Dynamic noise reduction in multilevel lasers: Nonlinear theory and the pump-operator approach

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We present a general theory that connects the pump process in multilevel lasers with the statistics of the laser field. The key ingredient in our approach is the derivation of an effective master equation that involves only the two laser levels and that contains a non-Markovian pump term. This pump term gives rise to a narrowing of the photon-number distribution in steady state. The qualitative features of this dynamic noise reduction can be inferred from the eigenvalues of the pump term, which are known analytically. As the mechanism that is responsible for the noise reduction, we identify a correlated excitation process in which the effective excitation rate from the lower to the upper laser level depends on the photon number. This correlated excitation process is very different from an excitation at equidistant times, such as in lasers with periodic external injection. We stress that our approach is general and the treatment nonlinear because we do not resort to approximations such as an adiabatic elimination of atomic variables or the linearization of a Fokker-Planck equation. It is therefore particularly relevant for the description of systems where the active medium consists of a few atoms only and where the field losses are not negligible on the time scale of the atomic relaxations. This would be the case in an ion-trap laser. For such a situation our results are substantially different from those obtained by a linearized treatment.

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I. INTRODUCTION

In recent years, the possibility of intensity-noise reduction in lasers below the shot-noise limit has received considerable attention. A particularly interesting mechanism where the pump-noise suppression is generated by the lasing medium itself rather than by some external control has been studied by Ritsch *et al.* [1], Ralph and Savage [2], and others [3].

In their work, Ritsch *et al.* [1] investigate the ''recycling'' process of the active electron in a multilevel medium from the lower to the upper laser level. The excitation process via a sequence of intermediate levels is identified as a mechanism that leads to amplitude squeezing. In the limit of infinitely many intermediate levels, the system is found to behave similar to a maser with a regular (external) injection of atoms in the upper state, for which sub-Poissonian noise reduction has been found earlier [4–6]. This result is particularly relevant for active media such as semiconductors, which involve a cascade-type relaxation when being pumped.

In this paper, we present a different approach, which provides new insights into the problem of noise reduction in general. We find that the key mechanism responsible for the intensity-noise quenching is given by a correlated excitation scheme where the effective excitation rate of the atom from the lower to the upper level depends on the number of photons in the field. The quantitative dependence of the excitation rate on the field variables is obtained in terms of the eigenvalues of a general pump operator [7] for an effective two-level laser.

We find that the concept of correlated atomic excitation explains the mechanism of noise reduction inherently, that is from the dynamic equations themselves rather than assuming the pump process to be a separate stochastic process that can be more or less regular. This mechanism is quite different from lasers with a regular injection of atoms in the excited state.

Apart from its mathematical transparence, this approach is also quite general insofar as no assumptions or approximations such as an adiabatic elimination or the linearization of a Fokker-Planck equation are necessary. This is particularly relevant for the treatment of the laser process with single trapped ions [8–10], where the recycling process, the loss process, and the emission process may have comparable time scales. Even for high-finesse mirrors the loss out of the mirrors during the ''dead period'' of the ion, when it is reexcited, is appreciable. Here, a weak-pump-field approximation is not applicable either, and the ground state will be depleted most of the time. In such a situation, the atom and the field have to be treated as a single coherent object, and the full density matrix equation has to be considered.

In a direct numerical treatment of the density matrix equations, Mu and Savage [8] showed that a four-level pump scheme can lead to sub-Poissonian output for such a one-atom laser. In our approach, we can give an analytical explanation for this numerical result. In addition, we are able to treat the general multilevel system on the same footing and without an approximation. By this we confirm the predicted enhancement of noise reduction for the multilevel excitation on the basis of the exact single-atom master equation. In particular, we can investigate the system for any parameters and are not restricted to the situation far above threshold. If one wishes to go beyond the single-atom case, one could employ the standard method by Risken [11], where correlations between different atoms are neglected consistently;

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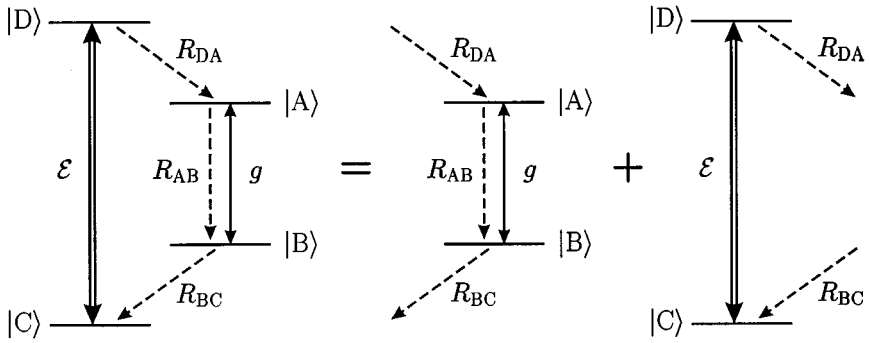

FIG. 1. Schematic representation of the four-level atom consisting of the laser system and the pump system.

other approximations would not be required.

The paper is organized as follows. In Sec. II we start with the full density-matrix description of the four-level laser and introduce the notion of the pump system and the laser system as two open subsystems of the composed laser model. The pump system involves the classical external field that drives the atom. It replaces the ''hot reservoir'' in standard reservoir theory [12] and can be reduced to such a simplified description under certain conditions to be stated. The laser system describes the coupling of the laser levels to the quantized field mode. It is coupled both to a cold reservoir, due to the cavity losses, and to the pump system. In Sec. II C we eliminate the variables of the pump system from the description without resorting to any approximations. In the master equation for the laser system, this leads to a non-Markovian term that accounts for the atomic memory during the reexcitation. In steady state, this non-Markovian term reduces to a pump operator, which is a function of the field variables. The eigenvalues of this operator can easily be found in the damping-basis representation. From the spectrum of eigenvalues, we can draw qualitative conclusions on noise reduction. In addition, we see how the non-Markovian master equation contains corrections to standard reservoir theory and in which limits the standard theory is recovered. In Sec. III we look at the pump operator in the number-state representation and identify correlated excitation as the key mechanism responsible for dynamic noise reduction. The elimination of the pump levels into a pump operator can also be done in the general case of arbitrarily many incoherent segments involved in the pumping process, which is done in Sec. IV. There, we also treat the limit of infinitely many intermediate levels, in which the non-Markovian pump term of the master equation reduces to an explicit retardation term. This is an appropriate place to come back to the interpretation of the mechanism of noise reduction. In Sec. V we summarize the results. In three appendices, we supply additional material and state some of the results of the main text in more generality.

II. THE FOUR-LEVEL LASER

The scheme to which we refer in the following is shown in Fig. 1. The laser transition is between the levels $|\mathrm{A}\rangle$ and $|\mathrm{B}\rangle$ and couples to a quantized laser mode of frequency ω, which is described by the ladder operators $a^\dagger$ and a. The lower level spontaneously decays to the ground state $|\mathrm{C}\rangle$ with the rate R_{BC} from where the atom is excited to an upper (pump) level $|\mathrm{D}\rangle$ by a classical coherent field with a harmonic time dependence $\exp(-i\omega_{\mathrm{DC}}t)$. From this level, which lies energetically higher than the upper laser level $|\mathrm{A}\rangle$, there is again a spontaneous decay to $|\mathrm{A}\rangle$. Together, this constitutes an effective pump process from $|\mathrm{B}\rangle$ to $|\mathrm{A}\rangle$. The frequency spacing between the levels as well as the decay rates are indicated by the symbols $\omega_{\mathrm{LL}'}$ and $R_{\mathrm{LL}'}$, respectively, with subscripts $\mathrm{L,L}'=\mathrm{A,B,C,D}$. For simplicity, we assume that the laser transition is resonant with the mode, $\omega_{\mathrm{AB}}=\omega$, and the pump transition is resonant with the classical field, $\omega_{\mathrm{DC}}=\omega_{\mathrm{DA}}+\omega+\omega_{\mathrm{BC}}$. Also, we do not consider spontaneous decay from $|\mathrm{D}\rangle$ to $|\mathrm{C}\rangle$ in order to keep the notation transparent. None of these restrictions is essential for the following. In Appendix B, we state some of the results for more general parameters. In the following, we will refer to the levels $|\mathrm{A}\rangle$ and $|\mathrm{B}\rangle$ as *laser* levels in contrast to the *pump* levels $|\mathrm{C}\rangle$ and $|\mathrm{D}\rangle$.

A. Pump levels and laser levels

The dynamics of the composed atom-field system is described by a master equation of the form

$$\frac{\partial}{\partial t}\mathrm{P}=\frac{1}{i\hbar}[H,\mathrm{P}]+L_a\mathrm{P}+L_\Sigma\mathrm{P}, \tag{2.1}$$

with a unitary and a nonunitary part.

The unitary part contains the coupling of the laser levels and the pump levels to the laser and the pump field, respectively. The Hamilton operator H for this interaction can, in an interaction picture, be written as

$$H/\hbar=-g(a|\mathrm{A}\rangle\langle\mathrm{B}|+a^\dagger|\mathrm{B}\rangle\langle\mathrm{A}|)-\mathscr{E}|\mathrm{D}\rangle\langle\mathrm{C}|-\mathscr{E}^*|\mathrm{C}\rangle\langle\mathrm{D}|, \tag{2.2}$$

where the Rabi frequencies g and $\mathscr{E}$ measure the coupling strengths to the laser transition and the pump transition, respectively.

The nonunitary part involves the photon damping L_a and the spontaneous decays L_Σ between the atomic levels. The first term reads

$$L_a\mathrm{P}=-\frac{A}{2}(\nu+1)(a^\dagger a\mathrm{P}-2a\mathrm{P}a^\dagger+\mathrm{P}a^\dagger a)$$

$$-\frac{A}{2}\nu(aa^\dagger\mathrm{P}-2a^\dagger\mathrm{P}a+\mathrm{P}aa^\dagger), \tag{2.3}$$

where A denotes the free relaxation rate of the mean photon number $\langle a^\dagger a\rangle$ towards its thermal equilibrium value $\langle a^\dagger a\rangle_\infty=\nu$. The second term has the form

$$L_\Sigma \mathrm{P} = -\frac{R_{\mathrm{AB}}}{2}\Big(|\mathrm{A}\rangle\langle\mathrm{A}|\mathrm{P}+\mathrm{P}|\mathrm{A}\rangle\langle\mathrm{A}| - 2|\mathrm{B}\rangle\langle\mathrm{A}|\mathrm{P}|\mathrm{A}\rangle\langle\mathrm{B}|\Big)$$
$$-\frac{R_{\mathrm{BC}}}{2}\Big(|\mathrm{B}\rangle\langle\mathrm{B}|\mathrm{P}+\mathrm{P}|\mathrm{B}\rangle\langle\mathrm{B}| - 2|\mathrm{C}\rangle\langle\mathrm{B}|\mathrm{P}|\mathrm{B}\rangle\langle\mathrm{C}|\Big)$$
$$-\frac{R_{\mathrm{DA}}}{2}(|\mathrm{D}\rangle\langle\mathrm{D}|\mathrm{P}+\mathrm{P}|\mathrm{D}\rangle\langle\mathrm{D}| - 2|\mathrm{A}\rangle\langle\mathrm{D}|\mathrm{P}|\mathrm{D}\rangle\langle\mathrm{A}|), \qquad (2.4)$$

in which the three terms on the right-hand side describe the relaxations from $|\mathrm{A}\rangle$, $|\mathrm{B}\rangle$, and $|\mathrm{D}\rangle$ into the lower-lying levels $|\mathrm{B}\rangle$, $|\mathrm{C}\rangle$, and $|\mathrm{A}\rangle$, respectively, by emission of photons into the modes of free space.

In the following, we will not make further assumptions on the relative strengths of the parameters g, $\mathscr{E}$, A, and $R_{\mathrm{LL}'}$. For the ion-trap laser, it will be a challenge to implement mirrors with a finesse that is sufficiently high. The lifetime of the photon has to be at least so long that it does not get lost during the finite reexcitation time of the ion. This is, in addition to a strong atom-field coupling, necessary to keep the microlaser going. Theoretically, this situation corresponds neither to a good-cavity limit nor a bad-cavity limit. Rather, the cavity is ''just good enough.'' The parameters will, therefore, typically be all of the same order of magnitude.

In a first step, we expand the composed state P into the bare atomic states, that is

$$\mathrm{P} = \sum_{\mathrm{LL}'} \rho_{\mathrm{LL}'} |\mathrm{L}\rangle\langle\mathrm{L}'|, \qquad (2.5)$$

with indices L,L′=A,B,C,D, wherein the coefficients $\rho_{\mathrm{LL}'}=\rho_{\mathrm{LL}'}(t,a^\dagger,a)$ are still functions of the photon variables $a^\dagger$ and a. For instance, $\rho_{\mathrm{AA}}=\langle\mathrm{A}|\mathrm{P}|\mathrm{A}\rangle$ describes the state of the laser field *given* that the atom is in state $|\mathrm{A}\rangle$. We therefore sometimes call the quantities $\rho_{\mathrm{LL}'}$ conditional states of the field.

