021$ meV, which significantly differs from (25). It shows the importance of the two-body treatment. As a test we checked that in the limit $m_\mu/m_p=0$, Eq. (22) coincides with that obtained directly from the Dirac equation.

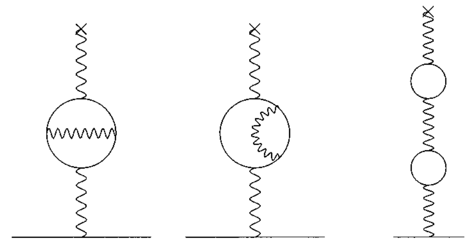

FIG. 3. Two-loop vacuum polarization contribution.

The correction to the Uehling term, i.e., the vacuum polarization with four vertices, can be described as an additional effective potential $V_{\mathrm{rel}}(r)$. This correction has been analyzed in detail by Borie and Rinker in [4]. This potential can be interpolated by a rational function as defined in [4]. The resulting energy correction is

$$E=\int d^3r V_{\mathrm{rel}}(r)\rho(r)=-0.0005\text{ meV}, \tag{26}$$

which is negligible in the case of muonic hydrogen.

The next to the leading contribution is the double VP term, presented in Fig. 2,

$$E=\left\langle\phi\left|V_{\mathrm{VP}}\frac{1}{(E-H)'}V_{\mathrm{VP}}\right|\phi\right\rangle. \tag{27}$$

This matrix element is calculated using (23) and (24) with the result

$$E(2P-2S)=\mu(Z\alpha)^2\left(\frac{\alpha}{\pi}\right)^2\frac{4}{9}\,0.0124=0.151\text{ meV}. \tag{28}$$

Further two-loop diagrams are presented in Fig. 3. The two-loop vacuum polarization has been calculated analytically by Källen and Sabry in [8]:

$$\bar{\omega}^{(2)}(-p^2)=\left(\frac{\alpha}{\pi}\right)^2(-p^2)\int_4^\infty d(q^2)\frac{1}{q^2(m_e^2q^2+p^2)}u^{(2)}(q^2), \tag{29}$$

where $u^{(2)}$ is given by Eq. (49) in [8]. This formula includes also the reducible part presented in Fig. 3(c). The energy shift

$$E=\int\frac{d^3p}{(2\,\pi)^3}\rho(p)\frac{4\,\pi\alpha}{p^2}\bar{\omega}^{(2)}(-p^2) \tag{30}$$

is calculated by analytical integration in p and numerical integration in q. The result is

$$\begin{aligned}E(2P-2S)=&\mu(Z\alpha)^2\left(\frac{\alpha}{\pi}\right)^2\int_4^\infty\frac{d(q^2)}{q^2}u^{(2)}(q^2)\frac{(\beta q)^2}{2\,(1+\beta q)^4}\\=&\mu(Z\alpha)^2\left(\frac{\alpha}{\pi}\right)^2\ 0.055\,266\,7=1.508\text{ meV}.\end{aligned} \tag{31}$$

B. Muon self-energy, vacuum polarization, and recoil corrections

For the calculation of muon self-energy and vacuum polarization we rewrite the corresponding formula known from the electronic hydrogen [10]. It includes only one-loop corrections (the two-loop is negligible, approximately equal to $-0.000\,06$ meV)

$$E(2S)=\frac{1}{8}m_\mu\frac{\alpha}{\pi}(Z\alpha)^4\left(\frac{\mu}{m_\mu}\right)^3\left\{\frac{10}{9}-\frac{4}{15}-\frac{4}{3}\ln k_0(2S)+\frac{4}{3}\ln\left(\frac{m_\mu}{\mu(Z\alpha)^2}\right)+4\,\pi Z\alpha\left(\frac{139}{128}+\frac{5}{192}-\frac{\ln(2)}{2}\right)\right\},\tag{32}$$

$$E(2P_{1/2})=\frac{1}{8}m_\mu\frac{\alpha}{\pi}(Z\alpha)^4\left(\frac{\mu}{m_\mu}\right)^3\left\{-\frac{1}{6}\frac{m_\mu}{\mu}-\frac{4}{3}\ln k_0(2P)\right\},\tag{33}$$

where $\ln k_0(n,l)$ is Bethe logarithm

$$\ln k_0(2S)=2.811\,769\,893\,120\,5,\tag{34}$$

$$\ln k_0(2P)=-0.030\,016\,708\,9.\tag{35}$$

The correction to the Lamb shift is

$$E(2P_{1/2}-2S_{1/2})=-0.668\text{ meV}.\tag{36}$$

Further corrections are due to the modification of muon self-energy by the electron vacuum polarization. They are described by the two diagrams shown in Fig. 4. Both of these corrections are very small, so we evaluate them only approximately. The first one is the muon self-energy in the Coulomb field V and $V_{\rm VP}$. We calculate the leading term logarithmic in α. For this we use a nonrelativistic formula for the energy shift

