Retardation effects in nonrelativistic two-photon electron bremsstrahlung in the Coulomb field

M. Dondera and Viorica Florescu
Faculty of Physics, University of Bucharest, P.O. Box 5211, 76900 Bucharest-Magurele, Romania

R. H. Pratt
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
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We investigate the deviations from the dipole approximation for the two-photon bremsstrahlung cross section σ_4, corresponding to the observation of the emitted photons (but not the scattered electron) in coincidence, which has been the subject of recent experiments. Our analysis is for the Coulomb field, and it is done in the Born approximation. We find that retardation effects are important, even in the lower energy range (8–12 keV) of the Hippler experiments, but not in the particular configuration for which experiments were performed. Using the simple analytic expression that we derive, including retardation, in the nonrelativistic Born approximation, we show that, in the incident electron energy range 10–50 keV, the differences between the relativistic Born (Smirnov's approach) and nonrelativistic dipole approximation results are mainly retardation effects.

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I. INTRODUCTION

The emission of two photons in coincidence in an electron-atom collision, not corresponding to transition radiation, detected for the first time by Altman and Quarles [1], is the subject of new experimental investigations [2–4]. The present situation has been reviewed recently [5,6]. The results for the $\pm45°$ geometry [3], and the new data for the $\pm90°$ geometry [4], at incident electron energy of 75 keV, were found to be consistent with the predictions of the relativistic Born approximation (BR) for the case of the Coulomb field (Smirnov's equations [7]). The use of a nonrelativistic dipole approximation approach (CNRD), treating exactly the Coulomb field effects, led to a different situation: the theoretical results are in the same range as the experimental data for the $\pm90°$ geometry [8,9] but not for the $\pm45°$ geometry [10]. Experiment thus suggests that in these situations relativistic or retardation effects are more important than Coulomb field effects. It has been suggested [10] that, in the electron energy range investigated experimentally, retardation effects might be important. The strong dependence of the difference between BR and CNRD results at these energies on the photon detection geometry raises the question of the relative importance of retardation and relativistic effects, particularly as one begins to consider regimes in which Coulomb field effects should not be neglected.

We note that a large discrepancy between theory and experiment was found in the case of Hippler's experiment [2], at the much lower electron energies of 8–12 keV, where Coulomb field effects beyond the Born approximation are expected to be larger. The experimental values are much larger than the theoretical values obtained utilizing full nonrelativistic dipole Coulomb predictions. Two very recent independent dipole calculations [11,12], which treat the case of the Kr atom (using a screened potential in Born approximation or in radial integrals), obtain even lower results than in the Coulomb case. The contribution from the target electron excitation, evaluated in Ref. [11], is small and does not explain the discrepancy between theory and experiment. We also note that early large discrepancies at high energies [1] were later established to represent consequences of other physical processes.

This situation indicates that further theoretical investigation is needed in order to understand the different competing effects: screening, retardation, relativity, and the description of the atomic field beyond the Born approximation. The analysis in this paper is limited to the Coulomb field case. At the energies involved in the experiments performed up to now, to judge from the one-photon cases, screening effects would not be expected to be of dominant importance. Here we will exhibit situations in which retardation effects are important and dominate the relativistic effects.

In one-photon processes relativistic and retardation effects have been studied somewhat systematically, particularly in the case of photoionization, but also in single-photon bremsstrahlung. As the number of independent parameters in a two-photon bremsstrahlung experiment is much larger, a systematic analysis here would require extensive investigations, which is probably premature in a situation in which experiments are still scarce but there is some prospect for additional results. Our purpose here is to explore some sample cases, including some relevant to the existing experiments, to obtain a preliminary idea as to the nature of the effects.

The main tool of our analysis is the Born approximation. The Born approximation is appropriate if both incident and scattered electrons have energies for which $\alpha Z/p$, with p the electron momentum, is much smaller than 1. However, our expectation is that the insights we gain regarding the relative importance of retardation effects and relativity will be helpful beyond the Born regimes. In particular we will demonstrate that, as in one-photon bremsstrahlung, the region of validity of Born approximation is extended by the Elwert factor (see Sec. III). Depending on the electron and photon energies, we compare three versions of the Born approximation:

(i) full-relativistic Born approximation (BR), which also includes retardation;
(ii) nonrelativistic Born approximation, with retardation included (BNRR);
(iii) nonrelativistic Born dipole (without retardation) approximation (BNRD).