Upon inserting (2.5) into (2.1), one obtains the following equations

$$\left(\frac{\partial}{\partial t} - L_a + R_{\mathrm{AB}}\right)\rho_{\mathrm{AA}} = -ig(\rho_{\mathrm{AB}}a^\dagger - a\rho_{\mathrm{BA}}) + R_{\mathrm{DA}}\rho_{\mathrm{DD}},$$
$$\left(\frac{\partial}{\partial t} - L_a + R_{\mathrm{BC}}\right)\rho_{\mathrm{BB}} = -ig(\rho_{\mathrm{BA}}a - a^\dagger\rho_{\mathrm{AB}}) + R_{\mathrm{AB}}\rho_{\mathrm{AA}},$$
$$\left(\frac{\partial}{\partial t} - L_a + \frac{R_{\mathrm{AB}}+R_{\mathrm{BC}}}{2}\right)\rho_{\mathrm{AB}} = -ig(\rho_{\mathrm{AA}}a - a\rho_{\mathrm{BB}}),$$
$$\left(\frac{\partial}{\partial t} - L_a + \frac{R_{\mathrm{AB}}+R_{\mathrm{BC}}}{2}\right)\rho_{\mathrm{BA}} = ig(a^\dagger\rho_{\mathrm{AA}} - \rho_{\mathrm{BB}}a^\dagger), \qquad (2.6)$$

as well as

$$\left(\frac{\partial}{\partial t} - L_a\right)\rho_{\mathrm{CC}} = -i\pi_{\mathrm{CD}} + R_{\mathrm{BC}}\rho_{\mathrm{BB}},$$
$$\left(\frac{\partial}{\partial t} - L_a + R_{\mathrm{DA}}\right)\rho_{\mathrm{DD}} = i\pi_{\mathrm{CD}},$$
$$\left(\frac{\partial}{\partial t} - L_a + \frac{1}{2}R_{\mathrm{DA}}\right)i\pi_{\mathrm{CD}} = 2|\mathscr{E}|^2(\rho_{\mathrm{CC}} - \rho_{\mathrm{DD}}), \qquad (2.7)$$

where we have introduced the anti-Hermitean polarization term $\pi_{\mathrm{CD}} = \mathscr{E}\rho_{\mathrm{CD}} - \mathscr{E}^*\rho_{\mathrm{DC}}$. The corresponding Hermitean combination satisfies the separate equation

$$\left(\frac{\partial}{\partial t} - L_a + \frac{1}{2}R_{\mathrm{DA}}\right)(\mathscr{E}\rho_{\mathrm{CD}} + \mathscr{E}^*\rho_{\mathrm{DC}}) = 0 \qquad (2.8)$$

and vanishes for large times. The equations for ρ_{AC}, ρ_{AD}, ρ_{BC}, ρ_{BD} and their adjoints are decoupled from the equations stated above. These ρ's involve pairs of levels between which there is no coherent connection, so the dissipation makes them vanish for large times, too, and they are not of interest for the present purpose.

In summary, the problem reduces to the solution of the seven coupled equations (2.6) and (2.7). The four ''laser equations'' (2.6) describe the interaction of the laser transition $|\mathrm{A}\rangle\leftrightarrow|\mathrm{B}\rangle$ with the quantized field. The system of these equations is closed except for the inhomogeneous term $R_{\mathrm{DA}}\rho_{\mathrm{DD}}$ in the first equation, which involves the upper pump level. This gain term, which is proportional to ρ_{DD}, describes the rate of change of the conditional state ρ_{AA} due to transitions from the pump level $|\mathrm{D}\rangle$ to the laser level $|\mathrm{A}\rangle$.

The three ''pump equations'' (2.7) describe the interaction of the pump transition $|\mathrm{C}\rangle\leftrightarrow|\mathrm{D}\rangle$ with the classical field. These are the Bloch equations for the conditional states of the laser field that involve the pump levels. Again, these equations are closed except for an inhomogeneous term $R_{\mathrm{BC}}\rho_{\mathrm{BB}}$ in the first equation of (2.7), which accounts for transitions from the lower laser level to the ground state.

Equations (2.6) and (2.7) thus describe two subsystems, the laser system and the pump system, which are coupled. The losses out of the laser system, with rate R_{BC}, appear as an inhomogeneous gain term for the pump system, and the losses out of the pump system, with rate R_{DA}, give an inhomogeneous gain term for the laser system.

Since the details of the pump process are generally not of interest, it would be natural to eliminate the equations of (2.7) from the description by solving them for ρ_{DD} and inserting the result into the first equation of (2.6). This is, in fact, what we are going to do.

In the situation of conventional laser theory where the field does not change appreciably on the time scale of the atomic excitation and, in particular, the losses out of the mirrors are negligible, this elimination can easily be done. Upon setting $L_a=0$ in (2.7) (which means $L_a' \ll R_{\mathrm{DA}}, |\mathscr{E}|$ for all relevant eigenvalues L_a' of L_a), we find in steady state $\rho_{\mathrm{DD}}=(R_{\mathrm{BC}}/R_{\mathrm{DA}})\rho_{\mathrm{BB}}$. The corresponding time-dependent relation holds if we further assume that $g, R_{\mathrm{BC}} \ll R_{\mathrm{DA}}, |\mathscr{E}|$ holds, which means that the excitation from $|\mathrm{C}\rangle$ to $|\mathrm{A}\rangle$ is faster than all other time scales. The first two laser equations of (2.6) then read

$$\left(\frac{\partial}{\partial t} - L_a + R_{\mathrm{AB}}\right)\rho_{\mathrm{AA}} = -ig(\rho_{\mathrm{AB}}a^\dagger - a\rho_{\mathrm{BA}}) + R_{\mathrm{BC}}\rho_{\mathrm{BB}},$$
$$\left(\frac{\partial}{\partial t} - L_a + R_{\mathrm{BC}}\right)\rho_{\mathrm{BB}} = -ig(\rho_{\mathrm{BA}}a - a^\dagger\rho_{\mathrm{AB}}) + R_{\mathrm{AB}}\rho_{\mathrm{AA}}, \qquad (2.9)$$

and the equations for ρ_{BA} and ρ_{AB} remain unchanged.

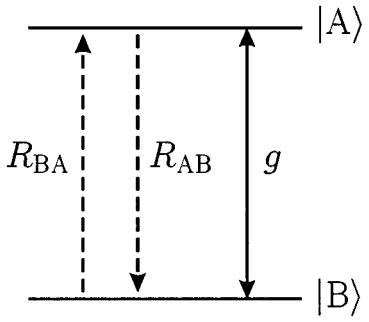

FIG. 2. Schematic representation of the standard heat-bath model for the two-level laser.

Now the system of equations involving the levels $|A\rangle$ and $|B\rangle$ is closed. The substitution $R_{DA}\rho_{DD}=R_{BC}\rho_{BB}$ means that the losses out of $|B\rangle$ appear as simultaneous gain into the upper level $|A\rangle$ without a further change in the field. This we call the limit of ''instantaneous pumping.'' It reduces the four-level model of Fig. 1 to the standard ''heat-bath model'' for the laser, where the pump process is determined by a single excitation rate $R_{BA}\equiv R_{BC}$, given phenomenologically, as shown in Fig. 2. When $R_{BA}>R_{AB}$, the atom can, due to an external heat bath, be inverted and the lasing process is possible [13].

In the present paper we are, however, interested in systems in which it is *not* possible to neglect photon damping in (2.7), since the losses out of the mirrors during the time of recycling may be significant. When this is the case, we cannot set $L_a=0$ in (2.7). Please note that L_a in (2.7) is an operator acting on conditional states $\rho_{LL'}$. The dependence of all functions $\rho_{LL'}$, including L,L′=C,D, on the photon variables $a^\dagger$ and a indicates that the dynamics of the pump process depends on the state of the laser field. If L_a were a number, we could again easily integrate (2.7) and solve for ρ_{DD}.

At this point it is helpful to remember that L_a, in fact, is essentially a number on certain states, namely its eigenstates. As has been shown in [14], these eigenstates form a complete set into which any function of the photon variables, so, for instance, the functions $\rho_{LL'}$, may be expanded. Consequently, any operator function of L_a is well defined in terms of the spectral representation of L_a. In the following treatment, we will therefore not care about the operator nature of L_a and treat it as if it were a number.

Before we proceed, let us briefly review the damping-basis formalism in the next section. For a detailed treatment the reader should consult Ref. [14].

B. The damping basis

The damping operator satisfies the following eigenvalue equations

$$L_a\rho_n^{(k)}=-A(n+|k|/2)\rho_n^{(k)},$$

$$\check{\rho}_n^{(k)}L_a=-A(n+|k|/2)\check{\rho}_n^{(k)}, \tag{2.10}$$

for $n=0, 1, 2, \ldots$ and $k=0, \pm1, \pm2, \ldots$ with right and left eigenstates $\rho_n^{(k)}$ and $\check{\rho}_n^{(k)}$, whose explicit forms involve normally ordered Laguerre polynomials as functions of $a^\dagger a$ as is shown in Ref. [14]. For zero temperature, $\nu=0$, these eigenstates are explicitly given by

$$\rho_n^{(k)}=a^{\dagger(|k|+k)/2}(-1)^{a^\dagger a+n}\binom{n+|k|}{a^\dagger a+|k|}a^{(|k|-k)/2} \tag{2.11}$$

and

$$\check{\rho}_n^{(k)}=\frac{n!}{(n+|k|)!}a^{\dagger(|k|-k)/2}\binom{a^\dagger a}{n}a^{(|k|+k)/2}. \tag{2.12}$$

Please note that the definition of a left eigenstate in (2.10) refers to the trace, analogous to an inner product for ''bras'' and ''kets'' in unitary quantum mechanics [15].

The set of these states is complete in the sense that we may write

$$\rho(t)=\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\alpha_{nk}(t)\rho_n^{(k)} \tag{2.13}$$

with coefficients

$$\alpha_{nk}(t)=\mathrm{tr}\{\check{\rho}_n^{(k)}\rho(t)\} \tag{2.14}$$

for any state $\rho(t)$ which is a function of $a^\dagger$ and a. The basis states also satisfy the duality relation

$$\mathrm{tr}\{\check{\rho}_n^{(k)}\rho_{n'}^{(k')}\}=\delta_{nn'}\delta_{kk'}, \tag{2.15}$$

and special examples are the thermal state $\rho_0^{(0)}$ and its dual $\check{\rho}_0^{(0)}=1$.

With the eigenstates of (2.10) at hand, we find the action of L_a on the conditional states $\rho_{LL'}$ in Eqs. (2.6) and (2.7). In particular, expansions of the form (2.13) are employed when an arbitrary function of L_a is acting, as is exemplified by

$$f(L_a)\rho_{LL'}(t)=\sum_{n,k}f(-A(n+|k|/2))\alpha_{nk}^{LL'}(t)\rho_n^{(k)}. \tag{2.16}$$

When we further modify Eqs. (2.6) and (2.7) in the next section, we will treat L_a as if it were an ordinary number, remembering that functions of L_a are evaluated in accordance with Eq. (2.16).

C. Elimination of the pump levels

For further treatment we write (2.7) in the form

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho_{CC}\\ \rho_{DD}\\ i\pi_{CD}\end{pmatrix}=\mathcal{M}\begin{pmatrix}\rho_{CC}\\ \rho_{DD}\\ i\pi_{CD}\end{pmatrix}+\begin{pmatrix}R_{BC}\rho_{BB}\\ 0\\ 0\end{pmatrix} \tag{2.17}$$

with the (Bloch) matrix

$$\mathcal{M}\equiv\begin{pmatrix}L_a & 0 & -1\\ 0 & L_a-R_{DA} & 1\\ 2|\mathcal{E}|^2 & -2|\mathcal{E}|^2 & L_a-\frac{1}{2}R_{DA}\end{pmatrix}. \tag{2.18}$$

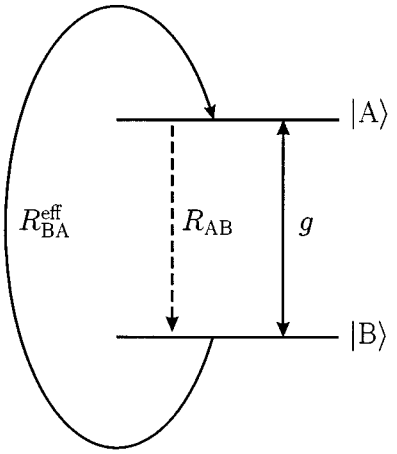

FIG. 3. Schematic representation of the multilevel atom as an effective two-level atom with pumping described by an operator.