$$E=\frac{2\,\alpha}{3\,\pi m_\mu^2}\int d\omega\left\langle\phi\left|\mathbf{p}\frac{(H-E)}{H-E+\omega}\mathbf{p}\right|\phi\right\rangle,\tag{37}$$

where H includes $V_{\rm VP}$. We also limit the integration range from $\mu\alpha^2$ to m_μ and neglect $(H-E)$ in the denominator:

$$E=\frac{2\,\alpha}{3\,\pi m_\mu^2}\ln\left(\frac{m_\mu}{\mu\alpha^2}\right)\langle\phi|\mathbf{p}(H-E)\mathbf{p}|\phi\rangle.\tag{38}$$

Using the identity $\langle\phi|\mathbf{p}(H-E)\mathbf{p}|\phi\rangle=1/2\langle\phi|\Delta(V+V_{\rm VP})|\phi\rangle$ and expanding in $V_{\rm VP}$, one obtains

$$E=\frac{\alpha}{3\,\pi m_\mu^2}\ln\left(\frac{m_\mu}{\mu\alpha^2}\right)\left(\langle\phi|\Delta(V_{\rm VP})|\phi\rangle+2\left\langle\phi\left|V_{\rm VP}\frac{1}{(E-H)'}\Delta\left(-\frac{\alpha}{r}\right)\right|\phi\right\rangle\right).\tag{39}$$

The energy shift obtained is

$$E(2P-2S)=-0.005\text{ meV}.\tag{40}$$

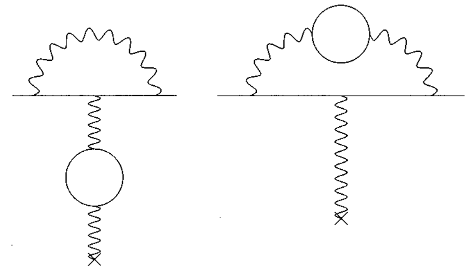

FIG. 4. Self-energy with the vacuum polarization corrections.

For the second diagram in Fig. 4 we can use the on-shell approximation for the external muon legs. The energy shift is then given by

$$E=\frac{\mu^3}{m_\mu^2}\frac{(Z\alpha)^4}{n^3}\left(4m_\mu^2F_1'(0)\delta_{l0}+F_2(0)\frac{c_{jl}}{2l+1}\right),\tag{41}$$

where

$$c_{jl}=\delta_{l0}+(1-\delta_{l0})\frac{j(j+1)-l(l+1)-\frac{3}{4}}{l(l+1)}\tag{42}$$

and F_1 and F_2 are two-loop muon form factors. These form factors have been calculated by Barbieri *et al.* [9]

$$m_\mu^2F_1'(0)=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{1}{9}\ln^2\frac{m_\mu}{m_e}-\frac{29}{108}\ln\frac{m_\mu}{m_e}+\frac{1}{9}\zeta(2)+\frac{395}{1296}+O\left(\frac{m_e}{m_\mu}\right)\right\}=\left(\frac{\alpha}{\pi}\right)^2\left\{2.216\,56+O\left(\frac{m_e}{m_\mu}\right)\right\},\tag{43}$$

$$F_2(0)=\left(\frac{\alpha}{\pi}\right)^2\left\{\frac{1}{3}\ln\frac{m_\mu}{m_e}-\frac{25}{36}+O\left(\frac{m_e}{m_\mu}\right)\right\}=\left(\frac{\alpha}{\pi}\right)^2\left\{1.082\,75+O\left(\frac{m_e}{m_\mu}\right)\right\}.\tag{44}$$

The correction to the Lamb shift is

$$E(2P\text{-}2S)=-0.001\text{ meV}.\tag{45}$$

The last correction from this class that we leave unevaluated is due to the virtual Delbrück scattering. We expect this correction to be below 0.001 meV, similarly to that in Eq. (26).

