Quantum-limited linewidth of a bad-cavity laser with inhomogeneous broadening

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(Received 15 June 1995)

We derive a general expression for the linewidth of inhomogeneously broadened traveling-wave gas lasers, without any assumption on the relative magnitude of the atomic and field decay constants, and for several types of atomic pumping statistics, ranging from Poissonian to regular. For small inhomogeneous broadening, an important linewidth narrowing occurs in the bad-cavity limit. This effect disappears, however, in the large inhomogeneous broadening limit (Doppler limit).

PACS number(s): 42.50.Dv, 42.50.Lc, 42.55.−f, 42.65.Sf

The quantum-limited linewidth of a fully inverted single-mode laser with on-resonance homogeneously broadened medium was originally derived by Schawlow and Townes in Ref. [1] as

$$\Delta \nu_{\rm ST}=\frac{\kappa}{2I_0}, \tag{1}$$

where κ is the cold-cavity loss rate and I_0 is the intracavity intensity of the laser light in units of number of photons. However, the result of Schawlow and Townes is valid only for a *good-cavity* laser for which the decay rate γ_{ab} of the atomic polarization is much larger than the cavity loss rate κ. In the last few years there has been a certain revival of interest about the quantum limitations on the laser linewidth in the so-called *bad-cavity* regime where the above assumption is not valid.

This interest is related to recent developments in semiconductor [2] and gas [3] microlasers that operate in the bad-cavity regime. On the other hand, several authors [4–7] have generalized the theory of Schawlow and Townes for both good- and bad-cavity regimes. These generalized theories predict the appearance of an additional factor in the linewidth formula equal to $[\gamma_{ab}/(\gamma_{ab}+\kappa/2)]^2$. While for good-cavity lasers ($\kappa/2\ll\gamma_{ab}$) this factor reduces to unity, it can be very small for bad-cavity lasers ($\kappa/2\gg\gamma_{ab}$), leading to the quenching of the linewidth. This theoretical prediction was recently experimentally confirmed by Woerdman and collaborators [8].

The theories in Refs. [4–7] are applicable only for a homogeneously broadened laser medium. In the experiment [8], however, the inhomogeneous broadening was of the same order of magnitude as the homogeneous one. Hence a natural question appears about the influence of the inhomogeneous broadening on the quantum-limited linewidth in the bad-cavity regime. The standard textbook theories (see, for example, Ref. [9]) of inhomogeneously broadened lasers do not give the answer to this question since they are valid only for good-cavity lasers.

In this paper, we solve the linewidth problem by extending the theory of quantum fluctuations in lasers with on-resonance homogeneously broadened media developed in Ref. [6]. This theory is applicable for lasers with any relative magnitude of the atomic and cavity decay rates. Moreover, it allows for variable atomic pumping statistics, ranging from Poissonian to completely regular one. The aim of the present paper is to generalize this theory for inhomogeneously broadened media and to evaluate the quantum-limited linewidth for both good- and bad-cavity regimes.

For the sake of simplicity we shall assume that (i) we have a single *traveling-wave* cavity mode, so that the spatial hole burning effect is absent, and (ii) the laser frequency is tuned to the center of the symmetrical atomic line, so that there is no frequency pulling.

Thus, we consider a laser medium consisting of moving two-level atoms that interact with a single traveling-wave cavity mode (Fig. 1). When the atoms move, they see an electric field with shifted frequency due to the Doppler effect. Each individual atom has its own resonance frequency ω_a determined by the value of its velocity component along the laser mode axis. The frequency distribution function $\rho(\omega_a)$ of atoms over the resonance frequencies is obtained from a Maxwell-Boltzmann velocity distribution [9,10],

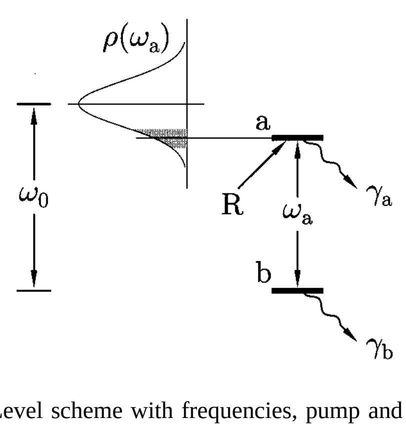

FIG. 1. Level scheme with frequencies, pump and decay rates.