This matrix, which contains as an entry also the damping operator L_a, is defined as an ordinary number matrix on the eigenstates of L_a. Any operations on $\mathscr{M}$ are therefore operations with an ordinary matrix when applied to these states. To integrate (2.17) explicitly,

$$\begin{pmatrix} \rho_{CC}(t) \\ \rho_{DD}(t) \\ i\pi_{CD}(t) \end{pmatrix} = e^{\mathscr{M}(t-t_0)} \begin{pmatrix} \rho_{CC}(t_0) \\ \rho_{DD}(t_0) \\ i\pi_{CD}(t_0) \end{pmatrix} + \int_{t_0}^{t} dt' e^{\mathscr{M}(t-t')} \times \begin{pmatrix} R_{BC}\rho_{BB}(t') \\ 0 \\ 0 \end{pmatrix}, \tag{2.19}$$

one has to diagonalize $\mathscr{M}$.

For the component $\rho_{DD}(t)$, in particular, one obtains

$$\rho_{DD}(t) = \int_{-\infty}^{t} dt' \Gamma_{BD}(t-t') \rho_{BB}(t') \tag{2.20}$$

with a kernel

$$\Gamma_{BD}(t-t') = \frac{|\mathscr{E}|^2 R_{BC}}{|\mathscr{E}|^2 - R_{DA}^2/16} \exp[(L_a - R_{DA}/2)(t-t')] \times \sin^2[\sqrt{|\mathscr{E}|^2 - R_{DA}^2/16}(t-t')], \tag{2.21}$$

which does not vanish identically for $t > t'$. In doing this, we choose the initial state at time $t_0 = -\infty$, whose contribution vanishes since the matrix $\mathscr{M}$ has only eigenvalues with negative real parts, as can be seen in (2.26) below. Upon inserting (2.20) in (2.6), we arrive at the laser equations

$$\left(\frac{\partial}{\partial t} - L_a + R_{AB}\right)\rho_{AA} = -ig(\rho_{AB}a^\dagger - a\rho_{BA}) + \int_{-\infty}^{t} dt' R_{BA}^{eff}(t-t')\rho_{BB}(t'),$$

$$\left(\frac{\partial}{\partial t} - L_a + R_{BC}\right)\rho_{BB} = -ig(\rho_{BA}a - a^\dagger\rho_{AB}) + R_{AB}\rho_{AA}, \tag{2.22}$$

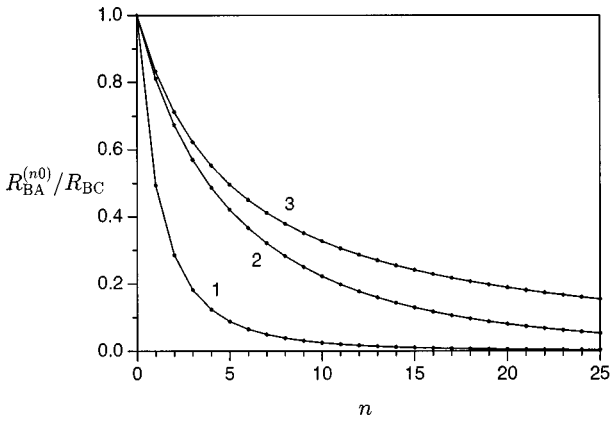

FIG. 4. Eigenvalue spectrum of the pump operator for the parameters $R_{DA}/A = 10$ and (1) $|\mathscr{E}|/A = 2$, (2) $|\mathscr{E}|/A = 10$, and (3) $|\mathscr{E}|/A = 50$.

supplemented by the equations for ρ_{AB} and ρ_{BA}, which are the same as in (2.6). The time integral in (2.22) accounts for the delay involved in pumping the atom from the lower to the upper laser level. Roughly speaking, the temporally nonlocal kernel

$$R_{BA}^{eff}(t-t') = \Gamma_{BD}(t-t')R_{DA} \tag{2.23}$$

represents an effective transition rate into the level $|A\rangle$ at time t, which depends on the population in $|B\rangle$, and thus on the state of the field, at the earlier time t'. This temporally nonlocal or non-Markovian behavior is significant in a regime where the time scales of the pump process and the laser process are comparable. If on the other hand the pump acts very fast compared to the laser transition, it can be adiabatically eliminated from the description, which is done in Appendix A.

In steady state, when $\rho_{BB}(t') = \rho_{BB}^{(SS)}$, the time integral in (2.22) can be evaluated and yields

$$R_{BA}^{eff} = \int_0^\infty dt' R_{BA}^{eff}(t') = R_{BC}\frac{2|\mathscr{E}|^2}{-\det\mathscr{M}}R_{DA} \tag{2.24}$$

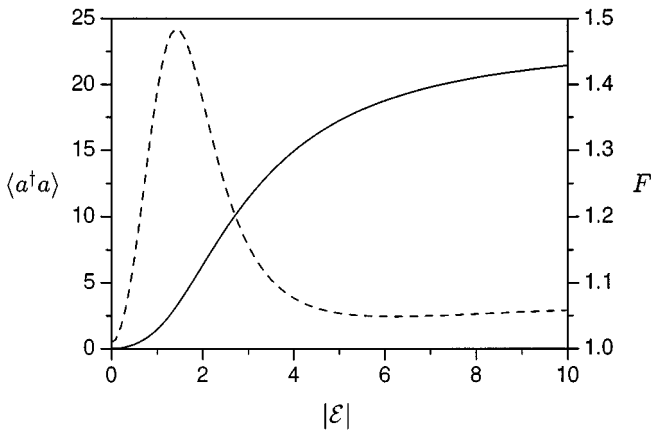

FIG. 5. The mean photon number $\langle a^\dagger a\rangle$ (solid curve) and the Fano factor F (dashed curve) as functions of the pump strength $|\mathscr{E}|$ for the four-level atom with the parameters $\nu = 0.01$, $A = 0.07$, $R_{AB} = 1$, $R_{BC} = R_{DA} = 10$, and a decay rate $R_{DC} = 1$ on the pump transition (as in Appendix B). All rates are given in units of g.

with $\mathscr{M}$ of (2.18). The evaluation of the integral can be seen most easily from (2.19) directly, which involves the inverse of the Bloch matrix $\mathscr{M}$. The inverse of the determinant is explicitly given by

$$\frac{1}{\det\mathscr{M}}\rho_n^{(k)}=\frac{1}{\lambda_{nk}^{(0)}\lambda_{nk}^{(-)}\lambda_{nk}^{(+)}}\rho_n^{(k)} \tag{2.25}$$

with the eigenvalues

$$\lambda_{nk}^{(0)}=-A(n+|k|/2)-(1/2)R_{\mathrm{DA}},$$
$$\lambda_{nk}^{(\pm)}=-A(n+|k|/2)-(1/2)R_{\mathrm{DA}}\pm\sqrt{(1/4)R_{\mathrm{DA}}^2-4|\mathscr{E}|^2} \tag{2.26}$$

of $\mathscr{M}$. For R_{DA}, $|\mathscr{E}|>0$, all eigenvalues are surely nonzero and $R_{\mathrm{BA}}^{\mathrm{eff}}$ is well defined in (2.24). Putting things together, we find the steady-state equations

$$(-L_a+R_{\mathrm{AB}})\rho_{\mathrm{AA}}=-ig(\rho_{\mathrm{AB}}a^\dagger-a\rho_{\mathrm{BA}})+R_{\mathrm{BA}}^{\mathrm{eff}}\rho_{\mathrm{BB}}$$
$$(-L_a+R_{\mathrm{BC}})\rho_{\mathrm{BB}}=-ig(\rho_{\mathrm{BA}}a-a^\dagger\rho_{\mathrm{AB}})+R_{\mathrm{AB}}\rho_{\mathrm{AA}},$$
$$[-L_a+(R_{\mathrm{AB}}+R_{\mathrm{BC}})/2]\rho_{\mathrm{AB}}=-ig(\rho_{\mathrm{AA}}a-a\rho_{\mathrm{BB}}),$$
$$[-L_a+(R_{\mathrm{AB}}+R_{\mathrm{BC}})/2]\rho_{\mathrm{BA}}=ig(a^\dagger\rho_{\mathrm{AA}}-\rho_{\mathrm{BB}}a^\dagger), \tag{2.27}$$

with $\rho_{\mathrm{LL}'}=\rho_{\mathrm{LL}'}^{(\mathrm{SS})}$. The excitation process via the levels $|\mathrm{C}\rangle$ and $|\mathrm{D}\rangle$ is now comprised in the *pump operator* $R_{\mathrm{BA}}^{\mathrm{eff}}$, which effectively describes the transition from the lower to the upper laser level [16] as illustrated in Fig. 3. The rate of these transitions, however, depends also on the state of the photon field as can be seen from the eigenvalue spectrum

$$R_{\mathrm{BA}}^{\mathrm{eff}}\rho_n^{(0)}=\frac{R_{\mathrm{BC}}2|\mathscr{E}|^2R_{\mathrm{DA}}}{2|\mathscr{E}|^2R_{\mathrm{DA}}+[4|\mathscr{E}|^2+(1/2)R_{\mathrm{DA}}^2]An+(3/2)R_{\mathrm{DA}}(An)^2+(An)^3}\rho_n^{(0)}\equiv R_{\mathrm{BA}}^{(n0)}\rho_n^{(0)}, \tag{2.28}$$

here explicitly reported for $k=0$. For $k\neq0$, the corresponding expression for $R_{\mathrm{BA}}^{(nk)}$ has the same appearance with An replaced by $A(n+|k|/2)$ everywhere in the denominator.

The appearance of A in the denominator indicates that the pump is sensitive to the losses of the field during the reexcitation. Since n labels the statistical moments of the field in the expansion (2.13), (2.28) means that the effective pump rates $R_{\mathrm{BA}}^{(n0)}$ from level $|\mathrm{B}\rangle$ to level $|\mathrm{A}\rangle$ is correlated with the state of the laser field. Roughly speaking, different statistical moments (or components) of the field feel a pump that acts with different strength. In particular, higher statistical moments (related to larger photon numbers) are less supported than lower moments. A quantity such as the Fano factor for the photon-number distribution, which involves ratios of its statistical moments, can thereby become smaller than one, indicating sub-Poissonian statistics.

In Fig. 4, the $k=0$ part of the spectrum (2.28) is shown. One observes that the dependence of the pump rates on n is strongest for small values of $|\mathscr{E}|$. This gives rise to the existence of a minimum observed in the curves for the Fano factor when plotted against $|\mathscr{E}|$, as is further pointed out in the discussion of Figs. 5 and 6 in the following section.

To get a feeling for the consequences of (2.28), let us look at the special case $g=0$ first. For $g=0$ when the atom and the field are decoupled, the field relaxes towards the vacuum or the thermal state $\rho_0^{(0)}$, on which the operator $R_{\mathrm{BA}}^{\mathrm{eff}}$ is identical to the number R_{BC}. In this situation, the pump does not feel the presence of the field nor its losses. The inversion of the atom is then uncorrelated with the state of the field and only determined by the relative strengths of the atomic relaxation rates and the pump.

For $g\neq0$, however, the atom and the field constitute a single coherent object and the transition rates from $|\mathrm{B}\rangle$ to $|\mathrm{A}\rangle$ and the state of the photon field are correlated according to (2.28). The effective pump rates of (2.28), in some sense, account for the "dead period" of the laser process during which the atom stays in one of the pump levels $|\mathrm{C}\rangle$ or $|\mathrm{D}\rangle$. During that time, the field is damped, and this damping is stronger for components with a larger n, corresponding to a larger photon number. This dead time constitutes a certain fraction of the whole process whose size is measured by the relative strengths of the rates An on one side, and $|\mathscr{E}|$, R_{DA} on the other side. If the pump process is fast compared to the relaxation of the field, that is $An\ll|\mathscr{E}|,R_{\mathrm{DA}}$, then the denominator of (2.28) can be replaced, in zeroth order, by the leading term, and one obtains again the heat-bath model

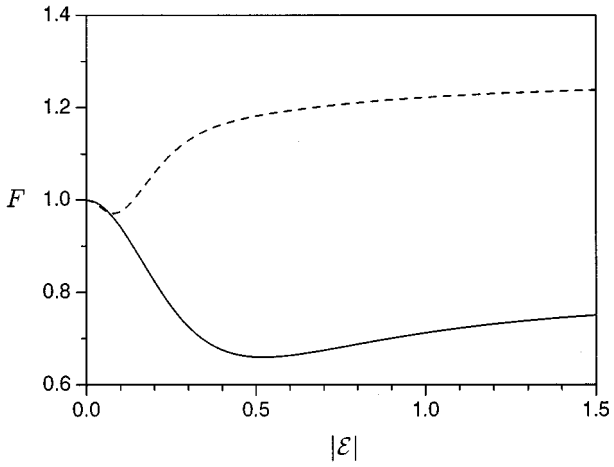

FIG. 6. The Fano factor F as a function of the pump strength $|\mathscr{E}|$ for the four-level atom with the parameters $\nu=0$, $A=0.02$, $R_{\mathrm{AB}}=0.01$, $R_{\mathrm{BC}}=1$, and $R_{\mathrm{DA}}=2$. All rates are given in units of g (solid curve) and in units of $10g$ (dashed curve).