The formulas (32) and (33) do not incorporate the recoil corrections that are beyond the reduced mass scaling. The recoil correction [10] of order $(Z\alpha)^4$ is obtained by taking the matrix element of Breit interaction Eqs. (18) and (19)

$$E(l,j)=\frac{(Z\alpha)^4\mu^3}{2\,n^3m_p^2}\left(\frac{1}{j+\frac{1}{2}}-\frac{1}{l+\frac{1}{2}}\right)(1-\delta_{l0}).\tag{46}$$

It gives a correction to the Lamb shift of

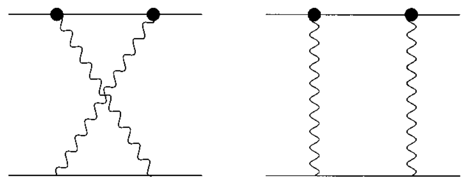

FIG. 5. Finite-size correction of order $(Z\alpha)^5$.

$$E(2P\text{-}2S)=\frac{\alpha^4\mu^3}{48\,m_p^2}=0.057\text{ meV}. \tag{47}$$

The recoil correction in order of $(Z\alpha)^5$ is [10]

$$E(n,l)=\frac{\mu^3}{m_\mu m_p}\frac{(Z\alpha)^5}{\pi n^3}\left\{\frac{2}{3}\delta_{l0}\ln\left(\frac{1}{Z\alpha}\right)-\frac{8}{3}\ln k_0(n,l)-\frac{1}{9}\delta_{l0}-\frac{7}{3}a_n-\frac{2}{m_p^2-m_\mu^2}\delta_{l0}\left[m_p^2\ln\left(\frac{m_\mu}{\mu}\right)-m_\mu^2\ln\left(\frac{m_p}{\mu}\right)\right]\right\}, \tag{48}$$

where

$$a_n=-2\left[\ln\left(\frac{2}{n}\right)+\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)+1-\frac{1}{2\,n}\right]\delta_{l0}+\frac{1-\delta_{l0}}{l(l+1)(2\,l+1)}. \tag{49}$$

It contributes the amount

$$E(2P\text{-}2S)=-0.045\text{ meV}. \tag{50}$$

We neglect higher-order corrections (i.e., $m_\mu/m_p\alpha^6$).

There is an additional recoil correction that has been neglected so far, but is significant for muonic hydrogen. It is due to the proton self-energy. It has been recently analyzed in the case of electronic hydrogen [11]. The difficulty in the interpretation of this correction lies in the fact that it overlaps with the nuclear finite-size effects. The conclusion from that paper is that there is an extra logarithmic term that could not be included in the proton radius definition

$$E(n,l)=\frac{4\,\mu^3(Z^2\alpha)(Z\alpha)^4}{3\,\pi n^3 m_p^2}\left[\delta_{l0}\ln\left(\frac{m_p}{\mu(Z\alpha)^2}\right)-\ln k_0(n,l)\right]. \tag{51}$$

It contributes

$$E(2P\text{-}2S)=-0.010\text{ meV}. \tag{52}$$

C. Nuclear finite-size corrections

In muonic atoms finite-size effects give a large contribution to energy levels. In the leading order the nuclear size correction is

$$E=\frac{2}{3\,n^3}\mu^3(Z\alpha)^4\langle r_p^2\rangle\delta_{l0}, \tag{53}$$

which is proportional to the mean-square nuclear charge radius. The main goal of this paper is to summarize various corrections to muonic hydrogen energy levels, which would allow one to extract the proton radius from future experiments. Nevertheless, we can use here the proton radius obtained from the electron scattering measurement [12]

$$r_p=0.862(12)\ \text{fm} \tag{54}$$

to get the approximate value

$$E(2P\text{-}2S)=[-5.147(\text{meV fm}^{-2})]\langle r_p^2\rangle=-3.862(108)\ \text{meV}. \tag{55}$$

There are various corrections beyond this leading finite-size contribution. We divide them into relativistic and QED corrections. The relativistic corrections based on the Dirac equation and external field approximation have been studied in detail by Friar [13]. He has calculated all the contributions in the order of $(Z\alpha)^6$. In the case of muonic hydrogen they give a negligible correction to the Lamb shift (less than 0.001 meV), therefore we consider only the $(Z\alpha)^5$ correction. We calculate it without the external field approximation. This is necessary because the proton mass is only 9 times larger than the muon mass. The finite-size correction in $(Z\alpha)^5$ is given by Fig. 5, which describes the forward-scattering amplitude. We use the on-mass-shell approximation for the external momenta. This amplitude is infrared divergent and we have to extract the lower-order terms, namely, the scattering amplitude for pointlike fermions and the leading finite-size contribution. The proton vertices are given by two form factors F_1 and F_2 [7]

$$\gamma^\mu\rightarrow\Gamma^\mu=\gamma^\mu F_1+\frac{i}{2\,m_p}\sigma^{\mu\nu}q_\nu F_2. \tag{56}$$