$$\rho(\omega_a)=\frac{1}{\sqrt{\pi}}\frac{1}{\Delta\omega_a}\exp\left[-\frac{(\omega_a-\omega_0)^2}{\Delta\omega_a^2}\right], \tag{2}$$

with the Doppler width $\Delta\omega_a$ equal to

$$\Delta\omega_a=\omega_0\sqrt{\frac{2kT}{Mc^2}}, \tag{3}$$

where M is the atomic mass and T is the gas kinetic temperature.

To incorporate the Doppler effect into the theory we shall separate the atoms into individual groups with different resonance frequencies ω_a. Within each group we shall consider the macroscopic atomic populations $\mathcal{N}_a(t,\omega_a)$ of the upper and $\mathcal{N}_b(t,\omega_a)$ of the lower level, and polarization $\mathcal{M}(t,\omega_a)$. These individual quantities now play the role of the *spectral densities* of the corresponding collective variables that are obtained by integration over all groups, so that

$$\mathcal{N}_a(t)=\int_{-\infty}^{\infty}d\omega_a\mathcal{N}_a(t,\omega_a), \tag{4}$$

similar equations holding for other atomic variables.

The c-number Langevin equations for these spectral densities of atomic variables and for the laser field is a straightforward generalization of the corresponding equations from [6]

$$\dot{\mathcal{N}}_a(t,\omega_a)=R\rho(\omega_a)-\gamma_a\mathcal{N}_a(t,\omega_a)-g[\mathcal{A}^*(t)\mathcal{M}(t,\omega_a)+\mathcal{M}^*(t,\omega_a)\mathcal{A}(t)]+\mathcal{F}_a(t,\omega_a), \tag{5a}$$

$$\dot{\mathcal{N}}_b(t,\omega_a)=-\gamma_b\mathcal{N}_b(t,\omega_a)+g[\mathcal{A}^*(t)\mathcal{M}(t,\omega_a)+\mathcal{M}^*(t,\omega_a)\mathcal{A}(t)]+\mathcal{F}_b(t,\omega_a), \tag{5b}$$

$$\dot{\mathcal{M}}(t,\omega_a)=-[\gamma_{ab}+i(\omega_a-\omega_0)]\mathcal{M}(t,\omega_a)+g\mathcal{D}(t,\omega_a)+\mathcal{F}_{\mathcal{M}}(t,\omega_a), \tag{5c}$$

$$\dot{\mathcal{A}}(t)=-\left[\frac{\kappa}{2}+i(\omega_c-\omega_0)\right]\mathcal{A}(t)+g\mathcal{M}(t). \tag{5d}$$

Here $\mathcal{D}(t,\omega_a)=\mathcal{N}_a(t,\omega_a)-\mathcal{N}_b(t,\omega_a)$ is the spectral density of the atomic population difference; $\mathcal{A}(t)$ is a c-number stochastic variable corresponding to the laser field operator inside the cavity; for a chosen (normal) ordering of operators (see Ref. [6] for details) and zero temperature of the external thermal reservoir the corresponding Langevin force in the equation for $\mathcal{A}(t)$ vanishes; γ_a, γ_b, and γ_{ab} are the decay rates of the atomic populations and polarization, respectively; $\kappa/2$ is the cold-cavity decay rate, the corresponding cold-cavity mode frequency is ω_c; ω_0 is the lasing frequency; g is the coupling constant of the atom-field interaction. As mentioned above, we shall assume that the laser frequency is tuned to the central frequency of the atomic frequency distribution function $\rho(\omega_a)$ as written in Eq. (2).