(2.9). The first-order correction to this approximation is proportional to the ratio of these rates and introduces an n dependence into the effective pump rates: For components of the field corresponding to larger indices n, the pump has to be stronger (faster) in order to compensate for the losses.

The correlation of the effective pump rates with different statistical components of the field is ultimately responsible for the phenomenon of noise reduction in multilevel lasers. In Sec. III the physical interpretation of this correlation is further illuminated by looking at the number-state representation of the pump operator.

D. Numerical treatment

To evaluate (2.27), we expand the conditional density operators into the damping basis, as explained in Sec. II B. The expansions

$$\rho_{\mathrm{AA}}=\sum_{n,k}\alpha_{nk}\rho_n^{(k)},$$

$$\rho_{\mathrm{BB}}=\sum_{n,k}\beta_{nk}\rho_n^{(k)},$$

$$\rho_{\mathrm{AB}}=\sum_{n,k}\gamma_{nk}\rho_n^{(k-1)},$$

$$\rho_{\mathrm{BA}}=\sum_{n,k}\eta_{nk}\rho_n^{(k+1)}, \tag{2.29}$$

which are similar to those in Ref. [14], produce a matrix equation that is the coordinate representation of (2.27). In steady state, the only nonvanishing components are those for $k=0$, for which we obtain [17]

$$\begin{aligned}
(An+R_{\mathrm{AB}})\alpha_n &= -ig(n+1)(\gamma_n-\eta_n)-ign\frac{\nu}{\nu+1}(\gamma_{n-1}-\eta_{n-1})+R_{\mathrm{BA}}^{(n0)}\beta_n,\\
(An+R_{\mathrm{BC}})\beta_n &= ig(n+1)(\gamma_n-\eta_n)+ign(\gamma_{n-1}-\eta_{n-1})-R_{\mathrm{AB}}\alpha_n,\\
[A(n+1/2)+(R_{\mathrm{AB}}+R_{\mathrm{BC}})/2]\gamma_n &= -ig(\nu+1)\alpha_n+ig\nu\beta_n-ig(\nu+1)(\alpha_{n+1}-\beta_{n+1}),\\
[A(n+1/2)+(R_{\mathrm{AB}}+R_{\mathrm{BC}})/2]\eta_n &= ig(\nu+1)\alpha_n-ig\nu\beta_n+ig(\nu+1)(\alpha_{n+1}-\beta_{n+1}),
\end{aligned} \tag{2.30}$$

where we have suppressed the index $k=0$. These are the equations that we can solve with a matrix eigenvalue algorithm along the lines of Ref. [9].

The difference to a real two-level laser is here given by the fact that the total trace of P involves also the intermediate pump levels. In (2.30) this has to be taken into account as a subsidiary condition, which can be written in the form

$$1=\mathrm{Tr}\{\mathrm{P}\}=\alpha_0+\beta_0\left[1+R_{\mathrm{BC}}\left(\frac{2}{R_{\mathrm{DA}}}+\frac{R_{\mathrm{DA}}}{4|\mathscr{E}|^2}\right)\right]. \tag{2.31}$$

Let us emphasize that the numerical effort to solve the steady-state equations (2.27) does not depend on the number N of pump levels involved. The dimension of the matrices describing (2.30) is the same for a thousand levels as for two levels, since the number N of levels only enters the definition of the rates $R_{\mathrm{BA}}^{(n0)}$ in (2.30) and does not influence the number of equations as will be shown in Sec. IV. This represents a substantial simplification as compared to a brute-force numerical treatment, for which the expense increases with the number of levels.

In Fig. 5, the lasing property of a single atom is demonstrated. As for a conventional four-level laser one can see a threshold, a region of linear gain, and saturation. At threshold, which is accompanied by large fluctuations in the photon number, the Fano factor

$$F=\frac{\langle(a^\dagger a)^2\rangle-\langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle} \tag{2.32}$$

has a maximum. Above threshold the mean photon number $\langle a^\dagger a\rangle$ increases first linearly with the strength of the pump field and eventually shows saturation. The Fano factor then approaches a value that exceeds the Poissonian value $F=1$ by a few percent. As one can see, these features are not as pronounced as for a conventional laser. In particular, the threshold is not as clearly defined as in the macroscopic case. The fact that F has a minimum after which it increases again slightly has its origin in the non-Markovian character of the pump. Qualitatively, this effect can be explained from the eigenvalue spectrum of the pump operator, Eq. (2.28) and Fig. 4. As discussed above, the dependence of the pump rates on the state of the field, which is the origin of noise reduction, is most pronounced for low values of $|\mathscr{E}|$. On the other hand, $|\mathscr{E}|$ has to be large enough to get above threshold. The competition between these two tendencies gives rise to a minimum at some intermediate region.

This can be seen more clearly in Fig. 6, where the Fano factor is plotted for two different sets of parameters. For a strong atom-field coupling the value of F remains below unity, whereas for a weak coupling there is only a small region in which the statistics is sub-Poissonian. Both cases exhibit, however, a minimum as in Fig. 5. The parameters in Fig. 6 agree with those in Fig. 7 of Ref. [8], where the output field was plotted.

III. CORRELATED EXCITATION SCHEME

To gain more intuitive insight into the dynamic action of the pump, let us look at the rate of change of the conditional

state ρ_{AA} in the number representation. The change $\delta\rho_{AA}$ due to the action of the pump during the time δt is given by

$$\delta\rho_{AA}=R_{BA}^{\text{eff}}\rho_{BB}\delta t, \tag{3.1}$$

with R_{BA}^{eff} as in (2.24). Suppose R_{BA}^{eff} were a number, R_{BA}, as in the heat-bath model. Then, in number representation, we would find

$$\delta\rho_{Am,Am}=R_{BA}\rho_{Bm,Bm}\delta t, \tag{3.2}$$

where

$$\rho_{Am,Am}=\langle m|\rho_{AA}|m\rangle \tag{3.3}$$

is the joint probability for the atom being excited in the upper state $|A\rangle$ and the field being in a number state $|m\rangle$ with exactly m photons. Equation (3.2) expresses that the number of transitions from $|B\rangle$ to $|A\rangle$ is proportional to the population in the state $|B\rangle$, and the proportionality factor does not depend on the state of the field, that is on m in (3.2).

In the present model, however, R_{BA}^{eff} is not a number but an operator. Correspondingly, (3.2) changes into a matrix equation

$$\delta\rho_{Am,Am}=\sum_{m'} R_{BA}^{[mm']}\rho_{Bm',Bm'}\delta t \tag{3.4}$$

with matrix elements

$$R_{BA}^{[mm']}=\sum_n \langle m|\rho_n^{(0)}|m\rangle R_{BA}^{(n0)}\langle m'|\check{\rho}_n^{(0)}|m'\rangle$$

$$=\sum_n R_{BA}^{(n0)}(-1)^{m+n}\binom{n}{m}\binom{m'}{n}, \tag{3.5}$$

as implied by (2.11) and (2.12) for $\nu=0$. Equation (3.4) now states that the number of transitions from level $|B\rangle$ to level $|A\rangle$ within the time increment δt depends on the number of photons in the field. More precisely, there are also transitions between the product states $|B\rangle|m'\rangle$ and $|A\rangle|m\rangle$ for $m<m'$, that is, transitions between the laser levels that are accompanied by a decrease in the number of photons in the field due to the cavity losses. In Fig. 7, the correlated excitation rates are plotted as a function of m and m' for some fixed values of the other parameters. The diagonal of this plot, specified by $m=m'$, reproduces curve 2 in Fig. 4, since for $m=m'$ (3.5) reduces to $R_{BA}^{[mm']}=R_{BA}^{(m0)}$. The first, second, etc. side diagonal in Fig. 7 corresponds to the loss of 1, 2, ... photons out of the cavity during the time the atom spends in the pump levels. Please note that for the standard two-level laser the values of $R_{BA}^{[mm']}$ vanish for $m\neq m'$ and are constant on the main diagonal. This can be seen from (3.5) by replacing the rates $R_{BA}^{(n0)}$ by the constant $R_{BA}\equiv R_{BC}$ and the remaining summation of the binomials yields Kronecker's $\delta_{mm'}$.

Consider now the first-order expansion of the rate matrix (3.5) if the damping rate A is small compared to the rates $|\mathscr{E}|$, R_{BC}, and R_{DA} involved in the recycling process. For weak damping (2.28) reduces to

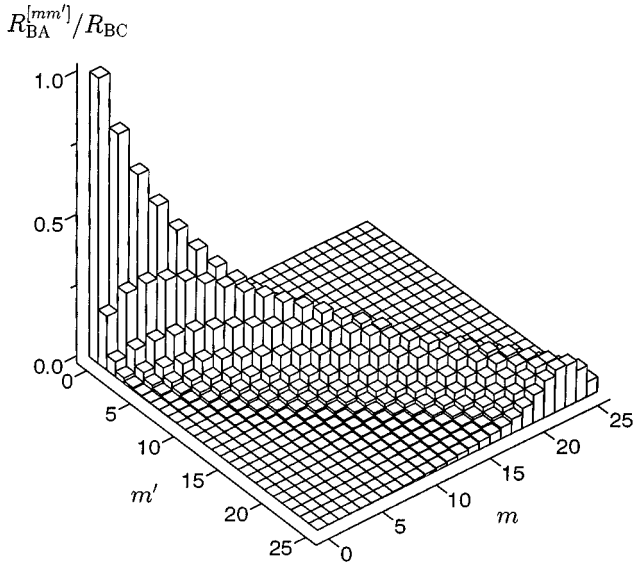

FIG. 7. Matrix elements of the pump operator in the number representation for the parameters of Fig. 4, curve 2: $|\mathscr{E}|=R_{DA}=10A$.

$$R_{BA}^{(n0)}=R_{BC}(1-\epsilon n), \tag{3.6}$$

where

$$\epsilon=\frac{2A}{R_{DA}}+\frac{AR_{DA}}{4|\mathscr{E}|^2}. \tag{3.7}$$

Inserting (3.6) into (3.5) and employing the relation

$$\sum_n x^n(-1)^{m+n}\binom{n}{m}\binom{m'}{n}=x^m(1-x)^{m'-m}\binom{m'}{m}$$

$$\equiv f_{mm'}(x), \tag{3.8}$$

one finds

$$R_{BA}^{[mm']}=R_{BC}\left[\left(1-\epsilon x\frac{\partial}{\partial x}\right)f_{mm'}(x)\right]_{x=1}$$

$$=R_{BC}[(1-\epsilon m)\delta_{mm'}+\epsilon(m+1)\delta_{m,m'-1}]. \tag{3.9}$$

This correlates, to first order in ϵ, the pump rates between $|B\rangle$ and $|A\rangle$ to field states with neighboring photon-number components m and $m+1$. The first-order approximation in (3.9) means that we consider a situation where the loss of the cavity during the excitation amounts to maximally one photon; see Fig. 8.

It is illuminating to establish a connection between the pump-operator approach for a multilevel laser used in this paper and the standard theory of an atomic-injection laser. It turns out that we can include (3.9) in a detailed-balance equation and thereby find an analytical expression for the first-order noise quenching.