After angular integration in Euclidean momentum space the correction to energy could be written as

$$E(S)=-\frac{\mu^3}{\pi n^3}\delta_{l0}(Z\alpha)^5\int_0^\infty\frac{dp}{p}T(p^2), \tag{57}$$

$$\begin{aligned}T(p^2)=&\frac{2\,(F_1^2-1)}{m_\mu m_p}+\frac{8\,m_\mu[F_2(0)+4\,m_p^2F_1'(0)]}{m_p(m_\mu+m_p)p}\\&+\frac{m_\mu}{m_p(m_\mu^2-m_p^2)p(p+\sqrt{4\,m_\mu^2+p^2})}\\&\times\left\{-16\,F_1F_2m_\mu^2+\frac{32}{p^2}(F_1^2-1)m_\mu^2m_p^2+8\,F_1F_2p^2+6\,F_2^2p^2+\frac{4}{m_\mu^2}(F_1^2-1)m_p^2p^2\right\}\\&-\frac{m_\mu}{m_p(m_\mu^2-m_p^2)p(p+\sqrt{4\,m_p^2+p^2})}\\&\times\left\{-16\,F_1F_2m_p^2+\frac{32}{p^2}(F_1^2-1)m_p^4+8\,F_1F_2p^2+6\,F_2^2p^2+4\,(F_1^2-1)p^2\right\},\end{aligned} \tag{58}$$

where $F_i = F_i(-p^2)$. T is proportional to the forward-scattering amplitude, with lower-order terms subtracted out. The first term in the expansion of (58) in the mass ratio m_μ/m_p is simple,

$$T^{(0)} = \frac{16\, m_\mu}{p^3}(G_E^2(-p^2) - 1 + 2\, G_E'(0)p^2). \tag{59}$$

We use here the Sachs form factors G_E and G_M because the experimental data for the proton structure are given in terms of these functions, defined by

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4\, m_p^2} F_2(q^2), \tag{60}$$

$$G_M(q^2) = F_1(q^2) + F_2(q^2). \tag{61}$$

For the calculation of the corresponding correction to the $(2P\text{-}2S)$ transition energy we use the dipole parametrization of the proton form factors

$$G_E(-p^2) = \frac{\Lambda^4}{(\Lambda^2 + p^2)^2}, \tag{62}$$

$$G_M(-p^2) = (1+\kappa) G_E(-p^2), \tag{63}$$

where κ is the proton anomalous magnetic moment $\kappa = 1.792\,847$. The dipole parametrization is known to describe the proton form factors well in a wide momentum range [14]. The parametrization with $\Lambda = 0.898 m_p = 842.6$ MeV was used previously by Bodwin and Yennie [15] in the calculation of finite-size effects in the hyperfine structure of hydrogen. We use the parametrization from [14] $\Lambda = 848.5$ MeV and obtain the value

$$E(2P\text{-}2S) = 0.018 \text{ meV}. \tag{64}$$

If instead of T one uses $T^{(0)}$, the result would be $E(2P\text{-}2S) = 0.021$ meV.

The QED corrections to the leading finite-size contribution are due to vacuum polarization effects. They are described by two diagrams in Fig. 6. The analogous modification of muon self-energy due to finite-size effects is negligible. The expression corresponding to the first diagram is

$$E(2P\text{-}2S) = -\left(\frac{2}{3}\pi Z \alpha r_p^2\right) \int \frac{d^3 q}{(2\,\pi)^3}\, \bar{\omega}(-q^2) \rho(q)$$
$$= -0.008 \text{ meV}, \tag{65}$$

where $\bar{\omega}$ was defined in (9) and ρ in (13). This is the only nuclear finite-size correction that is significant for the $2P$ state.

The expression corresponding to the second diagram is

$$E = -2\left(\frac{2}{3}\pi Z \alpha r_p^2\right) \int d^3 r\, \phi(r) V_{\mathrm{VP}}(r) G(r,0) \phi(0), \tag{66}$$

where G is a reduced Coulomb Green function for the $2S$ state given in Eq. (23). It gives a correction to the $2P\text{-}2S$ Lamb shift of

$$E(2P\text{-}2S) = -0.013 \text{ meV}. \tag{67}$$

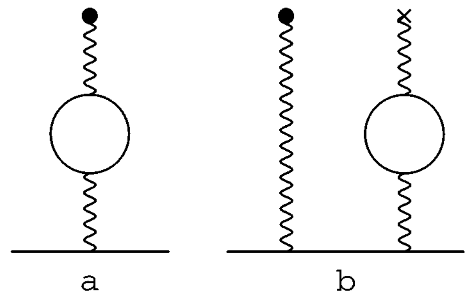

FIG. 6. Finite size with vacuum polarization correction.

III. FINE AND HYPERFINE STRUCTURE

For the determination of the Lamb shift from the $2P\text{-}2S$ frequency measurement, one needs the precise values for the fine and hyperfine structures of the $n=2$ energy shell. This requires the study of the vacuum polarization effects in the fine structure (fs) and hyperfine structure (hfs) and also of the nuclear finite-size effect to the $2S$ hfs. The hfs in the muonic hydrogen is more complicated due to the $2P_{1/2}$ and $2P_{3/2}$ mixing [16]; therefore there is an extra correction to the energy of P levels that has to be taken into account for the determination of absolute energy levels.