The functions $\mathcal{F}_\mu(t,\omega_a)$, $\mu=a,b$, or $\mathcal{M}$, are the c-number Langevin forces for the respective atomic variables. Having vanishing mean values and Gaussian statistics, these functions are completely described by their second-order correlation functions:

$$\langle\mathcal{F}_\mu(t,\omega_a)\mathcal{F}_\nu(t',\omega_a')\rangle=2\mathcal{D}_{\mu\nu}(\omega_a)\,\delta(\omega_a-\omega_a')\,\delta(t-t'). \tag{6}$$

The diffusion coefficients $\mathcal{D}_{\mu\nu}$ for the on-resonance homogeneously broadened medium were calculated in Ref. [6]. The new coefficients $\mathcal{D}_{\mu\nu}(\omega_a)$ in Eq. (6) are obtained from $\mathcal{D}_{\mu\nu}$ upon replacing the collective atomic variables by their corresponding spectral densities:

$$2\mathcal{D}_{aa}(\omega_a)=\gamma_a\langle\mathcal{N}_a(t,\omega_a)\rangle+R(1-p)\rho(\omega_a)-g[\langle\mathcal{M}^*(t,\omega_a)\mathcal{A}(t)\rangle+\langle\mathcal{A}^*(t)\mathcal{M}(t,\omega_a)\rangle], \tag{7a}$$

$$2\mathcal{D}_{bb}(\omega_a)=\gamma_b\langle\mathcal{N}_b(t,\omega_a)\rangle-g[\langle\mathcal{M}^*(t,\omega_a)\mathcal{A}(t)\rangle+\langle\mathcal{A}^*(t)\mathcal{M}(t,\omega_a)\rangle], \tag{7b}$$

$$2\mathcal{D}_{ab}(\omega_a)=g[\langle\mathcal{M}^*(t,\omega_a)\mathcal{A}(t)\rangle+\langle\mathcal{A}^*(t)\mathcal{M}(t,\omega_a)\rangle], \tag{7c}$$

$$2\mathcal{D}_{\mathcal{M}\mathcal{M}}(\omega_a)=2g\langle\mathcal{M}(t,\omega_a)\mathcal{A}(t)\rangle, \tag{7d}$$

$$2\mathcal{D}_{\mathcal{M}^*\mathcal{M}}(\omega_a)=(2\gamma_{ab}-\gamma_a)\langle\mathcal{N}_a(t,\omega_a)\rangle+R\rho(\omega_a), \tag{7e}$$

$$2\mathcal{D}_{b\mathcal{M}}(\omega_a)=\gamma_b\langle\mathcal{M}(t,\omega_a)\rangle. \tag{7f}$$

Here the angle brackets denote the quantum mechanical mean value of the corresponding variables. A point worth comment is the modification of the first diffusion coefficient $\mathcal{D}_{aa}(\omega_a)$, which depends on the pumping statistics of the atomic medium (distribution of time intervals for the successive atomic excitations). This statistics is determined by the parameter p, $0\leqslant p\leqslant 1$, and can vary from a Poissonian distribution of excited atoms, when $p=0$, to a completely regular one, when $p=1$. Writing the corresponding term in $\mathcal{D}_{aa}(\omega_a)$ as $R(1-p)\rho(\omega_a)$, we assume that the parameter p is the same for different atomic groups. Clearly, R is now the total pumping rate averaged over all groups. It is easy to show that if the pumping is Poissonian for the totality of atoms, independently of their resonance frequencies, then it will also be Poissonian for each frequency group. On the other hand, regular pumping of the inhomogeneously broadened atoms would require an incoherent pumping mechanism with the bandwidth larger than the inhomogeneously broadened linewidth, so that each group is pumped with the same degree of regularity and independently of all other groups. Although this is not an easy experimental endeavor, we keep here $p\neq 0$ for the sake of generality.

Setting the time derivatives to zero in Eqs. (5a)–(5d) and dropping off the Langevin forces we arrive at the steady-state mean values for the atomic spectral densities. In what follows these steady-state values will be identified with a zero subscript. They are conveniently expressed in terms of four dimensionless parameters:

(i) The dimensionless intensity $I = I_0/I_s$ of the laser field inside the cavity, where I_s is the homogeneous saturation intensity,

$$I_s = \frac{\gamma_{ab}}{2g^2}\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}, \tag{8}$$

(ii) The dimensionless pumping rate $r = R/R_{\rm th}$, where $R_{\rm th}$ is the homogeneous threshold pumping rate,

$$R_{\rm th} = \frac{\gamma_{ab}\gamma_a\kappa}{2g^2}, \tag{9}$$

(iii) The dimensionless atomic detuning $x = (\omega_a - \omega_0)/\gamma_{ab}$, and (iv) the dimensionless inhomogeneous broadening width $\alpha = \Delta\omega_a/\gamma_{ab}$.