For the first case, the equations derived by Smirnov express the cross section σ_5, corresponding to the observation of all three final particles [see Eq. (9) here for exact definition] in two forms: (a) as a trace of a product of γ matrices [Eq. (A1) of [7]] and (b) as an elementary but extremely long algebraic expression obtained after performing the trace [Eq. (3) of [7]]. An analytic calculation of the cross section σ_4 [defined in Eq. (10) here], which refers to the detection of the two photons only, is not feasible. The BNRR approximation is presented in this paper: the most differential cross section, σ_5, has an extremely simple analytic expression in comparison with Smirnov's equations; nevertheless, the angular integration required when the scattered electron is not observed is done numerically. The analytic expressions in the BNRD are very simple for both σ_5 and σ_4. The equations are those derived in Ref. [10].

By comparing the three versions of the Born approximation, we can identify the region of energy parameters in which the dipole approximation is not valid and yet relativistic effects are not too important, so that the BNRR approach is acceptable. At the present stage of theory, this identification of regimes is important, drawing attention to the fact that the BNRD approximation has a very limited range of validity and should not generally be used in examining experimental results.

In Sec. II we consider the nonrelativistic matrix element for two-photon bremsstrahlung in the Coulomb field in the first-order Born approximation (BNRR), with retardation effects included. A simple analytic formula, Eq. (2), together with Eqs. (4)–(7), specifies the amplitude in this approximation. We define various differential cross sections. We analyze only the cross section σ_4 [see Eq. (10)], corresponding to observation of the two photons but not the scattered electron. This is the only cross section measured up to now [1–4]. In Sec. III we compare first BNRD with BNRR results at the incident electron energy of 10 keV, then BNRR and BR results at 10 keV and higher energies. We conclude that BNRD is not adequate at 10 keV, and we argue that dipole approximation will probably not work even at lower energies. The equations expressing BNRR can be used in the range 10–50 keV, to the extent to which Born approximation is valid, to describe the predictions of theory for the Coulomb field case.

When real atoms are under consideration, screening effects should be included. As a first guide for their order of magnitude, one can think of the situation of one-photon bremsstrahlung, as presented in [13]. The differences between screened and Coulomb results for the electron spectrum depend on Z and on the electron kinetic energies, for 50 keV electrons, for instance, far from the low-frequency region, the relative difference is of the order of several percent for $Z=13$ and about 20% for $Z=92$ [13]. In all cases screening reduces the cross section. In the Born approximation this will be the case for two-photon bremsstrahlung, too [12]. We thus anticipate that the Coulomb results discussed here can indicate the expected magnitude of cross sections in current experiments.

II. THE MATRIX ELEMENT AND THE DIFFERENTIAL CROSS SECTION σ_4

The amplitude $\mathcal{M}$ of the two-photon bremsstrahlung of an electron in a potential V is given by the Kramers-Heisenberg-Waller $\mathcal{M}$ matrix element, between initial and final electron full-continuum states with well-defined asymptotic behavior, corresponding to outgoing and incoming spherical waves, respectively. The electron energies are denoted by E_1, E_2 and the asymptotic electron momenta by $\vec{p}_1, \vec{p}_2$. We use electron wave functions normalized in the energy and solid angle scales. The photon momenta are denoted by $\vec{\kappa}_1$ and $\vec{\kappa}_2$ and the polarization vectors by $\vec{s}_1$ and $\vec{s}_2$. The electron spin is not considered.

The amplitude can always be written as

$$\mathcal{M}=\sum_{i,j} M_{ij} s_{1i}^{*} s_{2j}^{*}. \tag{1}$$

In the Born approximation, we obtain

$$\begin{aligned} M_{ij}=&\mathcal{O}\delta_{ij}+B_1(\Omega_1)(p_{1i}-\kappa_{1i})(p_{1j}-\kappa_{1j})+\tilde{B}_1(\Omega_2)(p_{1i}\\ &-\kappa_{2i})(p_{1j}-\kappa_{2j})+B_2(\Omega_1)(p_{2i}+\kappa_{2i})(p_{2j}+\kappa_{2j})\\ &+\tilde{B}_2(\Omega_2)(p_{2i}+\kappa_{1i})(p_{2j}+\kappa_{1j})+C(\Omega_1)(p_{1i}-\kappa_{1i})\\ &\times(p_{2j}+\kappa_{2j})+\tilde{C}(\Omega_2)(p_{2i}+\kappa_{1i})(p_{1j}-\kappa_{2j}), \end{aligned} \tag{2}$$