A. Hyperfine structure of $2S_{1/2}$

The hyperfine structure in μH is described by the Fermi splitting

$$E_{\mathrm{hfs}}(2S) = \frac{1}{3}\alpha^4 \frac{\mu^3}{m_\mu m_p}(1+\kappa)(1+a_\mu) = 22.832 \text{ meV}, \tag{68}$$

where a_μ is the anomalous magnetic moment of the muon, $a_\mu = 0.001\,166$. There are two kinds of corrections that contribute at the 0.01-meV precision level: the vacuum polarization and the binding $O(\alpha)$ nuclear-structure-dependent correction. The relativistic corrections of order $O(\alpha^2)$ are negligible (≈ 0.0004 meV).

The vacuum polarization effect is calculated in a way similar to (25). The spin-spin interaction (a_μ is neglected)

$$V = \frac{8}{3}\frac{\alpha}{m_\mu m_p}\pi \delta^3(r)(1+\kappa)\frac{\sigma_\mu \sigma_p}{4} \tag{69}$$

in the case of a massive photon takes the form

$$V_{\mathrm{VP}} = \frac{8}{3}\frac{\alpha}{m_\mu m_p}(1+\kappa)\frac{\sigma_\mu \sigma_p}{4}\left(\pi \delta^3(r) - \frac{1}{4}\frac{e^{-\rho r}}{r^3}(\rho r)^2\right). \tag{70}$$

The correction to hfs splitting is given by

$$\delta E_{\mathrm{hfs}}(\rho) = \langle \phi | V_{\mathrm{VP}} | \phi \rangle + 2\left\langle \phi \left| V \frac{1}{(E-H)'}\left(-\frac{\alpha}{r}\right) e^{-\rho r} \right| \phi \right\rangle. \tag{71}$$

The ρ integral is the same as in (25). The result is

$$\delta E_{\mathrm{hfs}}(2S) = \frac{1}{3}\alpha^4 \frac{\mu^3}{m_\mu m_p}(1+\kappa) 0.0025 = 0.058 \text{ meV}. \tag{72}$$

The calculation of the binding $O(\alpha)$ nuclear-structure-dependent correction hfs is performed in a way similar to the Lamb shift. One considers the on-shell spin-dependent part of the scattering amplitude as presented in Fig. 5. The correction to the hfs could then be written as

$$\delta E_{\rm hfs}=-\frac{1}{3}\frac{\alpha^5\mu^3}{\pi n^3}\delta_{l0}\frac{\langle\sigma_\mu\sigma_p\rangle}{4}\int\frac{dp}{p}T_{\rm hfs}(p^2), \tag{73}$$

$$\begin{aligned}T_{\rm hfs}(p^2)=&\frac{2F_2^2p^2}{m_\mu^2m_p^2}+\frac{64[1+F_2(0)]}{(m_\mu+m_p)p}\\&+\frac{1}{(m_\mu^2-m_p^2)p(p+\sqrt{4m_\mu^2+p^2})}\bigg(-128F_1^2m_\mu^2\\&-128F_1F_2m_\mu^2+16F_1^2p^2+64F_1F_2p^2+16F_2^2p^2\\&+\frac{32F_2^2m_\mu^2p^2}{m_p^2}+\frac{4F_2^2p^4}{m_\mu^2}-\frac{4F_2^2p^4}{m_p^2}\bigg)\\&+\frac{1}{(m_\mu^2-m_p^2)p(p+\sqrt{4m_p^2+p^2})}\\&\times(128F_1^2m_p^2+128F_1F_2m_p^2-16F_1^2p^2\\&-64F_1F_2p^2-48F_2^2p^2).\end{aligned} \tag{74}$$

In the limit of large m_p

$$T_{\rm hfs}=\frac{64}{pm_p}[G_M(0)G_E(0)-G_M(-p^2)G_E(-p^2)]. \tag{75}$$

$T_{\rm hfs}$ coincides with the result obtained by Zemach [10], and for the pointlike proton, i.e., $F_1=1,\ F_2=0$, $\delta E_{\rm hfs}$ becomes a known recoil correction

$$\delta E_{\rm hfs}=-8\frac{\alpha^5\mu^3}{\pi n^3}\frac{m_\mu m_p}{(m_p^2-m_\mu^2)}\ln\left(\frac{m_p}{m_\mu}\right)\frac{\langle\sigma_\mu\sigma_p\rangle}{4}. \tag{76}$$

Having done these checks, we integrate Eq. (74) numerically and obtain the value

$$\begin{aligned}\delta E_{\rm hfs}(2S)&=-\frac{1}{3}\frac{\alpha^5\mu^3}{\pi n^3}\delta_{l0}\frac{\langle\sigma_\mu\sigma_p\rangle}{4}6.893\,96\ \text{meV}^{-2}\\&=-0.145\ \text{meV}.\end{aligned} \tag{77}$$

If we take the Zemach part only, we get the value $\delta E_{\rm hfs}(2S)=-0.183$ meV.