We shall also use the dimensionless frequency distribution function $w(x)$ instead of $\rho(\omega_a)$:

$$w(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\alpha}\exp[-x^2/\alpha^2], \quad \int_{-\infty}^{\infty} dx\, w(x) = 1. \tag{10}$$

The steady-state spectral densities of atomic variables in terms of these parameters are

$$\mathcal{N}_{a0}(x) = \frac{Rw(x)}{\gamma_a}\left[\frac{1+x^2}{1+I+x^2} + \frac{\gamma_a}{\gamma_a+\gamma_b}\frac{I}{1+I+x^2}\right], \tag{11a}$$

$$\mathcal{N}_{b0}(x) = \frac{Rw(x)}{\gamma_a}\frac{\gamma_a}{\gamma_a+\gamma_b}\frac{I}{1+I+x^2}, \tag{11b}$$

$$\mathcal{M}_0(x) = \frac{Rw(x)}{\gamma_a}\frac{g\mathcal{A}_0}{\gamma_{ab}}\frac{1-ix}{1+I+x^2}, \tag{11c}$$

where we have assumed the steady-state field amplitude $\mathcal{A}_0$ to be real.

Equations (11a)–(11c) express the steady-state atomic variables in terms of the yet unknown dimensionless intensity I of the laser field. To determine this intensity itself we may replace $\mathcal{M}_0(x)$ given by Eq. (11c) into Eq. (5d) for the steady-state field amplitude:

$$\kappa/2 + i(\omega_c - \omega_0) = \frac{g^2 R}{\gamma_a\gamma_{ab}}\int_{-\infty}^{\infty} dx\, w(x)\frac{1-ix}{1+I+x^2}. \tag{12}$$

Since $w(x)$ is an even function, the imaginary part of the integral vanishes providing $\omega_c = \omega_0$. Thus, as mentioned above, there is no frequency pulling in our model. The real part of Eq. (12) gives us the equation for I:

$$r\int_{-\infty}^{\infty} dx \frac{w(x)}{1+I+x^2} = 1. \tag{13}$$

For the Gaussian function $w(x)$, given by Eq. (10), this integral can be evaluated analytically, yielding

$$r = \frac{\alpha v}{\sqrt{\pi}}\frac{\exp[-v^2/\alpha^2]}{\mathrm{erfc}[v/\alpha]}, \tag{14}$$

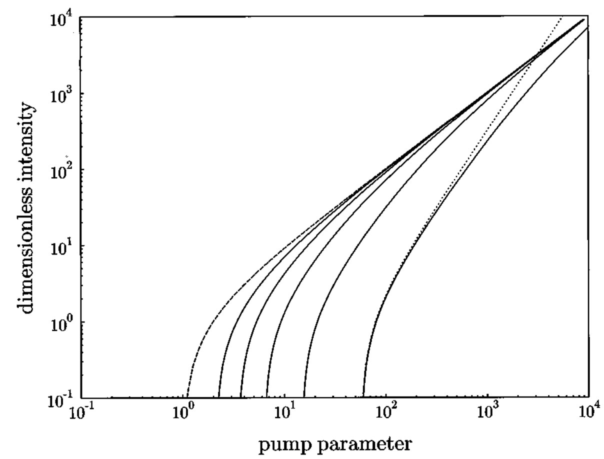

FIG. 2. Stationary dimensionless intensity I as a function of the pump parameter r for different values of the inhomogeneous broadening; the uppermost curve (broken) corresponds to the homogeneous case, $\alpha = 0$; then in descending order $\alpha = 2.5, 5, 10, 25$, and 100; dotted line is the approximate solution given by Eq. (16).

where $v = \sqrt{1+I}$, and $\mathrm{erfc}(z)$ is the complementary error function,

$$\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z dt \exp(-t^2). \tag{15}$$

While in general the transcendental equation (14) for $I(r)$ can be solved only numerically, two limiting cases allow for simple analytical results.