where the invariant amplitudes denoted by $\tilde{B}_1$, $\tilde{B}_2$, and $\tilde{C}$ are obtained from B_1, B_2, and C, respectively, by interchanging the vectors $\vec{\kappa}_1$ and $\vec{\kappa}_2$. The values of the parameter Ω are

$$\Omega_1=E_1-k_1, \quad \Omega_2=E_1-k_2, \tag{3}$$

where k_1 and k_2 are the photon energies.

For the nuclear point Coulomb potential, we find

$$\mathcal{O}=-m_e^2\,\frac{2\alpha Zc}{\pi^2}\,\frac{\sqrt{p_1p_2}}{\Delta^2}\,\frac{\vec{\Delta}\cdot(\vec{\kappa}_1+\vec{\kappa}_2)}{(\Delta^2-2\vec{p}_1\cdot\vec{\Delta})(\Delta^2+2\vec{p}_2\cdot\vec{\Delta})}, \tag{4}$$

$$B_1=\frac{\alpha Zc}{2\pi^2}\,\frac{\sqrt{p_1p_2}}{\Delta^2}\,\frac{1}{E_2-\dfrac{(\vec{p}_1-\vec{\kappa}_1-\vec{\kappa}_2)^2}{2m_e}}\,\frac{1}{\Omega-\dfrac{(\vec{p}_1-\vec{\kappa}_1)^2}{2m_e}}, \tag{5}$$

$$C=\frac{\alpha Zc}{2\pi^2}\,\frac{\sqrt{p_1p_2}}{\Delta^2}\,\frac{1}{\Omega-\dfrac{(\vec{p}_2+\vec{\kappa}_2)^2}{2m_e}}\,\frac{1}{\Omega-\dfrac{(\vec{p}_1-\vec{\kappa}_1)^2}{2m_e}}, \tag{6}$$

$$B_2=\frac{\alpha Zc}{2\pi^2}\,\frac{\sqrt{p_1p_2}}{\Delta^2}\,\frac{1}{\Omega-\dfrac{(\vec{p}_2+\vec{\kappa}_2)^2}{2m_e}}\,\frac{1}{E_1-\dfrac{(\vec{p}_2+\vec{\kappa}_1+\vec{\kappa}_2)^2}{2m_e}}, \tag{7}$$

where we denote by $\vec{\Delta}$ the momentum transfer from the initial electron to the nucleus:

$$\vec{\Delta}=\vec{p}_1-\vec{p}_2-\vec{\kappa}_1-\vec{\kappa}_2. \tag{8}$$

In the dipole approximation, where we neglect the photon wave functions in the matrix element, the amplitudes B_1, B_2, and C reduce to expressions in agreement with our previous calculation [10].

Since our calculation has used the first Born approximation, the only Z dependence of the amplitude is contained in the factor αZ, which leads to differential cross sections, as for instance σ_5 and σ_4, described below, proportional to Z^2. Our discussions of the importance of relativistic and retardation effects in the Born approximation hence apply for any Z, and they continue to apply in the Elwert-Born approximation (see next section), assuming the same version of the Elwert factor is used throughout.

We list two of the multiply differential cross sections that may be met in double bremsstrahlung experiments. The most completely differential cross section, but with no observation of electron spin and no photon polarization detection, is

$$\sigma_5=\frac{d^5\bar{\bar{\sigma}}}{dk_1dk_2d\Omega_1d\Omega_2d\Omega_e}=\frac{1}{2}\,\frac{r_0^2}{E_1m_ec^2}\,k_1k_2\sum_{s_1,s_2}|\mathscr{M}|^2. \tag{9}$$

The double bar means summation over the polarizations of the two photons. To measure σ_5, a triple coincidence experiment (the scattered electron in coincidence with the emitted photons) is needed, in which the directions of the three particles, and the energies of two of the particles, are recorded.