The effects of proton polarizability are very difficult in the evaluation. There exists in the literature the upper bounds for the hfs in electronic hydrogen. It is of the order of 10% of the elastic correction. Since this effect only weakly depends on the lepton mass, we assume the same estimate for μH, namely, 0.015 meV. We obtain for the final result for the hfs splitting

$$E_{\rm hfs}(2S)=22.745(15)\ \text{meV}. \tag{78}$$

This relatively large error suggests that the proton radius determination will be more reliable when the hfs for $2S$ or $1S$ is measured experimentally.

B. Fine structure of 2*P*

The fine structure is mostly accounted for by the relevant term from the Breit Hamiltonian

$$V=\frac{\alpha}{r^3}\left(\frac{1+2a_\mu}{4m_\mu^2}+\frac{1+a_\mu}{2m_\mu m_p}\right)L\sigma_\mu. \tag{79}$$

The energy splitting is

$$\begin{aligned}E_{\rm fs}&=E(2P_{3/2})-E(2P_{1/2})\\&=\frac{1}{32}\frac{\mu^3}{m_\mu^2}\alpha^4\left(1+2a_\mu+(1+a_\mu)\frac{2m_\mu}{m_p}\right)\frac{\langle L\sigma_\mu\rangle}{3}\\&=8.347\ \text{meV}.\end{aligned} \tag{80}$$

There is a small correction due to the vacuum polarization. It is calculated from the massive photon analog of (79),

$$V=\frac{\alpha}{r^3}\left(\frac{1+2a_\mu}{4m_\mu^2}+\frac{1+a_\mu}{2m_\mu m_p}\right)e^{-\rho r}(1+\rho r)L\sigma_\mu \tag{81}$$

and gives a correction

$$\begin{aligned}\delta E&=\frac{1}{32}\frac{\mu^3}{m_\mu^2}\alpha^4\left(1+2a_\mu+(1+a_\mu)\frac{2m_\mu}{m_p}\right)\frac{\langle L\sigma_\mu\rangle}{3}0.0006\\&=0.005\ \text{meV}.\end{aligned} \tag{82}$$

The final value for the fine structure in μH is

$$E_{\rm fs}=8.352\ \text{meV}. \tag{83}$$

C. Hyperfine structure of $2P_{1/2}$ and $2P_{3/2}$

The relevant operator from Breit Hamiltonian is

$$\begin{aligned}V=&\frac{\alpha}{2m_\mu m_p}\left(1+\kappa+\frac{m_\mu}{2m_p}(1+2\kappa)\right)\frac{\sigma_pL}{r^3}\\&-\frac{\alpha}{4m_\mu m_p}(1+\kappa)(1+a_\mu)\frac{\sigma_\mu^i\sigma_p^j}{r^3}\left(\delta^{ij}-3\frac{r^ir^j}{r^2}\right).\end{aligned} \tag{84}$$

Since this operator does not commute with $J=L+\frac{1}{2}\sigma_\mu$, the states $P_{1/2}$ and $P_{3/2}$ are mixed and the hyperfine structure is more complicated. The off-diagonal matrix element reads (a_μ is neglected)

$$\begin{aligned}&\langle{}^3P_{1/2}|V|{}^3P_{3/2}\rangle\\&\quad=\frac{1}{3}\alpha^4\frac{\mu^3}{m_\mu m_p}(1+\kappa)\left(1+\frac{m_\mu}{m_p}\frac{1+2\kappa}{1+\kappa}\right)\left(-\frac{\sqrt{2}}{48}\right)\end{aligned} \tag{85}$$

and the diagonal ones are

$$E_{\rm hfs}(P_{1/2})=\frac{1}{3}\alpha^4\frac{\mu^3}{m_\mu m_p}(1+\kappa)\left(\frac{1}{3}+\frac{a_\mu}{6}+\frac{1}{12}\frac{m_\mu}{m_p}\frac{1+2\kappa}{1+\kappa}\right), \tag{86}$$

$$E_{\rm hfs}(P_{3/2}) = \frac{1}{3}\alpha^4 \frac{\mu^3}{m_\mu m_p}(1+\kappa)\left(\frac{2}{15} - \frac{a_\mu}{30} + \frac{1}{12}\frac{m_\mu}{m_p}\frac{1+2\kappa}{1+\kappa}\right). \quad (87)$$