To do so, we proceed from the stationary version of the Scully-Lamb equation for the maser [18],

$$0=[rK+L_a]\rho. \tag{3.10}$$

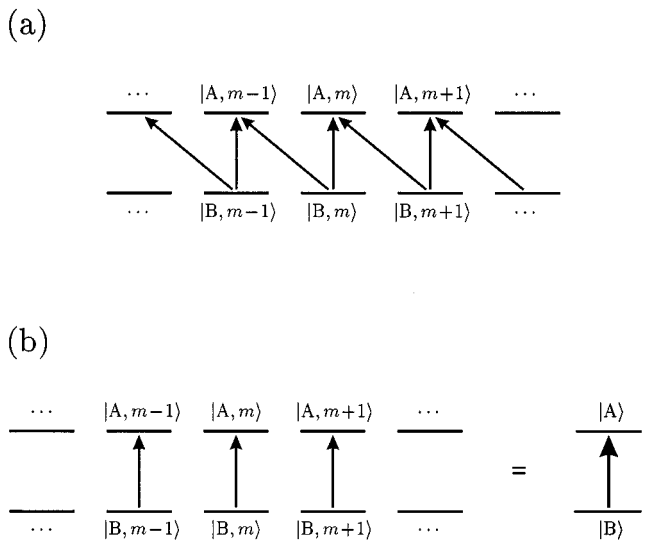

FIG. 8. (a) Schematic representation of the pump transitions to first order in the feedback parameter ϵ, i.e., at most one photon leaves the cavity during the recycling time of the atom. (b) In the good-cavity limit, the pump transitions can be described by a single pump rate between the lower and upper lasing level.

Here, r describes the injection rate of the atoms into the cavity and the change of the field state due to the interaction with a single atom is given by $K\rho$. For a fixed interaction time τ, the number representation of (3.10) reads

$$0=-r\,\sin^2(g\tau\sqrt{m+1})\rho_{mm}+A(m+1)\rho_{m+1,m+1}+r\,\sin^2(g\tau\sqrt{m})\rho_{m-1,m-1}-Am\rho_{mm}, \tag{3.11}$$

where we have again restricted ourselves to zero temperature. Applied to the atomic-injection model, a correlated excitation scheme means that the state of the maser field gives a feedback on the injection rate. Formally, the rate r in (3.10) is then an operator as in (3.9)

$$r_{mm'}=r(1-\epsilon m)\delta_{mm'}+r\epsilon(m+1)\delta_{m,m'-1}, \tag{3.12}$$

and ϵ measures the strength of the feedback. We take for granted that $\epsilon m\ll 1$ holds for all relevant values of m. Using (3.12) in the number representation of (3.10), we obtain

$$0=-r[1-\epsilon(m+1)]\sin^2(g\tau\sqrt{m+1})\rho_{mm}+[A-\epsilon r\,\sin^2(g\tau\sqrt{m+2})](m+1)\rho_{m+1,m+1}+r(1-\epsilon m)\sin^2(g\tau\sqrt{m})\rho_{m-1,m-1}-[A-\epsilon r\,\sin^2(g\tau\sqrt{m+1})]m\rho_{mm} \tag{3.13}$$

instead of (3.11). If the interaction time τ is a fixed quantity, as in the situation of the micromaser [19], (3.13) leads to a detailed-balance equation. If the atoms decay with a rate γ before they leave the cavity, as is typical for optical transitions, one should take the average of (3.13) over the atomic lifetime. Detailed balance then yields

$$\left[\mathscr{C}-\epsilon\frac{\mathscr{A}(m+2)}{1+(\mathscr{B}/\mathscr{A})(m+2)}\right](m+1)\rho_{m+1,m+1}=\frac{\mathscr{A}(m+1)}{1+(\mathscr{B}/\mathscr{A})(m+1)}[1-\epsilon(m+1)]\rho_{mm}, \tag{3.14}$$

where we have introduced the linear-gain coefficient

$$\mathscr{A}=2r(g/\gamma)^2 \tag{3.15}$$

and the self-saturation coefficient

$$\mathscr{B}=4(g/\gamma)^2\mathscr{A} \tag{3.16}$$

as well as $\mathscr{C}\equiv A$ for notational analogy with Ref. [18].

Far above threshold, the solution to (3.14) is given by

$$\rho_{mm}=\rho_{00}\left(\frac{\mathscr{A}^2}{\mathscr{B}\mathscr{C}}\right)^m\frac{1}{m!}\left(1-\epsilon\left[\frac{m(m+1)}{2}-m\frac{\mathscr{A}^2}{\mathscr{B}\mathscr{C}}\right]\right) \tag{3.17}$$

to first order in ϵ. The normalization of ρ to unit trace determines ρ_{00}. This corresponds to a quenched Poisson-type distribution with a mean photon number

$$\langle a^\dagger a\rangle=\langle a^\dagger a\rangle_{\mathrm{p}}(1-\epsilon) \tag{3.18}$$

and a Fano factor

$$F\equiv\frac{\langle(a^\dagger a)^2\rangle-\langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle}=1-\epsilon\langle a^\dagger a\rangle_{\mathrm{p}}, \tag{3.19}$$

where $\langle a^\dagger a\rangle_{\mathrm{p}}=\mathscr{A}^2/(\mathscr{B}\mathscr{C})$ denotes the Poissonian mean for $\epsilon=0$. This result tells us that the quenching of the photon number fluctuations gets stronger for an increasing feedback parameter ϵ and is proportional to the zeroth-order photon number $\langle a^\dagger a\rangle_{\mathrm{p}}$.

Please note that the derivation of the detailed-balance equation relies on the first-order expansion of the pump operator, so all results given here are only meaningful for small ϵ. The purpose of the previous derivations was mainly to establish a link to the well-known Scully-Lamb approach and to provide further insight into the correlated excitation scheme when regarded as a feedback-type pump control.

IV. THE MULTILEVEL LASER

In the multilevel configuration, as depicted in Fig. 9, the ''pump electron'' [20] will undergo a cascade of transitions before it reaches the upper laser level $|\mathrm{A}\rangle$. In the work of Ritsch *et al.* [1], it is argued that (a simplified version of) such a multistep recycling process leads to a regularization of the electron's arrival times at $|\mathrm{A}\rangle$. In analogy to a maser with regular external injection, this would then explain the noise-reduction effect in the multilevel scheme.

In our present approach, where we do not use an adiabatic approximation, our strategy is again to eliminate the whole cascade of intermediate levels and find the corresponding effective pump operator for the reduced laser system. The correlated excitation rates, which we obtain as eigenvalues, will then contain the information about the influence of the cascade.

The same applies to the limit of infinitely many interme-

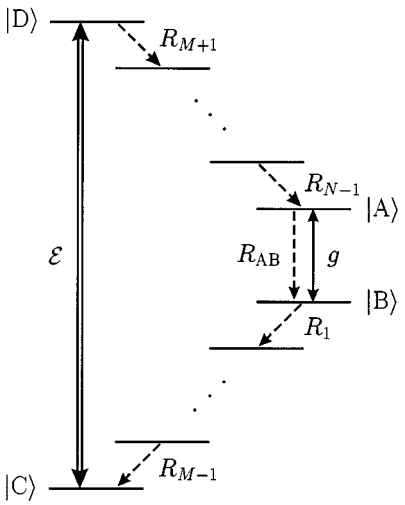

FIG. 9. Schematic representation of the multilevel atom.

diate levels. The pump operator in steady state acquires then a particularly simple exponential form. In the time-dependent equations, the corresponding non-Markovian term introduces a time retardation into the pump dynamics. The interpretation of this time retardation shows that the analogy of the multilevel laser with the regular injection laser, where ''fresh'' atoms are injected in a well-prepared state at equidistant times, is very vague. Different from an injection laser, the responsible mechanism for noise reduction in the multilevel laser is a correlated excitation process.

A. Simple cascades

The dependence of the conditional state $\rho_{M-1,M-1}$ — the bottom rung of the lower cascade in Fig. 9 — on the state $\rho_{BB}=\rho_{11}$ — the top rung of that cascade — can easily be calculated. Since there are no coherences in steady state, the equations for the cascade levels read

$$\begin{aligned}(-L_a+R_{M-1})\rho_{M-1,M-1}&=R_{M-2}\rho_{M-2,M-2},\\ (-L_a+R_{M-2})\rho_{M-2,M-2}&=R_{M-3}\rho_{M-3,M-3},\\ &\vdots\\ (-L_a+R_2)\rho_{22}&=R_1\rho_{BB}.\end{aligned}\tag{4.1}$$

We solve this recursion iteratively and obtain

$$\begin{aligned}\rho_{M-1,M-1}&=\frac{R_{M-2}}{R_{M-1}-L_a}\rho_{M-2,M-2}=\cdots\\ &=\frac{R_{M-2}\cdots R_1}{(R_{M-1}-L_a)\cdots(R_2-L_a)}\rho_{BB}.\end{aligned}\tag{4.2}$$

Similarly, the state $\rho_{N-1,\,N-1}$ involving the level prior to the upper laser level $|A\rangle$ can be expressed in terms of the state $\rho_{DD}\equiv\rho_{M+1,M+1}$:

$$\rho_{N-1,\,N-1}=\frac{R_{N-2}\cdots R_{M+1}}{(R_{N-1}-L_a)\cdots(R_{M+2}-L_a)}\rho_{DD}.\tag{4.3}$$

Again, this straightforward elimination is possible since we know L_a in terms of its spectral decomposition, that is, we can treat L_a essentially like a number. At this point, we have reduced the description to four levels.

Now we proceed as in Sec. II to derive the pump operator corresponding to (2.24). The result is

$$R_{BA}^{\rm eff}=\frac{R_{BC}^{\rm eff}2|\mathscr{E}|^2R_{DA}^{\rm eff}}{-\det\mathscr{M}}\tag{4.4}$$

with the effective rates

$$R_{BC}^{\rm eff}\equiv R_1\prod_{j=2}^{M-1}\frac{R_j}{R_j-L_a}\tag{4.5}$$

and

$$R_{DA}^{\rm eff}\equiv R_{M+1}\prod_{j=M+2}^{N-1}\frac{R_j}{R_j-L_a}\tag{4.6}$$

and with $\mathscr{M}$ as in (2.18) if R_{DA} therein is replaced by R_{M+1}.

The eigenvalues of $R_{BA}^{\rm eff}$ for $k=0$, viz.,

$$\begin{aligned}R_{BA}^{\rm eff}\rho_n^{(0)}&=R_1/[1+c_1An+c_2(An)^2+\cdots\\ &\quad+c_{N-1}(An)^{N-1}]\rho_n^{(0)}\\ &=R_{BA}^{(n0)}\rho_n^{(0)},\end{aligned}\tag{4.7}$$

with some real constants $c_1,\ldots,c_{N-1}$, which depend on the electric field and the atomic relaxation rates, now involve a polynomial in An of degree $N-1$.

In (4.7), the dependence of these pump rates on the index n, and thus on the photon number, is enhanced with an increasing number of cascade levels. As for the four-level laser, the pump acts less efficiently for a large photon number. In addition, due to the power $N-1$ in the denominator, the dependence on the photon number is stronger in the multilevel situation and increases with the number N of atomic levels.

The steady-state equation for the laser is still given by (2.27) where now the pump operator (4.4) rather than (2.24) is to be used. In Fig. 10, we plot the dependence of the Fano factor on $|\mathscr{E}|$ for various numbers N of atomic levels. As one can see, the noise reduction becomes stronger with increasing N. For $N=50$, however, the process has reversed; see Fig. 10(b). This is because with a growing number of intermediate levels the dead time of the atom increases, so that the pump cannot compensate for the loss out of the cavity. The mean number of photons therefore drops and the region of linear gain shrinks to zero; see Fig. 11.

Figures 10 and 11 show the statistics of the laser field that is produced by a single atom. In particular, one can see how the choice of different excitation channels affects the noise of the output field. As a rule, a cascade of pump levels is favorable to a three- or four-level scheme, although if a certain intensity is required, there is an optimum value of pump levels. A more realistic model, which also considers parallel decay and pump channels, for instance, when magnetic sublevels are involved, is worked out in Sec. IV B.

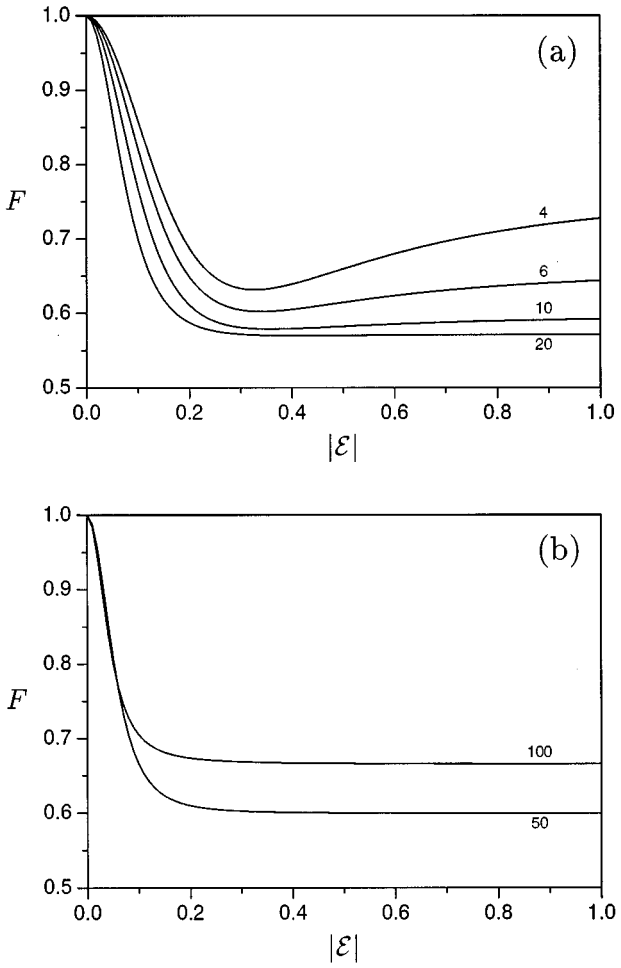

FIG. 10. The Fano factor F as a function of the pump strength $|\mathscr{E}|$ for the parameters $\nu=0$, $A=0.02$, $R_{\mathrm{AB}}=0.01$, $R_1=R_2=\cdots=R_{M-1}=1$, and $R_{M+1}=R_{M+2}=\cdots=R_{N-1}=1$ (in units of g). The curve labeled N corresponds to an N-level atom.