For $v/\alpha \gg 1$ we recover the homogeneous solution, $I(r) = r - 1$. It is worth noting that the condition $v/\alpha \gg 1$ can be satisfied not only for a small inhomogeneous broadening, $\alpha \ll 1$, but also for intermediate values of α but large dimensionless intensity I. In other words, the important parameter is not α itself but $\alpha/v = \Delta\omega_a/(\gamma_{ab}\sqrt{1+I})$ which is the ratio of the Doppler width to the *power broadened* homogeneous width.

Another limiting case is the opposite one of very large inhomogeneous broadening, $v/\alpha \ll 1$, i.e., the so-called Doppler limit. In this case the approximate solution of Eq. (14) is

$$I(r) = \frac{\pi r^2}{\alpha^2} - 1. \tag{16}$$

In contrast to the homogeneous case the intensity increases now quadratically with the dimensionless pumping rate. Setting $I(r)$ to zero we find the threshold value of r in the Doppler limit as $r_{\rm th} = \alpha/\sqrt{\pi}$. It is linearly proportional to the Doppler width $\Delta\omega_a$. The behavior of $I(r)$ for several different values of α is shown in Fig. 2.

To investigate the small fluctuations of the laser field and atomic variables around the steady-state solution we split each of the four variables $\{\mathcal{N}_a, \mathcal{N}_b, \mathcal{M}, \mathcal{A}\} \equiv Q$ into a sum $Q(t) = Q_0 + \delta Q(t)$ of a large steady-state value Q_0 and a small fluctuation $\delta Q(t)$. Performing the Fourier transform of the time-dependent fluctuating terms,

$$\delta Q(\Omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\exp[i\Omega t]\delta Q(t), \tag{17}$$

we arrive at a linear algebraic system of four complex equations,

$$(\gamma_a-i\Omega)\delta\mathcal{N}_a(\Omega,x)=-g\mathcal{A}_0[\delta\mathcal{M}(\Omega,x)+\delta\mathcal{M}^*(-\Omega,x)]-g[\mathcal{M}_0\delta\mathcal{A}^*(-\Omega)+\mathcal{M}_0^*\delta\mathcal{A}(\Omega)]+\mathcal{F}_a(\Omega,x), \tag{18a}$$

$$(\gamma_b-i\Omega)\delta\mathcal{N}_b(\Omega,x)=g\mathcal{A}_0[\delta\mathcal{M}(\Omega,x)+\delta\mathcal{M}^*(-\Omega,x)]+g[\mathcal{M}_0\delta\mathcal{A}^*(-\Omega)+\mathcal{M}_0^*\delta\mathcal{A}(\Omega)]+\mathcal{F}_b(\Omega,x), \tag{18b}$$

$$[\gamma_{ab}(1+ix)-i\Omega]\delta\mathcal{M}(\Omega,x)=g\mathcal{D}_0(x)\delta\mathcal{A}(\Omega)+g\mathcal{A}_0\delta\mathcal{D}(\Omega,x)+\mathcal{F}_{\mathcal{M}}(\Omega,x), \tag{18c}$$

$$-i\Omega\delta\mathcal{A}(\Omega)=-\kappa/2\delta\mathcal{A}(\Omega)+g\int_{-\infty}^{\infty}dx\,\delta\mathcal{M}(\Omega,x). \tag{18d}$$

Further, we split each of the four complex fluctuations into their real and imaginary parts, $\delta Q(\Omega)=\delta X_\alpha(\Omega)+i\delta Y_\alpha(\Omega)$, $\alpha=a,b,\mathcal{M},\mathcal{A}$, which correspond to the fluctuations of the amplitude and phase quadrature components. Here we shall be concerned with the fluctuation spectrum of the phase quadrature of the laser field,

$$\langle\delta Y_{\mathcal{A}}(\Omega)\delta Y_{\mathcal{A}}(\Omega')\rangle=\delta(\Omega+\Omega')(\delta Y^2_{\mathcal{A}})_\Omega, \tag{19}$$