For the two-photon-in-coincidence experiments, with the scattered electron not observed, the quantity of interest is

$$\sigma_4\equiv\frac{d^4\sigma}{dk_1dk_2d\Omega_1d\Omega_2}=\int\sigma_5d\Omega_e. \tag{10}$$

We will analyze this differential cross section, which depends on the two photon energies k_1 and k_2 and on the scalar products

$$\vec{n}_1\cdot\vec{n}_2=\cos\,\Theta,\quad \vec{n}_1\cdot\hat{\vec{p}}_1=\cos\,\theta_1,\quad \vec{n}_2\cdot\hat{\vec{p}}_1=\cos\,\theta_2. \tag{11}$$

We have denoted by $\vec{n}_1$ and $\vec{n}_2$ the unit vectors along the photon directions.

In dipole approximation the structure of the exact Coulomb cross section σ_4 is determined by five quantities:

$$\sigma_4=\sigma_0+\sigma_0(\cos^2\theta_1+\cos^2\theta_2)+\sigma_b\cos^2\Theta$$
$$+\sigma_c\cos\Theta\cos\theta_1\cos\theta_2+\sigma_d\cos^2\theta_1\cos^2\theta_2. \tag{12}$$

Simple dipole expressions are obtained in the Born approximation [10]. Recently Korol [14] has also derived an analytic expression for σ_4, in the dipole approximation, written in terms of associated Legendre polynomials; his results agree with ours [15].

The structure of σ_4 is much more complex when retardation effects are included, because of the existence of two new vectors, the two photon momenta, as can be judged from Eq. (2). Nevertheless, we can argue, based on the symmetry of the process amplitude to the interchange of $\vec{\kappa}_1$ and $\vec{\kappa}_2$, that linear terms in photon momenta in the cross section σ_4 will appear only in the scalar product $\vec{p}_1\cdot(\vec{\kappa}_1+\vec{\kappa}_2)$.

III. NUMERICAL RESULTS AND CONCLUSIONS

The main purpose of our numerical work was to establish the origin of the large discrepancies existing between BNRD [10] and BR, as reported in [3,4], and to show that there is an energy range in which this discrepancy is caused by retardation effects, which therefore should be included in any attempts to go beyond the Born approximation. To do this we have compared the three versions of the Born approximation, described in Sec. I, namely, BNRD, BNRR, and BR, for five detection configurations. In all of the configurations considered, the emitted photons and the incident electron momentum are coplanar, as in the present experiments, and make equal angles $\pm\theta$ with the incident electron. There are two reasons for analyzing only this type of configuration: (i) the experimental configurations used up to now are of this type, and (ii) little changes have been noticed in our calculations when other than coplanar geometries have been examined. The configurations we describe here, denoted by I to V, correspond to $\theta=10°$, 30°, 45°, 75°, and 90°, respectively. Use of this sequence permits some understanding of the angular dependence of the cross section. We have examined the range of incident electron energy, 1–100 keV, although corrections to the Born approximation should be considered below 10 keV and relativistic effects by 50 keV.

We first illustrate for a representative case the results of the comparison between BNRD and BNRR at relatively low energies, when relativistic effects can be neglected. The quantity represented in Fig. 1 is the ratio between BNRD and BNRR values for σ_4. The illustrated incident electron energy is 10 keV. The values for the ratio k_1/T_1 are 0.1 and 0.5, in Figs. 1(a) and 1(b), respectively. The five curves in each figure correspond to the five configurations mentioned before. The ratio is shown as a function of k_2/T_1, with k_2 the other photon energy. The range of abscissa is different in the two panels here and in the following figures because of the constraint imposed by the energy conservation. While at $\theta=90°$ the ratio is near 1, which means negligible retardation effects, for the other configurations these effects are important and change dramatically with the configuration, and in a rather complicated way [16]. Thus, in Fig. 1 we illustrate the inadequacy of the dipole approximation in any configuration but $\theta=90°$, which we generally find for electron energies above 10 keV. The special situation of the latter configuration can be explained by the fact that, as mentioned at the end of Sec. II, the linear terms in photon momenta in the cross section σ_4 are proportional to the scalar product $\vec{p}_1\cdot(\vec{\kappa}_1+\vec{\kappa}_2)$. When the photons' directions are orthogonal to the incident electron momentum, this quantity vanishes. The retardation effects will also be small for vanishing $\vec{\kappa}_1+\vec{\kappa}_2$. Finally, we mention that within Born approximation, however inadequate (excepting the case of very low Z) below $T_1=1$ keV, we have found relative errors in the range of 20%, suggesting that retardation effects could be important even at this low energy.