There also is a small correction coming from the vacuum polarization. The extension of (84) to a massive photon is

$$V_{\rm VP} = \frac{\alpha}{2 m_\mu m_p}\left(1+\kappa+\frac{m_\mu}{2 m_p}(1+2\kappa)\right)\frac{e^{-\rho r}}{r^3}(1+\rho r)\frac{\sigma_p L}{r^3} - \frac{\alpha}{4 m_\mu m_p}(1+\kappa)(1+a_\mu)\sigma_\mu^i \sigma_p^j \frac{e^{-\rho r}}{r^3} \times\left[(\rho r)^2\left(\delta_{ij} - \frac{r^i r^j}{r^2}\right) + (1+\rho r)\left(\delta^{ij} - 3\frac{r^i r^j}{r^2}\right)\right]. \quad (88)$$

It gives the corrections

$$\delta E_{\rm hfs}(P_{1/2}) = \frac{1}{3}\alpha^4 \frac{\mu^3}{m_\mu m_p}(1+\kappa)0.000\,22, \quad (89)$$

$$\delta E_{\rm hfs}(P_{3/2}) = \frac{1}{3}\alpha^4 \frac{\mu^3}{m_\mu m_p}(1+\kappa)0.000\,08. \quad (90)$$

To calculate the energy levels we form a matrix for an effective Hamiltonian in the basis of the states ${}^1P_{1/2}$, ${}^3P_{1/2}$, ${}^3P_{3/2}$, and ${}^5P_{3/2}$,

$$H = \begin{bmatrix} -\frac{3}{4}\beta_1 & & & \\ & \frac{1}{4}\beta_1 & \beta_2 & \\ & \beta_2 & -\frac{5}{8}\beta_3+\gamma & \\ & & & \frac{3}{8}\beta_3+\gamma \end{bmatrix}, \quad (91)$$

where

$$\beta_1 = E_{\rm hfs}(P_{1/2}) = 7.963 \text{ meV},$$

$$\beta_2 = \langle {}^3P_{1/2}|V|{}^3P_{3/2}\rangle = -0.796 \text{ meV},$$

$$\beta_3 = E_{\rm hfs}(P_{3/2}) = 3.393 \text{ meV},$$

$$\gamma = E_{\rm fs} = 8.352 \text{ meV}. \quad (92)$$

The mixing term β_2 shifts the energy levels, as shown in Fig. 7, by the amount

$$\Delta = 0.145 \text{ meV}. \quad (93)$$

A similar calculation in the usual (electronic) hydrogen atom gives a shift of $\Delta = 2.5$ kHz. This effect should be taken into account when the Lamb shift is determined experimentally from $2S$-$2P$ splitting.

IV. SUMMARY

We have calculated all contributions to the Lamb shift that could enter at the 0.01-meV precision level, except for the three-loop vacuum polarization. The results are presented in Table I. The theoretical predictions for the Lamb shift in muonic hydrogen are

$$E_L = E(2P_{1/2}\text{-}2S_{1/2}) = 205.932(10)\text{ meV} - [5.197(\text{meV fm}^{-2})]r_p^2 = 202.070(108)\text{ meV}. \quad (94)$$

The three-loop vacuum polarization is the most difficult in the evaluation and limits the precision of our result. There are also other corrections that should be considered for the improvements of theoretical predictions. One of them is related to the general problem of how well the proton can be described by the elastic form factors. We have already included radiative corrections (the proton self-energy), but there also could be further effects.

Once the Lamb shift in μH has been measured, the obtained result may allow for a tenfold improvement in the precision of the proton charge radius, as given by Eq. (54). There also is an older experiment [17] that predicted a different value for the proton charge radius $r_p = 0.805(11)$ fm. The recent calculation of two-loop corrections to the Lamb shift [18,19] in hydrogen suggest that the more recent measurement is correct. Otherwise there would be a large disagreement between theoretical predictions and several Lamb shift measurements [20].

TABLE I. Summary of results for corrections to the Lamb shift in muonic hydrogen.

Correction	Value (in meV)	Equation
leading order VP	205.006	(16)
relativistic correction to VP	0.059	(25)
double VP	0.151	(28)
two-loop VP	1.508	(31)
muon self-energy and VP	-0.668	(36)
muon self-energy with electron VP	-0.006	(40) and (45)
recoil of order α^4	0.057	(47)
recoil of order α^5	-0.045	(50)
proton self energy	-0.010	(52)
leading finite size of order α^4	$-r_p^2 5.197 = -3.862(108)$	(55)
finite size of order α^5	0.018	(64)
VP with finite size	-0.021	(65) and (67)
sum of corrections to the Lamb shift in muonic hydrogen	$205.932(10) - r_p^2 5.197 = 202.070(108)$	(94)