which is related to the linewidth of the laser. The corresponding fluctuation spectrum of the amplitude quadrature, related to the photocurrent noise spectrum under direct detection of the laser light, will be discussed elsewhere. The spectral density $(\delta Y^2_{\mathcal{A}})_\Omega$ in the low-frequency limit Ω is related as follows to the laser linewidth $\Delta\nu$:

$$(\delta Y^2_{\mathcal{A}})_\Omega=I_0\Delta\nu/\Omega^2, \tag{20}$$

where I_0 is a steady-state intracavity field intensity. The typical low-frequency divergence of $(\delta Y^2_{\mathcal{A}})_\Omega$ as $1/\Omega^2$ is a manifestation of phase diffusion under steady-state laser operation.

Incidentally, because of the detuning in Eq. (18c) for the atomic polarization, the amplitude and the phase fluctuations become coupled, in contrast to the case of the on-resonance homogeneous medium (see Ref. [6]). This implies that the expression for the phase fluctuation $\delta Y_{\mathcal{A}}(\Omega)$ is more complicated and contains all the atomic fluctuation forces. Luckily, for the calculation of $\Delta\nu$ we need to know only $\delta Y_{\mathcal{A}}(\Omega)$ in the low-frequency limit, which brings about a rather significant simplification:

$$-i\Omega\delta Y_{\mathcal{A}}(\Omega)=\left[-\kappa/2+\frac{g^2}{\gamma_{ab}}\int_{-\infty}^{\infty}dx\frac{\mathcal{D}_0(x)(1-i\Omega/\gamma_{ab})}{(1-i\Omega/\gamma_{ab})^2+x^2}\right]\delta Y_{\mathcal{A}}(\Omega)+g\int_{-\infty}^{\infty}dx\frac{(1+I)\xi_{\mathcal{M}}(x)-x\chi_{\mathcal{M}}(x)}{1+I+x^2}-g^2\mathcal{A}_0\int_{-\infty}^{\infty}dx\frac{x}{1+I+x^2}\left[\frac{\mathcal{F}_a(x)}{\gamma_a}-\frac{\mathcal{F}_b(x)}{\gamma_b}\right]. \tag{21}$$

Here we have splitted the Langevin forces into real and imaginary parts as $\mathcal{F}_\alpha(\Omega,x)=\xi_\alpha(\Omega,x)+i\chi_\alpha(\Omega,x)$, and have introduced a shorthand $\mathcal{F}_\alpha(x)=\mathcal{F}_\alpha(\Omega=0,x)$.

Using Eq. (21), and the diffusion coefficients of the Langevin forces from Eqs. (7a)-(7f), the linewidth $\Delta\nu$, as given by Eq. (20), can be calculated in a straightforward way. Quite amazingly, the final result looks very simple,

$$\Delta\nu=\frac{\kappa}{2I_0}\left(\frac{\gamma}{\gamma+\kappa/2}\right)^2\left[1+\frac{1}{2}\frac{\gamma_a}{\gamma_a+\gamma_b}I(1+c)+\frac{1}{2}\frac{\gamma_b}{\gamma_a+\gamma_b}I\left(1-\frac{1}{2}p\right)(1-c)\right], \tag{22}$$

where we have introduced a new parameter γ as $\gamma=\gamma_{ab}/c$ to make the result look more similar to the homogeneous case [6].

All the information about the inhomogeneous broadening is now contained in a single coefficient c, which is defined as follows,

$$c\int_{-\infty}^{\infty}dx\frac{w(x)}{1+I+x^2}=\int_{-\infty}^{\infty}dx\frac{w(x)}{1+I+x^2}\,\frac{1+I-x^2}{1+I+x^2}. \tag{23}$$

For the Gaussian distribution function given by Eq. (10) this coefficient can be evaluated analytically and is equal to

$$c=\frac{2}{\sqrt{\pi}}\frac{v}{\alpha}\frac{\exp[-v^2/\alpha^2]}{\mathrm{erfc}[v/\alpha]}-\frac{2v^2}{\alpha^2}. \tag{24}$$

Equations (22) and (24) are the main results of this paper. Before going into their detailed discussion we would like to compare the linewidth $\Delta\nu$ given by Eq. (22) with the Schawlow-Townes linewidth $\Delta\nu_{\mathrm{ST}}$ from Eq. (1). The difference is given by the two additional factors in Eq. (22), in curly and square brackets, respectively. The first one is always smaller or equal to unity and will be called the *quenching factor* as it leads to a decreasing of the linewidth. The second one, on contrary, is always bigger or equal unity and will be called the *enhancement factor*.