We now compare our results with a Fokker-Planck treatment as presented in Ref. [1] and sketched in Appendix C, where the atomic variables are adiabatically eliminated from the description. To facilitate the comparison with Ref. [1], we replace the Rabi frequency $\mathscr{E}$ that is associated with the coherent pump field by an incoherent pump rate Γ and vary all rates $(1/2)R_1=R_2=\cdots=R_{N-1}\equiv\Gamma$ simultaneously. This ensures that the mean number of photons does not drop with an increasing number of pump levels.

In Fig. 12, we see that the noise reduction increases monotonically with the number N of levels and the Fano factor appears to approach an optimal value of $F\cong 0.5$. Although this qualitative behavior is predicted also in Ref. [1], a detailed comparison shows a severe deviation of the exact curves from the results obtained within a linearized Fokker-Planck-type treatment.

In Fig. 13(a), we plot the curves of Fig. 12 for $N=4$ and $N=10$, which are calculated with the damping-basis method, and the corresponding curves obtained from a (linearized) Fokker-Planck treatment. There are several observations to be made. For strong damping as in Fig. 13(a), there is no match between the Fokker-Planck (dashed and dotted curves) and the damping-basis results (solid curves). This is

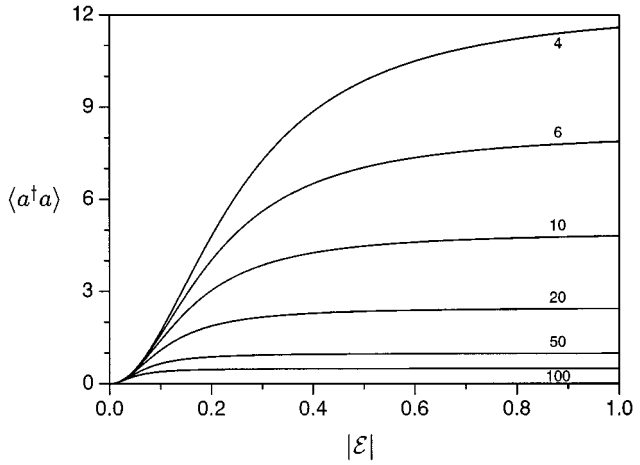

FIG. 11. The mean photon number $\langle a^{\dagger}a\rangle$ as a function of the pump strength $|\mathscr{E}|$ for the parameters of Fig. 10.

not surprising, since for small values of Γ the adiabatic elimination does not claim to hold in a regime where Γ/A is not very large compared to unity; for large Γ, on the other side, the cooperativity parameter $c=2g^2/\Gamma A$ goes to zero and a linearization in this regime is not adequate.

For weaker damping, shown in Fig. 13(b), the Fokker-Planck curves approximate the exact curves in a larger range of values of the pump strength Γ. In the semiclassical regime, with $A/g\ll 1$ and a correspondingly large mean photon number, all curves approach a (thresholdless) step function at the origin. Note, however, that the Fokker-Planck curves always have a singularity (pole) if $c=1$, that is $\Gamma=2g^2/A$, whereas the exact curves smoothly approach $F=1$ for $\Gamma\to\infty$ (and $\nu=0$).

Although it may appear that the system would saturate with increasing pump rate Γ, as is the case in Fig. 5, this is not true here as is demonstrated in Fig. 14. Since Γ also affects the decay of the optical polarization on the lasing transition, the laser gets below threshold again; this phenom-

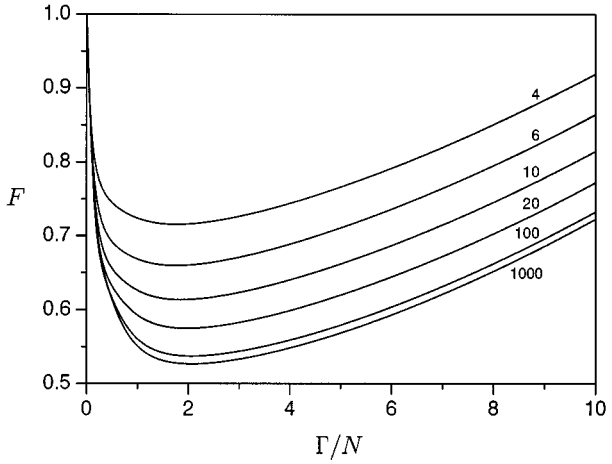

FIG. 12. The Fano factor F as a function of the pump and decay rate $\Gamma=(1/2)R_1=R_2=\cdots=R_{N-1}$ for $\nu=0$, $A=0.5$, and $R_{\mathrm{AB}}=0$. Here all rates are given in units of $g/\sqrt{N}$. The curve labeled N corresponds to an N-level atom.

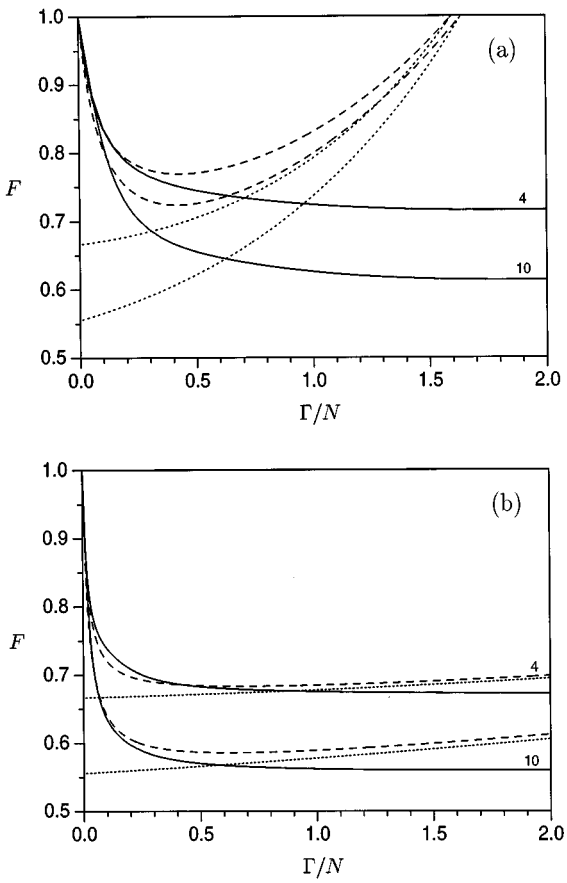

FIG. 13. The Fano factor F as a function of Γ/N with $N=4$ (upper curves) and $N=10$ (lower curves) for (a) $A=0.5$ and (b) $A=0.1$. The other parameters are the same as in Fig. 12 (rates in units of $g/\sqrt{N}$). The results of the damping-basis approach (solid curves) are compared with approximate Fokker-Planck solutions (dashed and dotted curves) given by Eqs. (C3) and (C5), respectively.

enon is called self-quenching by Mu and Savage [8]. In the present scheme this is only due to a simplified model where all rates are varied simultaneously. In a real laser, e.g., in an ion-trap laser, the relaxation rates would be fixed, and the pumping is controlled by an external field on one segment of the cycle. In this situation we will always see saturation as illustrated in Fig. 5.

B. General cascades

If for the transition $|\mathrm{C}\rangle\rightarrow|\mathrm{D}\rangle$ we choose an incoherent excitation rather than the coherent pump field, then the degree of the polynomial in (4.7) will be $N-2$ rather than $N-1$. In this situation, the degree $N-2$ is just determined by the number of pump levels (excluding $|\mathrm{A}\rangle$ and $|\mathrm{B}\rangle$). The additional power in the case of a coherent pump field arises from the elimination of the polarization π_{CD} as an additional variable in the Bloch equations (2.17) as compared to rate equations for an incoherent excitation.

As a further observation we may thus state that a coherent pump field will additionally contribute to noise reduction. From this it is also clear that several coherent segments involved in the pump process will influence the state of the

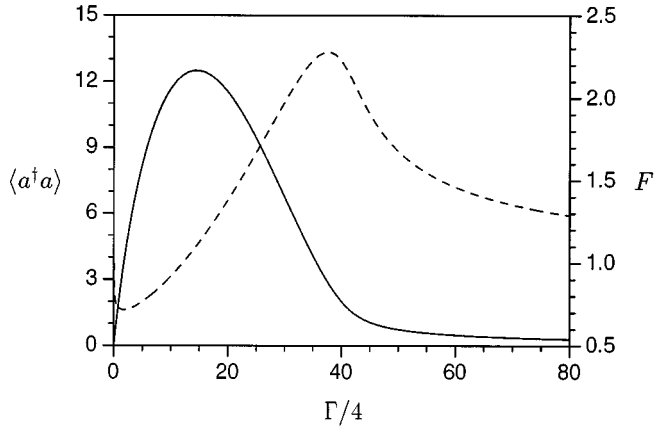

FIG. 14. The mean photon number $\langle a^{\dagger}a\rangle$ (solid curve) and the Fano factor F (dashed curve) as functions of Γ/N for $N=4$ and the parameters of Fig. 12. All rates are given in units of $g/\sqrt{N}=g/2$.

field and, in particular, its noise properties.

Our considerations are not restricted to simple cascades either, but may, as in real atoms, also involve several parallel decay channels, e.g., via magnetic sublevels. Similarly one can describe broadband excitation. These situations are illustrated in Fig. 15. Compared to the situation of Fig. 9, we now have to replace the product rate

$$\frac{R_{L-1}}{R_L-L_a}\,\frac{R_L}{R_{L+1}-L_a} \tag{4.8}$$

for a simple cascade $|L-1\rangle\rightarrow|L\rangle\rightarrow|L+1\rangle$ by the sum

$$\sum_I \frac{R_{L-1,I}}{R_I-L_a}\,\frac{R_I}{R_{L+1}-L_a} \tag{4.9}$$

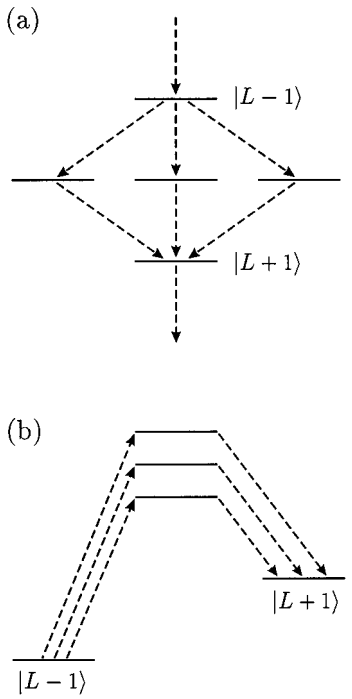

FIG. 15. Schematic representation of parallel decay channels (a) and broadband excitation (b).

in which the index I runs over all sublevels $|I\rangle$ within the intermediate manifold. An application of these results to a model of an experiment with an ion-trap laser is the intended subject of a forthcoming paper.

C. The limit of infinitely many pump levels

In order to see how the system is described in the limit $N\to\infty$, we go back to the laser equations (2.6) supplemented by the time-dependent version of (4.1) and the corresponding equations for the other cascade levels. The elimination of the pump equations in the N-level situation can be done in a similar fashion as in Sec. II C.

For the kernel $R_{\rm BA}^{\rm eff}(t-t')$ in (2.22) one obtains

$$R_{\rm BA}^{\rm eff}(t-t')=R_{\rm BC}\Gamma\frac{[\Gamma(t-t')]^{N-3}}{(N-3)!}e^{(L_a-\Gamma)(t-t')} \tag{4.10}$$

if one assumes an incoherent pump from $|{\rm C}\rangle$ to $|{\rm D}\rangle$ (see Fig. 9) and for simplicity sets $R_2=R_3=\cdots=R_{N-1}=\Gamma$, as well as $R_{\rm BC}\equiv R_1$.