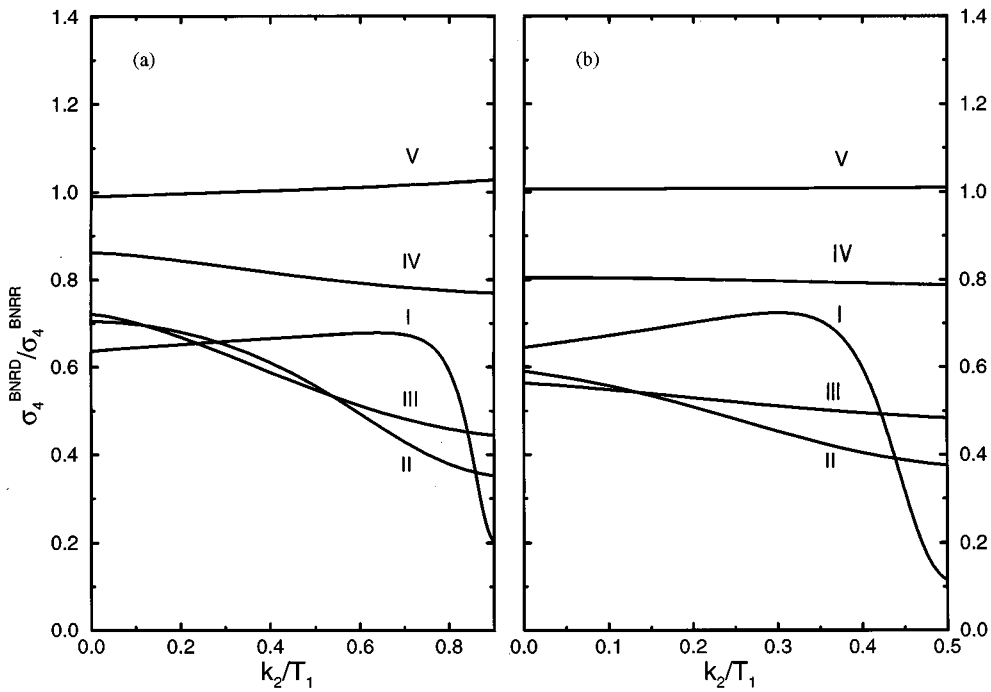

FIG. 1. The ratio between the values of σ_4 in the BNRD and BNRR calculations for $T_1=10$ keV and the five configurations I to V, corresponding to $\theta=10°$, 30°, 45°, 75°, and 90°, respectively, as a function of k_2/T_1, for (a) $k_1/T_1=0.1$ and (b) $k_1/T_1=0.5$.

We have to be cautious about using Born approximation in the low-energy range. To illustrate this aspect, we present Fig. 2, showing the ratio between the values of σ_4 in Born approximation and σ_4 with Coulomb effects included, for $Z=13$, as in Ref. [9], both in the dipole approximation. The configurations considered are III and V; the upper solid curves correspond to $k_1/T_1=0.1$ and the lower solid ones to $k_1/T_1=0.5$. One sees that Born approximation predicts lower values than the exact Coulomb calculation. For the Born approximation to apply, it is necessary for $\eta_1=\alpha Z/p_1$ and $\eta_2=\alpha Z/p_2$ to be much smaller than 1; also the final electron energy should be high enough. In the case of Fig. 2, $\eta_1=0.48$, and for $k_1/T_1=0.5$ η_2 increases from 0.715 to 2.15 with k_2/T_1 increasing from 0.05 to 0.45; for $k_1/T_1=0.1$, η_2 increases from 0.52 to 2.15 with k_2/T_1 increasing from 0.05 to 0.85. This explains the trend of the curves and also the difference between the upper and lower curves. The dashed curves show how the use of Elwert factor