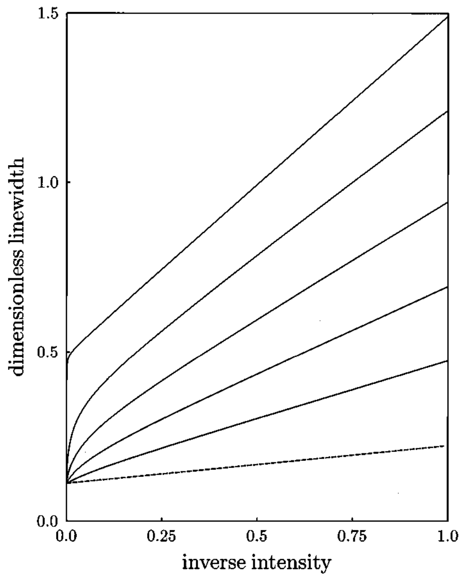

FIG. 3. Dimensionless linewidth $\Delta\tilde{\nu}=2I_s\Delta\nu/\kappa$ as a function of the inverse dimensionless intensity $1/I$ for different values of the inhomogeneous broadening; the topmost curve is for the Doppler limit, $\alpha=1000$; then in descending order, $\alpha=25,10,5,2.5$; the broken line is for the homogeneous limit, $\alpha=0$; $\gamma_a/\gamma_b=100$, $\kappa/2\gamma_{ab}=2$.

Let us first consider, as we did for the steady-state solution, the homogeneous and the Doppler limits. In the homogeneous limit, $\alpha=0$, we have $c=1$. This is easy to see from Eq. (23), taking into account that for $\alpha=0$ the atomic frequency distribution becomes the δ function, $w(x)=\delta(x)$. For the linewidth $\Delta\nu$ in this limit we recover the result from [6]

$$\Delta\nu=\frac{\kappa}{2I_0}\left(\frac{\gamma_{ab}}{\gamma_{ab}+\kappa/2}\right)^2\left[1+\frac{\gamma_a}{\gamma_a+\gamma_b}I\right]. \qquad (25)$$

The quenching factor is determined by the ratio of the cavity loss rate to the polarization decay rate, $\kappa/2\gamma_{ab}$, and becomes very small in the bad-cavity regime, $\kappa/2\gamma_{ab}\gg 1$. The enhancement factor in the homogeneous limit is due to incomplete inversion of the laser medium and depends on the the ratio of the relaxation constants of the upper and lower levels, γ_a/γ_b. When this ratio is small, the lower atomic level is almost empty [see Eqs. (11a), (11b)] and the enhancement factor is equal to unity. Thus, for the fully inverted good-cavity laser the linewidth $\delta\nu$ from Eq. (25) coincides with the Schawlow-Townes linewidth $\Delta\nu_{\mathrm{ST}}$ from Eq. (1).

In the opposite Doppler limit, $\alpha=\infty$, we have $c=0$, as follows from Eq. (24), and the linewidth $\Delta\nu$

$$\Delta\nu=\frac{\kappa}{2I_0}\left[1+\frac{1}{2}I-\frac{1}{4}p\,\frac{\gamma_b}{\gamma_a+\gamma_b}I\right]. \qquad (26)$$

The quenching factor becomes unity in the Doppler limit, i.e., the line narrowing due to the bad-cavity effects disappears. This is the first important conclusion of our theory: large inhomogeneous broadening leads to suppression of the atomic memory effects in the linewidth. We should mention here that Benkert, Scully, and Süssmann [11] have come to a similar conclusion regarding the influence of the inhomogeneous broadening on the *short-time scale* evolution of the laser phase. However, this short-time scale behavior has no influence on the linewidth and is only visible in the far wings of the optical spectrum.