In the limit of very large N, the right-hand side of (4.10) turns into

$$R_{\rm BC}\delta(t-t'-\tau)e^{L_a\tau} \tag{4.11}$$

if the pump rates Γ are increased with the number of pump transitions, such that the average time $\tau=(N-2)/\Gamma$ for the excitation from level $|2\rangle$ to $|{\rm A}\rangle$ is kept constant. The time integral in (2.22) with the kernel (4.10) can then be evaluated, which yields

$$\left(\frac{\partial}{\partial t}-L_a+R_{\rm AB}\right)\rho_{\rm AA}=-ig(\rho_{\rm AB}a^\dagger-a\rho_{\rm BA})+R_{\rm BC}e^{L_a\tau}\rho_{\rm BB}(t-\tau),$$

$$\left(\frac{\partial}{\partial t}-L_a+R_{\rm BC}\right)\rho_{\rm BB}=-ig(\rho_{\rm BA}a-a^\dagger\rho_{\rm AB})+R_{\rm AB}\rho_{\rm AA}, \tag{4.12}$$

supplemented by the equations for $\rho_{\rm AB}$ and $\rho_{\rm BA}$, which are the same as in (2.6). The first equation of (4.12) implies that the rate of change of $\rho_{\rm AA}$ at time t is determined by the state $\rho_{\rm BB}$ and thus the population in $|{\rm B}\rangle$ at the retarded time $t-\tau$.

The reduced state of the photon field

$$\rho(t)=\sum_{L=1}^{N}\rho_{LL}(t) \tag{4.13}$$

is obtained by summing over the atomic levels; it obeys

$$\left(\frac{\partial}{\partial t}-L_a\right)\rho(t)=-ig([\rho_{\rm AB},a^\dagger]-[a,\rho_{\rm BA}]). \tag{4.14}$$

This equation can be integrated to produce

$$\rho(t)=\rho_{\rm AA}(t)+\rho_{\rm BB}(t)+R_{\rm BC}\int_{t-\tau}^{t}dt'e^{L_a(t-t')}\rho_{\rm BB}(t') \tag{4.15}$$

in the limit $N\to\infty$. One can show that the trace of $\rho(t)$ in (4.15) is conserved in time, a result which can be seen more easily in the differential equation (4.14). The time integral in (4.15) accounts for the fraction of time that the atom spends in the pump levels. Note that for $\tau\to 0$ the integral vanishes and (4.12) reduces to the standard heat-bath model discussed in Sec. II A.

The retardation in (4.12) is irrelevant in steady state, and the pump operator is simply given by

$$R_{\rm BA}^{\rm eff}=R_{\rm BC}e^{L_a\tau}. \tag{4.16}$$

Incidentally, the same result is obtained if in [the incoherent analog (B4) with (B6) of] (4.4) the same limit as above is performed. The origin of noise reduction is here again due to the fact that the operator $R_{\rm BC}\exp(L_a\tau)$ acts more weakly on components of the state $\rho_{\rm BB}$ corresponding to large photon numbers (or statistical moments) than on components corresponding to small photon numbers. This introduces the correlation between the pump rates and the state of the field. In other words: The noise reduction does not come about because the $\rho_{\rm BB}$ term in (4.12) is retarded, but because the operator $\exp(L_a\tau)$ suppresses the higher statistical moments in $\rho_{\rm BB}$.

Except for the exponential factor $\exp[L_a(t-t')]$, the kernel (4.10) is essentially equal to the conditional probabilities used by Marte and Zoller [5] in their model for lasers with a non-Poissonian pump and later by Ritsch *et al.* [1] to explain noise reduction in their treatment of multilevel lasers.

On the one hand, it is true that the N-dependent factor $\Gamma\exp[-\Gamma(t-t')][\Gamma(t-t')]^{N-3}/(N-3)!$ in (4.10) is the effective transition rate of the atom into level $|{\rm A}\rangle$ at time t *given* that the atom is in the lower state $|2\rangle$ at the initial time t'. According to (4.11), in the limit $N\to\infty$ this time-dependent transition rate is δ peaked at the time $t=t'+\tau$, and this suggests the self-regularization of the whole excitation process that is emphasized by Ritsch *et al.* [1].

On the other hand, the time integral in (2.22) reduces to the term involving a retarded time argument in (4.12). But this retardation process, which is interesting enough in itself, does not describe a periodicity in the actual excitation of the atom. This stands in marked contrast to a laser with explicit periodic injection where ''fresh'' atoms are injected in a well-prepared state and at certain externally controlled times [21]. To generate dynamics of this kind in the multilevel laser, one would have to force the atom into the state $|{\rm A}\rangle$ at regular, predetermined instants, perhaps by performing cleverly designed measurements at these moments. But this is not the physical situation we are considering.

In summary, we conclude that the multilevel laser with a closed pump cycle is not equivalent to an atomic beam laser with regular injection. The physical mechanism that we find responsible for noise reduction in the multilevel laser is correlated atomic excitation.

V. CONCLUSIONS

We have investigated a single-mode laser with multilevel excitation in a regime where the atomic dynamics is not necessarily fast compared to the field dynamics. The effect of dynamic noise reduction in multilevel lasers, which has been

predicted by Ritsch *et al.* [1] and others [2,3], is here treated within the framework of a theory that does not rely on adiabatic elimination or linearization of a Fokker-Planck equation.

The technical key point of our approach is the introduction of an effective pump operator and its analytical evaluation with the aid of the damping basis [14]. The eigenvalue spectrum of this pump operator provides the mathematical expression for a physical mechanism which we call correlated excitation. Possible photon losses during the excitation of the atom from the lower to the upper laser level give rise to a correlation of the excitation rates with the number of photons in the field. Instead of a single rate one obtains a rate matrix.

This correlated excitation mechanism is different from a regular pumping. If one wishes to establish a connection with an atomic-injection laser, the correlated excitation corresponds to a feedback mechanism which changes the atomic injection rate depending on the state of the field. Thereby, the photon number fluctuations can be reduced below the shot-noise level.

From the eigenvalues of the pump operator one can draw qualitative and quantitative conclusions about several features of the noise reduction. Examples are (i) the role of the number of pump levels; (ii) the existence of an optimal pump strength such that the noise is minimized; and (iii) the observation that coherent pumping leads to stronger noise reduction than incoherent pumping. The latter has already been noted in a numerical treatment [8].

When the atomic dynamics is not fast compared to the dynamics of the field and we are not necessarily far above threshold, our results differ substantially from a linearized treatment. An example for such a situation would be an ion-trap laser operating with single atoms as an active medium.

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APPENDIX A: EFFECTIVE LIOUVILLE OPERATOR

It is important for the consistency of the method that the system of equations (2.27) is a representation of a genuine Liouville operator, which conserves the trace and has no eigenvalues with a positive real part. First let us state that (2.27) may be written in the form

$$\mathscr{L}\mathrm{P}\equiv\frac{1}{i\hbar}[H,\mathrm{P}]+L_a\,\mathrm{P}+L_\sigma^{\mathrm{eff}}\mathrm{P}=0 \tag{A1}$$

with L_a as in (2.3) and

$$H=-\hbar g(a|\mathrm{A}\rangle\langle\mathrm{B}|+a^\dagger|\mathrm{B}\rangle\langle\mathrm{A}|), \tag{A2}$$

where the composed state

$$\mathrm{P}=\sum_{\mathrm{LL}'}\rho_{\mathrm{LL}'}|\mathrm{L}\rangle\langle\mathrm{L}'| \tag{A3}$$

only involves the levels L,L′=A,B. The ''atomic'' operator has the form

$$\begin{aligned}L_\sigma^{\mathrm{eff}}\mathrm{P}=&-\frac{R_{\mathrm{AB}}}{2}(|\mathrm{A}\rangle\langle\mathrm{A}|\mathrm{P}+\mathrm{P}|\mathrm{A}\rangle\langle\mathrm{A}|)+R_{\mathrm{AB}}|\mathrm{B}\rangle\langle\mathrm{A}|\mathrm{P}|\mathrm{A}\rangle\langle\mathrm{B}|\\&-\frac{R_{\mathrm{BC}}}{2}(|\mathrm{B}\rangle\langle\mathrm{B}|\mathrm{P}+\mathrm{P}|\mathrm{B}\rangle\langle\mathrm{B}|)+R_{\mathrm{BA}}^{\mathrm{eff}}|\mathrm{A}\rangle\langle\mathrm{B}|\mathrm{P}|\mathrm{B}\rangle\langle\mathrm{A}|.\end{aligned} \tag{A4}$$

The trace of this equation yields

$$\mathrm{Tr}\{L_\sigma^{\mathrm{eff}}\mathrm{P}\}=\mathrm{tr}\{R_{\mathrm{BA}}^{\mathrm{eff}}\rho_{\mathrm{BB}}\}-R_{\mathrm{BC}}\,\mathrm{tr}\{\rho_{\mathrm{BB}}\}=0, \tag{A5}$$

which is obtained from Eq. (2.28) in conjunction with $\mathrm{tr}\{\rho_n^{(k)}\}=\delta_{n0}\delta_{k0}$. Note that the partial (atomic) trace of (A4) does not vanish, since L_σ^{eff}, different from the standard phenomenological term, also acts on the photon variables in a nontrivial manner.

To show the positivity of $-L_\sigma^{\mathrm{eff}}$, we solve the eigenvalue equation

$$L_\sigma^{\mathrm{eff}}\mathrm{P}=\lambda\mathrm{P}, \tag{A6}$$

which can be cast into the form

$$\begin{pmatrix}-R_{\mathrm{AB}}&R_{\mathrm{BA}}^{\mathrm{eff}}&0&0\\R_{\mathrm{AB}}&-R_{\mathrm{BC}}&0&0\\0&0&-(R_{\mathrm{AB}}+R_{\mathrm{BC}})/2&0\\0&0&0&-(R_{\mathrm{AB}}+R_{\mathrm{BC}})/2\end{pmatrix}\begin{pmatrix}\rho_{\mathrm{AA}}\\\rho_{\mathrm{BB}}\\\rho_{\mathrm{AB}}\\\rho_{\mathrm{BA}}\end{pmatrix}=\lambda\begin{pmatrix}\rho_{\mathrm{AA}}\\\rho_{\mathrm{BB}}\\\rho_{\mathrm{AB}}\\\rho_{\mathrm{BA}}\end{pmatrix}. \tag{A7}$$

The characteristic polynomial has the zeros

$$\lambda_{nk}^{(1)}=\lambda_{nk}^{(2)}=-(1/2)(R_{\mathrm{AB}}+R_{\mathrm{BC}}),$$

$$\lambda_{nk}^{(3)}=-(1/2)(R_{\mathrm{AB}}+R_{\mathrm{BC}})-\sqrt{(1/4)(R_{\mathrm{AB}}+R_{\mathrm{BC}})^2-R_{\mathrm{AB}}(R_{\mathrm{BC}}-R_{\mathrm{BA}}^{(nk)})},$$

$$\lambda_{nk}^{(4)}=-(1/2)(R_{\mathrm{AB}}+R_{\mathrm{BC}})+\sqrt{(1/4)(R_{\mathrm{AB}}+R_{\mathrm{BC}})^2-R_{\mathrm{AB}}(R_{\mathrm{BC}}-R_{\mathrm{BA}}^{(nk)})} \tag{A8}$$

for $n=0,1,2,\ldots$ and $k=0,\pm1,\pm2,\ldots$ with $R_{\mathrm{BA}}^{(nk)}$ of (2.28). Since the inequalities $0<R_{\mathrm{BA}}^{(nk)}\leqslant R_{\mathrm{BC}}$ hold for all n,k, all eigenvalues are, indeed, negative or zero. The stationary state of this operator (alone) is given by $(R_{\mathrm{AB}}+R_{\mathrm{BC}})^{-1}(R_{\mathrm{BC}}|\mathrm{A}\rangle\langle\mathrm{A}|+R_{\mathrm{AB}}|\mathrm{B}\rangle\langle\mathrm{B}|)\rho_0^{(0)}$, that is, by the vacuum or thermal state of the field and an inversion of the atom that is determined by the ratio of the rates R_{BC} and R_{AB}.

Since L_σ^{eff} has been shown to be a genuine Liouville operator, $\mathscr{L}$ of (A1) is also one. The corresponding equation of motion thus obeys the formal criteria of a master equation. Note, however, that the elimination of the pump equations in the above manner was only possible in steady state, and insofar the Liouville operator $\mathscr{L}$ may only be used to describe the stationary properties of the system. Nevertheless, positivity and trace conservation are important for consistency. It means that one can also use $\mathscr{L}$ as the semigroup generator for the corresponding time-dependent equations of motion. However, as we shall see below, the transient description is only correct if the pump acts very fast. In that situation, an adiabatic elimination of the pump levels is possible, which will bring us back to the phenomenological heat-bath model.