$$f_E=\frac{\eta_2}{\eta_1}\frac{1-\exp(-2\pi\eta_1)}{1-\exp(-2\pi\eta_2)} \tag{13}$$

reduces the discrepancy between Born and Coulomb results in the nonrelativistic dipole approximation, as in the case of single-photon bremsstrahlung [13]. The Elwert factor was introduced by Sommerfeld [17] in order to take into account the deviations of the initial and final electron Coulomb continuum states from the plane waves. As these states are of the same type in two-photon bremsstrahlung, it is reasonable to use it in this case, too. Our results demonstrate the utility of the Elwert factor in two-photon bremsstrahlung. With increasing electron energies, the Born approximation becomes fairly good.

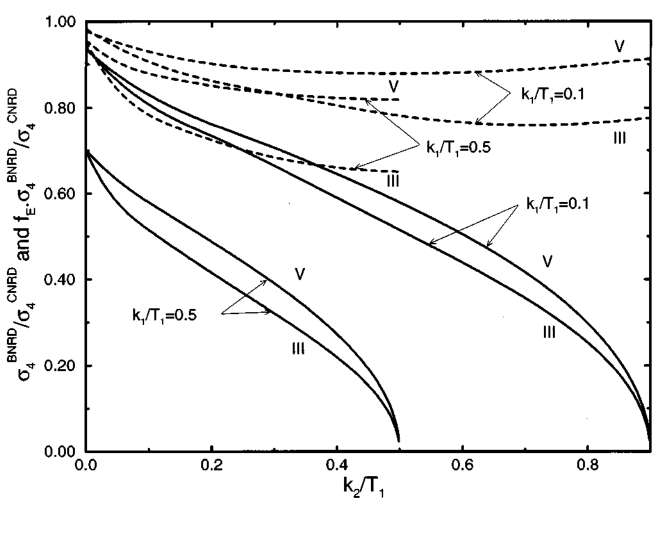

FIG. 2. The ratio between the values of σ_4 in the BNRD and CNRD approaches for $Z=13$, $T_1=10$ keV, as a function of k_2/T_1. The lower solid line curves correspond to $k_1/T_1=0.5$, the upper solid line curves to $k_1/T_1=0.1$. The labels III and V correspond to $\theta=45°$ and 90°, respectively. The dashed curves are obtained by multiplying with the electron Elwert factor f_E [see Eq. (13)].

Now we compare our BNRR results with those provided by the relativistic Born (BR) equations in order to illustrate the validity of our BNRR approximation and establish to what extent BNRR results can be used at higher energies. The numbers we use for BR, based on Smirnov's equation, Eq. (A1) in Ref. [7], were obtained with a numerical code described in [18]. This code was checked in a variety of situations by comparison with numbers provided by Quarles [19] and Scofield [20], based on independent numerical procedures. Retardation effects are, of course, included in such calculations. We present in Fig. 3 the ratio between results obtained with BNRR and with BR for σ_4 as a function of k_2/T_1. Each curve corresponds to a different value of T_1.