The enhancement factor in the Doppler limit is also different from the homogeneous case. It provides two intensity-independent contributions to the linewidth $\Delta\nu$ given by the second and third terms in the square brackets. The first contribution is positive but now independent of the relaxation constants of the atomic levels, i.e., of the degree of inversion. The third term contains the statistical parameter p, i.e., depends on the pumping statistics. This is the second important result of our theory: the influence of the pumping statistics of the atomic medium on the linewidth of the inhomogeneously broadened laser. Such an influence is absent for the on-resonance homogeneous laser. The physical reason for this effect is the coupling between the amplitude and the phase fluctuations through the detuning between the laser frequency and the resonance frequencies of the individual atomic groups. As was first pointed out by Golubev and Sokolov [12], such coupling is also present in the off-resonance homogeneous medium, and brings about a similar influence of the pumping statistics on the laser linewidth.

The statistics-dependent term in Eq. (26) brings a negative contribution into the enhancement factor, which is maximum for regular pumping statistics, $p=1$, and long lifetime of the upper level, $\gamma_a\ll\gamma_b$. In this optimum case the third term cancels out half of the second.

Figure 3 shows the dimensionless linewidth, defined as $\Delta\tilde{\nu}=2I_s\Delta\nu/\kappa$, as a function of the inverse dimensionless intensity $1/I$, for different values of the inhomogeneous broadening. Two features of these curves are worth pointing out. The first one is the linear asymptotic behavior of $\Delta\tilde{\nu}(1/I)$ for small dimensionless intensities. The slope of these asymptotic lines depends, among other parameters, on the inhomogeneous broadening, and can be found from Eqs. (22), (24), by making a Taylor expansion of the quenching and enchancement factors up to the first order in the dimensionless intensity I. The second feature of these curves is that all of them start from the same point of the homogeneous linewidth for very large dimensionless intensity

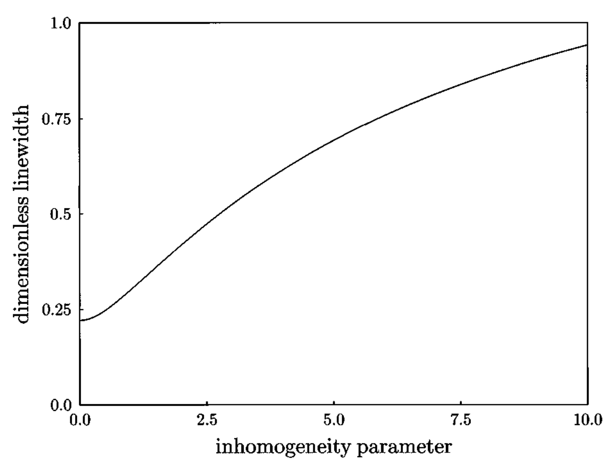

FIG. 4. Dimensionless linewidth $\Delta\tilde{\nu}$ as a function of the inhomogeneous parameter α; $\gamma_a/\gamma_b=100$, $\kappa/2\gamma_{ab}=2$, $I=1$.

$I \to \infty$. Hence, for nonzero inhomogeneity, there is always some region of intensities I where the linewidth shows nonlinear dependence on $1/I$. Such an effect would be interesting to observe experimentally. Unfortunately, as follows from Fig. 3, it takes place for rather high dimensionless intensities, which are experimentally hard to achieve.

Figure 4 illustrates the dependence of the dimensionless linewidth $\Delta \tilde{\nu}$ on the inhomogeneity parameter α for fixed intensity I.

In conclusion, we have obtained an analytical expression for the linewidth of an inhomogeneously broadened laser, irrespective of the relative magnitudes of atomic and field decay constants. Especially interesting results are obtained in the bad-cavity limit, where an important linewidth narrowing may occur. First experiments [13] performed with high-gain midinfrared gas lasers, e.g., HeNe $\lambda = 3.39 \mu$m and HeXe $\lambda = 3.51 \mu$m, operating in the bad-cavity regime and having gain profiles with comparable homogeneous and inhomogeneous widths, show the results that are in very good agreement with the theory presented above.

A. Z. K. and L. D. acknowledge the financial support of CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).

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