For time-dependent phenomena, the pump term involves a time integral as in (2.22), which stresses the non-Markovian character of this system-reservoir interaction. In this situation the pump levels, although eliminated in (2.22), still enter the initial state. To see this explicitly, it is expedient to look at the equations in Laplace space, defined by

$$\hat{\mathrm{P}}(s)=\int_0^\infty dt e^{-st}\mathrm{P}(t). \tag{A9}$$

Equations (2.22) then acquire the form

$$\begin{aligned}(s-L_a+R_{\mathrm{AB}})\hat{\rho}_{\mathrm{AA}}&=-ig(\hat{\rho}_{\mathrm{AB}}a^\dagger-a\hat{\rho}_{\mathrm{BA}})+R_{\mathrm{BA}}^{\mathrm{eff}}(s)\hat{\rho}_{\mathrm{BB}}(s)\\&\quad+\rho_{\mathrm{AA}}(0)\\&\quad+I[\rho_{\mathrm{CC}}(0),\rho_{\mathrm{DD}}(0),i\pi_{\mathrm{CD}}(0);s],\\(s-L_a+R_{\mathrm{BC}})\hat{\rho}_{\mathrm{BB}}&=-ig(\hat{\rho}_{\mathrm{BA}}a-a^\dagger\hat{\rho}_{\mathrm{AB}})+R_{\mathrm{AB}}\hat{\rho}_{\mathrm{AA}}+\rho_{\mathrm{BB}}(0)\end{aligned} \tag{A10}$$

with the pump operator

$$R_{\mathrm{BA}}^{\mathrm{eff}}(s)=R_{\mathrm{BC}}\frac{2|\mathscr{E}|^2}{\det(s-\mathscr{M})}R_{\mathrm{DA}} \tag{A11}$$

and an initial-value term

$$\begin{aligned}I[\rho_{\mathrm{CC}}(0),\rho_{\mathrm{DD}}(0),i\pi_{\mathrm{CD}}(0);s]=\frac{1}{\det(s-\mathscr{M})}&(2|\mathscr{E}|^2\rho_{\mathrm{CC}}(0)+[(s-L_a)(s-L_a+R_{\mathrm{DA}}/2)+2|\mathscr{E}|^2]\rho_{\mathrm{DD}}(0)\\&-(s-L_a)i\pi_{\mathrm{CD}}(0)).\end{aligned} \tag{A12}$$

For convenience, we have chosen here $t=0$ as the initial time rather than $t=-\infty$ as in (2.22). In general, the initial values of the conditional density operators involving the pump levels enter the dynamics of the (two-level) laser system via the term (A12). In this sense, the pump levels cannot be completely eliminated from the description. This is obvious in the situation when the atom is initially in the lower laser level $|\mathrm{B}\rangle$. A short time later, the pump levels will be populated, and so the trace over the laser (sub)system cannot be conserved.

However, when there are different time scales, a complete elimination is possible. If the pump acts very fast compared to the dynamics of the laser system, that is $\gamma\gg g,A$ (we use γ as a short-hand notation for Γ, $|\mathscr{E}|$, and all rates $R_{\mathrm{LL}'}$, which involve a pump level L or L′), then the time integral in (2.19) simplifies and yields

$$\begin{aligned}\int_{-\infty}^t dt' e^{\mathscr{M}(t-t')}\begin{pmatrix}R_{\mathrm{BC}}\rho_{\mathrm{BB}}(t')\\0\\0\end{pmatrix}&\\\simeq\mathscr{M}^{-1}\begin{pmatrix}R_{\mathrm{BC}}\rho_{\mathrm{BB}}(t)\\0\\0\end{pmatrix}.&\end{aligned} \tag{A13}$$

This equation is valid as long as terms of the order L_a/γ may be neglected (that is $nA/\gamma\ll1$ for all relevant $n=0,1,2,\ldots$). The L_a dependence of $\mathscr{M}^{-1}$ is therefore insignificant within this approximation and, instead of $\mathscr{M}^{-1}$, we could as well use $\mathscr{M}_{A\to0}^{-1}$, which gives

$$R_{\mathrm{DA}}\rho_{\mathrm{DD}}(t)=R_{\mathrm{BC}}\rho_{\mathrm{DD}}(t) \tag{A14}$$

and brings us back to the phenomenological heat-bath model (2.9), corresponding to the limit of instantaneous pumping. On the other hand, if the photon losses are appreciable and the slow time scale is set by g, that is γ, $A \gg g$, then it is necessary to keep the L_a dependence of $\mathscr{M}$ in (A13).

To account for both possible situations, we use the time-dependent equations

$$\left(\frac{\partial}{\partial t}-L_a+R_{\mathrm{AB}}\right)\rho_{\mathrm{AA}}=-ig(\rho_{\mathrm{AB}}a^\dagger-a\rho_{\mathrm{BA}})+R_{\mathrm{BA}}^{\mathrm{eff}}\rho_{\mathrm{BB}},$$

$$\left(\frac{\partial}{\partial t}-L_a+R_{\mathrm{BC}}\right)\rho_{\mathrm{BB}}=-ig(\rho_{\mathrm{BA}}a-a^\dagger\rho_{\mathrm{AB}})+R_{\mathrm{AB}}\rho_{\mathrm{AA}}, \tag{A15}$$

with the pump operator (2.24), when $\gamma \gg g$ holds (limit of fast pumping). In this regime, the fraction of time the atom spends in the pump levels is negligible compared to the time it spends in the two laser levels, and we can consistently neglect the initial-value term (A12) in (A10).

APPENDIX B: GENERALIZATIONS

In order to account for additional polarization decay on the laser transition, e.g., due to atomic collisions or stray electric fields, and for relaxations on the pump transition, the nonunitary part of the master equation has to be supplemented by

$$L_{\mathrm{DC}}\mathrm{P}=-\frac{R_{\mathrm{DC}}}{2}(|\mathrm{D}\rangle\langle\mathrm{D}|\mathrm{P}+\mathrm{P}|\mathrm{D}\rangle\langle\mathrm{D}|-2|\mathrm{C}\rangle\langle\mathrm{D}|\mathrm{P}|\mathrm{D}\rangle\langle\mathrm{C}|) \tag{B1}$$

and

$$L_{\Sigma}^{\perp}\mathrm{P}=-\frac{2R_{\mathrm{BA}}^{\perp}-R_{\mathrm{BA}}}{2}(|\mathrm{A}\rangle\langle\mathrm{A}|\mathrm{P}|\mathrm{B}\rangle\langle\mathrm{B}|+|\mathrm{B}\rangle\langle\mathrm{B}|\mathrm{P}|\mathrm{A}\rangle\langle\mathrm{A}|)$$
$$-\frac{2R_{\mathrm{DC}}^{\perp}-R_{\mathrm{DC}}}{2}(|\mathrm{D}\rangle\langle\mathrm{D}|\mathrm{P}|\mathrm{C}\rangle\langle\mathrm{C}|+|\mathrm{C}\rangle\langle\mathrm{C}|\mathrm{P}|\mathrm{D}\rangle\langle\mathrm{D}|). \tag{B2}$$

Here R_{DC} and $R_{\mathrm{DC}}^{\perp}$ are the longitudinal and transverse relaxation rate between $|\mathrm{D}\rangle$ and $|\mathrm{C}\rangle$, respectively, and $R_{\mathrm{BA}}^{\perp}$ is the transverse decay rate on the laser transition. For purely radiative decay $R_{\mathrm{BA}}^{\perp}$ is equal to $R_{\mathrm{BA}}/2$.

With these generalizations, the eigenvalues of the pump operator for the N-level laser of Fig. 9 are given by

$$R_{\mathrm{BA}}^{(nk)}=\frac{R_{\mathrm{BC}}^{(nk)}R_{\mathrm{CD}}R_{\mathrm{DA}}^{(nk)}}{M^{(nk)}}, \tag{B3}$$

with

$$R_{\mathrm{BC}}^{(nk)}=R_1\prod_{j=2}^{M-1}\frac{R_j}{R_j+A(n+|k|/2)},$$

$$R_{\mathrm{DA}}^{(nk)}=R_{M+1}\prod_{j=M+2}^{N-1}\frac{R_j}{R_j+A(n+|k|/2)}. \tag{B4}$$

We obtain for a coherent pump

$$R_{\mathrm{CD}}=2|\mathscr{E}|^2,$$

$$M^{(nk)}=[4|\mathscr{E}|^2+R_{\mathrm{DC}}^{\perp}(R_{M+1}+R_{\mathrm{DC}})]A(n+|k|/2)$$
$$+(R_{M+1}+R_{\mathrm{DC}}+R_{\mathrm{DC}}^{\perp})A^2(n+|k|/2)^2$$
$$+A^3(n+|k|/2)^3+2|\mathscr{E}|^2R_{M+1} \tag{B5}$$

and for an incoherent pump

$$R_{\mathrm{CD}}=\Gamma,$$

$$M^{(nk)}=[\Gamma+A(n+|k|/2)][R_{M+1}+R_{\mathrm{DC}}+A(n+|k|/2)]$$
$$-\Gamma R_{\mathrm{DC}}. \tag{B6}$$

The additional phase decay on the lasing transition is accounted for by including $R_{\mathrm{BA}}^{\perp}$ into the laser equations. In order to model incoherent pumping that can be realized by broadband excitation between $|\mathrm{C}\rangle$ and $|\mathrm{D}\rangle$, we simply have to set $R_{\mathrm{CD}}=\Gamma$ and $R_{\mathrm{DC}}=\Gamma+A_{\mathrm{DC}}$ with the Einstein coefficient A_{DC} describing the spontaneous emission rate and a bidirectional pump rate Γ.

APPENDIX C: FOKKER-PLANCK APPROACH

Adiabatic elimination of the atomic degrees of freedom, valid for $\Gamma \gg g$, A, leads to a Fokker-Planck equation [22,23]

$$\frac{\partial}{\partial t}P=\left(\frac{\partial}{\partial I}A_I+\frac{\partial}{\partial\varphi}A_\varphi\right)P+\left(\frac{\partial^2}{\partial I^2}D_{II}+\frac{\partial^2}{\partial\varphi^2}D_{\varphi\varphi}\right.$$
$$\left.+\frac{\partial^2}{\partial I\partial\varphi}D_{I\varphi}\right)P, \tag{C1}$$

with drift coefficients A_I, A_φ and diffusion coefficients D_{II}, $D_{\varphi\varphi}$, $D_{I\varphi}$ for the intensity I and phase φ of the laser field. Here, P is Glauber's P function of the photon state $\rho=\Sigma_L\rho_{LL}$. It is a function of φ and I.

Linearization around the steady-state laser intensity $\bar{I}\equiv\langle a^\dagger a\rangle$, valid far above threshold, leads to analytic expressions for the drift and diffusion coefficients, from which we can calculate the Mandel Q parameter,

$$Q\equiv\frac{\langle a^\dagger a(a^\dagger a-1)\rangle-\langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle}=\left.\frac{D_{II}}{I(dA_I/dI)}\right|_{I=\bar{I}}. \tag{C2}$$

The Fano factor $F=1+Q$ is then found,

$$F=1+\frac{\Gamma(2\Gamma-\bar{n})-\bar{n}^2(N-1)(N-2)}{(N-1)[2\bar{n}[\Gamma+(N-1)\bar{n}]+X]} \tag{C3}$$

with the laser-induced stimulated emission rate $\bar{n}=2g^2\bar{I}/\Gamma$ and a residual term

$$X=g^2\left(\frac{5\Gamma+\bar{n}(N-1)}{\Gamma}+\frac{3\Gamma-6\bar{n}}{\Gamma+\bar{n}(N-1)}\right). \tag{C4}$$

Far above threshold, X can be neglected, which has been done in Ref. [1], giving

$$F=1+\frac{\Gamma(2\Gamma-\bar{n})-\bar{n}^2(N-1)(N-2)}{2\bar{n}(N-1)[\Gamma+(N-1)\bar{n}]}. \tag{C5}$$

For small Γ, however, it is advantageous to keep the additional term (see Fig. 13).

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$$B(1-s)\equiv R_{\mathrm{AB}}, \quad Bs\equiv R_{\mathrm{BA}}\equiv R_{\mathrm{BC}}.$$

It is found that the photon statistics as well as the spectral properties of such a (two-level) one-atom laser are quite similar to a macroscopic laser, and only for weak pump strengths there are some differences [9]. In particular, there is no sub-Poissonian photon statistics in this model.

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[16] An analogous elimination yields the effective pump operator for the three-level laser. We have reproduced the corresponding results of Ref. [10] in this way.

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