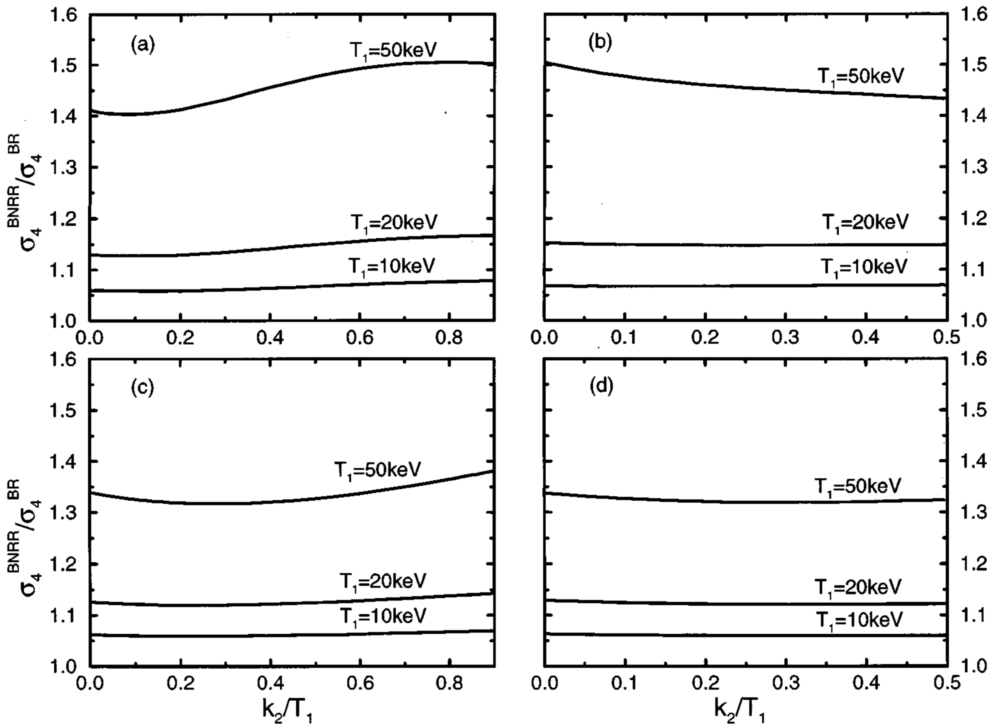

FIG. 3. The ratio between the values of σ_4 in BNRR and BR approximations, as a function of k_2/T_1, for $k_1/T_1=0.1$ and 0.5, for configurations III and V: (a) $k_1/T_1=0.1$, $\theta=45°$, (b) $k_1/T_1=0.5$, $\theta=45°$, (c) $k_1/T_1=0.1$, $\theta=90°$, and (d) $k_1/T_1=0.5$, $\theta=90°$. The three curves in each panel correspond to different values of T_1, indicated on the graphs.

Two configurations (III and V) and two values of k_1/T_1 are considered. Now, in contrast to Fig. 1, the change with the configuration has become less impressive, showing that retardation effects give the main difference between relativistic and nonrelativistic results. At $T_1=10$ keV, the relative error for $\theta=90°$ has become only 6%. The relative error is less than 15% at $T_1=20$ keV for both configurations. We notice the weak dependence of the ratio on all the parameters we have changed: k_1/T_1, k_2/T_1, and θ.

Now, we remember that the most striking fact in comparing BNRD and BR at energies of 70 keV, where the experiments of Quarles and co-workers are done, was the fact that BNRD predicts a larger cross-section for configuration V ($\theta=90°$) than for configuration III ($\theta=45°$), as can be seen from Figs. 3, 4, and 6 of Ref. [10]. This is also true for the CNRD results. By switching to the BNRR approach, the situation is reversed, in agreement with BR. While at 70 keV relativistic effects matter, they do not for the 8–12 keV incident electron energy range of Hippler's experiment [2]. A direct comparison with Hippler's data was presented in Fig. 1 of [10], based on a Coulomb nonrelativistic dipole approximation calculation, for the incident electron energy of 8.82 keV. But, as all these Hippler experiments were performed for $\theta=90°$, retardation effects are unimportant and do not explain the disagreement between theory and experiment. For other geometries, retardation effects must be included, even at these relatively low energies. We have studied the energy dependence of the ratio between the values of σ_4 at 90° and 45°. The inversion in magnitude between the values of σ_4 at 45° and 90° occurs at approximately 40 keV. We mention that BR calculations [18] show that at even higher energies near forward emission is favored.

In conclusion, our analysis shows the importance of the retardation effects in two-photon bremsstrahlung, and particularly their effect on the angular distribution of the emitted photons. Our comparisons suggest that the dipole approximation results for the distributions already cease to be correct at electron energies as low as 1 keV. Retardation effects may be suppressed by picking configurations for which $\vec{p}_1\cdot(\vec{\kappa}_1+\vec{\kappa}_2)=0$. Because our analysis is based on Born approximation, perhaps improved in Elwert-Born approximation, its conclusions should be checked also by a study including Coulomb field effects exactly. This seems to be feasible in the nonrelativistic case, but is a much more complex problem in a relativistic treatment.

The BNRR nonrelativistic Born approach including retardation, presented in Sec. II of this paper, provides very simple equations in comparison to the relativistic Born equations of Smirnov. These BNRR equations lead to numerical results in error by less than 20% in the energy range 10 to 30 keV and also at lower energies if the Elwert factor is included.

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