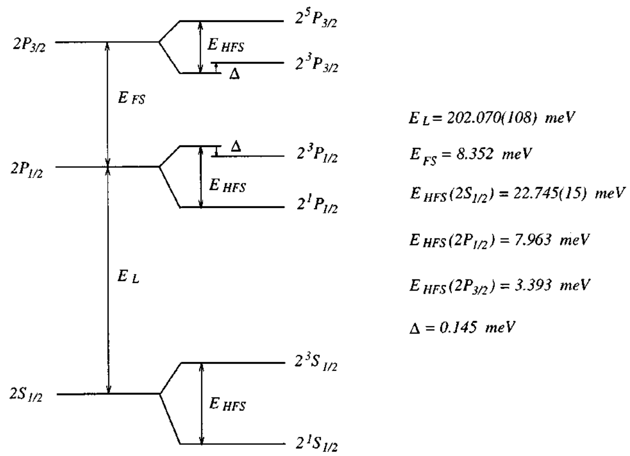

FIG. 7. Level scheme of muonic hydrogen for the $n=2$ shell.

Although the charge radius is well defined for a noninteracting particle, its precise definition for a charged, interacting particle is not unique. The radiative corrections to elastic form factors G_E and G_M are infrared divergent, which means that they depend logarithmically on the artificial photon mass μ. It is not possible to simply subtract out the radiative corrections because the effective QED vertices are modified by finite-size effects and moreover QED for higher spins is not renormalizable. In a recent paper on radiative recoil corrections to the Lamb shift in hydrogen [11], we made an attempt to propose one definition of the charge radius using the inelastic form factors. We use that definition here to identify the proton self-energy correction. A possible difference in the definition of charge radii does not matter at the current precision level, but in view of recent progress in the precision of Lamb shift [20] and isotope shift measurements in hydrogen [21] and helium [22], the reanalysis of nuclear structure effects could soon be necessary.

ACKNOWLEDGMENTS

The author wants to thank T.W. Hänsch for the continuous encouragement, E. Borie for sending copies of her Ph.D. dissertation, D. Taqqu and R. Rosenfelder for the invitation to Paul Scherrer Institute, J. Sapirstein and I. Sick for the information on the dipole parametrization of proton form factors, and M. Weitz and D. Leibfried for carefully reading the manuscript.

[1] K. Pachucki, D. Leibfried, M. Weitz, A. Huber, W. König, and T.W. Hänsch, J. Phys. B **29**, 177 (1996).
[2] D. Taqqu (private communication).
[3] A. Giacomo, Nucl. Phys. B **11**, 411 (1969).
[4] E. Borie and G.A. Rinker, Rev. Mod. Phys. **54**, 67 (1982).
[5] E. Borie (unpublished).
[6] G.W.F. Drake and L.L. Byer, Phys. Rev. A **32**, 713 (1985).
[7] C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1990).
[8] G. Källen and A. Sabry, Mat. Fys. Medd. **29**, 17 (1955).
[9] R. Barbieri, M. Caffo, and E. Remiddi, Nuovo Cimento Lett. **7**, 60 (1973).
[10] J.R. Sapirstein and D.R. Yennie, in *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990).
[11] K. Pachucki, Phys. Rev. A **52**, 1079 (1995).
[12] G.G. Simon, C. Schmidt, F. Borkowski, and V.H. Walther, Nucl. Phys. A **333**, 381 (1980).
[13] J.L. Friar, Ann. Phys. (N.Y.) **122**, 151 (1979).
[14] G. Höhler *et al.*, Nucl. Phys. B **114**, 505 (1976).
[15] G.T. Bodwin and D.R. Yennie, Phys. Rev. D **37**, 498 (1988).
[16] S.V. Romanov, Z. Phys. D **33**, 17 (1995).
[17] L.N. Hand, D.J. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).
[18] K. Pachucki, Phys. Rev. Lett. **72**, 3154 (1994).
[19] M. Eides and V. Shelyuto, Phys. Rev. A **52**, 954 (1995).
[20] S.R. Lundeen and F.M. Pipkin, Phys. Rev. Lett. **46**, 232 (1981); E.W. Hagley and F.M. Pipkin, *ibid.* **72**, 1172 (1994); M. Weitz, A. Huber, F. Schmidt-Kaler, D. Leibfried, and T.W. Hänsch, *ibid.* **72**, 328 (1994); D.J. Berkeland, E.A. Hinds, and M.G. Boshier, *ibid.* **75**, 2470 (1995).
[21] F. Schmidt-Kaler, D. Leibfried, M. Weitz, and T.W. Hänsch, Phys. Rev. Lett. **70**, 2261 (1993).
[22] D. Shiner, R. Dixson, and V. Vedantham, Phys. Rev. Lett. **74**, 3553 (1995).