

Formation of stable solitons in quadratic nonlinear media

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Dispersive quadratic media with wave mixing between the first- and second-harmonic modes due to a $\chi^{(2)}$ nonlinearity are shown to inhibit wave collapse and to support stable solitons. The stability of this coupled-soliton family is demonstrated by means of a Lyapunov analysis based on the energy integral of the wave-coupling equations. The dynamics of the coupled modes is finally studied using a virial identity, which predicts either a stable propagation of the mutually trapped solitons or a spreading of both waves, depending on the incident-beam power.

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Solitons play an important role in the dynamics of dispersive wave systems in nonlinear media where they arise as steadily propagating localized wave structures with finite energy. Well-known examples are the envelope soliton of electromagnetic waves propagating in dielectric media with a third-order nonlinear response, i.e., in the so-called Kerr or $\chi^{(3)}$ media characterized by an intensity-dependent refractive index. These solitons are usually solutions to the nonlinear Schrödinger equation (NSE), which is exactly solvable in the (1+1)-space-time-dimensional case for which solitons are stable. It turns out, however, that solitonlike solutions to the NSE in higher dimensions are unstable and may undergo a catastrophic collapse, provided the initial beam power exceeds a threshold value (see, e.g., [1]). In this case, the beam ultimately collapses by localizing its energy in essentially one point with strongly damaging effects. Thus these cubic media have limited applications in the guiding and steering of optical beams. Therefore, the propagation of intense light beams in quadratic nonlinear media (so-called $\chi^{(2)}$ materials) has attracted wide attention because of the possibility of forming multidimensional solitary waves that appear to be stable. In such media, solitonlike structures are formed by the interaction of the fundamental and second-harmonic waves that mutually trap each other and form a bound state. This effect is often referred to as the $\chi^{(2)}:\chi^{(2)}$ cascaded nonlinearity, which has been observed experimentally in several different materials (see, e.g., [2]). Very recently, a large number of theoretical works have described the formation and the dynamics of both temporal and spatial solitons [3–6], although the first investigation of soliton formation in such media dates back to Karamzin and Sukhorukov [7]. Further, Kanashov and Rubenichik [8] showed that (3+1)-dimensional solitons are stable in analogous systems.

In the following, we investigate the behavior of multidimensional localized structures in dispersive quadratic media. Our investigations apply to spatial soliton-type structures in

$(D+1)$ -dimensional space, i.e., a D -dimensional transverse “plane” orthogonal to a one-dimensional propagation direction, but may easily be generalized to temporal as well as spatiotemporal soliton structures by, e.g., replacing one of the transverse spatial coordinates by a retarded time variable. For the sake of convenience, the terminology “soliton” will henceforth be employed to mean stable-shaped solitary waves, as is often used in this context (see, e.g., [3–8]). We find that in $\chi^{(2)}$ media wave collapse will not take place for any dimension D of physical significance ($D \leq 3$). Furthermore, we prove that the solitonlike solution is generally stable in the Lyapunov sense for all $D < 4$, and we detail the possible motions of the stable coupled solitons in terms of the initial data. Here, we must specify that the stability of the $(D+1)$ -dimensional solitons should be interpreted as stability with respect to perturbations belonging to the same dimensional space. Thus it does not exclude the so-called waveguide instability; e.g., the instability of a (2+1)-dimensional stationary structure with respect to perturbations developing along the third dimension, as studied in [8].

We start our analysis from the set of equations describing the coupling of the slowly varying envelope of the fundamental wave with the second harmonic (see [4] or [6] for the details of the derivation):

$$i(\partial E_1/\partial z) + \gamma_1 \nabla_{\perp}^2 E_1 + \chi_1 E_1^* E_2 \exp(-i\delta kz) = 0, \quad (1)$$

$$i(\partial E_2/\partial z) + \gamma_2 \nabla_{\perp}^2 E_2 + \chi_2 E_1^2 \exp(i\delta kz) = 0, \quad (2)$$

where the diffraction operator ∇_{\perp}^2 acts in a D -dimensional transverse plane of spatial coordinate vector \vec{x}_{\perp} . The wave-space convection (also called “walk-off” effect [5]) has been disregarded. Here, E_1 and E_2 are the amplitudes of the fundamental and second-harmonic modes with wave numbers k_1 and k_2 , respectively. δk is the wave-vector mismatch $\delta k = 2k_1 - k_2$, $\gamma_{1,2} \equiv 1/2k_{1,2}$ denote the diffractive coefficients, and $\chi_{1,2}$ correspond to the appropriate components of the second-order nonlinear susceptibility tensor. By means of simple rescalings and phase-shifted transformations, Eqs. (1) and (2) can be brought into the canonical form

$$i(\partial w/\partial \zeta) + \Delta w + w^* v = 0, \quad (3)$$

$$2i(\partial v/\partial \zeta) + \Delta v - \beta v + w^2 = 0. \quad (4)$$

In Eqs. (3) and (4), the Laplacian Δ is expressed in terms of the transformed space variable $\vec{\xi} = \sqrt{k_1/\gamma_1} \vec{x}_{\perp}$; $\zeta = k_1 z$ denotes the new longitudinal length coordinate, and $\beta = \delta k \alpha / k_1$ where $\alpha \equiv \gamma_1 / \gamma_2 \approx 2$ justifies the factor 2 in

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front of (4). Finally, the normalized amplitudes w and v are related to E_1 and E_2 as $w = (\sqrt{\alpha\chi_1\chi_2}/k_1)E_1$, $v = (\chi_1/k_1)E_2 \exp(-i\delta kz)$. Equations (3) and (4) can be applied to model, for instance, nonlinear interactions in lithium niobate (LiNbO_3) materials [2, 3]: such materials have high second-order coupling coefficients, so that the nonlinear scale length, which the light wave has to propagate to achieve a 2π phase rotation, is of the order of a few millimeters at reasonable power levels and remains considerably shorter than typical waveguide lengths of the order of a few centimeters. Following the crystalline anisotropy, the field polarization, and the ambient temperature, it is further possible to select the incident fundamental and second-harmonic components of a laser beam in such a way that the wave-number mismatch δk is almost vanishing ($k_2 \sim 2k_1$) for absorption lengths of several centimeters. As already established in, e.g., [8], the system (3),(4) admits the following two invariants, namely the ‘‘mass’’ or ‘‘power’’:

$$N = \int (|v|^2 + \frac{1}{2}|w|^2) d^D \xi, \quad (5)$$

and the energy (Hamiltonian) integral

$$H = \int \left(|\vec{\nabla} w|^2 + \frac{1}{2} |\vec{\nabla} v|^2 + \frac{\beta}{2} |v|^2 - \text{Re}(w^2 v^*) \right) d^D \xi. \quad (6)$$

In order to investigate the various dynamical aspects of the coupled waves (w, v) , we construct a so-called ‘‘virial’’ identity, in analogy with the standard result of the NSE [1], consisting in the double derivative with respect to the longitudinal distance ζ of the mean square radius

$$I(\zeta) = \int \xi^2 (|v|^2 + \frac{1}{2}|w|^2) d^D \xi. \quad (7)$$

To compute the virial relation, we first multiply (3) by $(\xi^2 w^*/2)$ and (4) by $(\xi^2 v^*)$, then integrate the imaginary part of the sum of the resulting equations to obtain

$$\frac{\partial I}{\partial \zeta} = 2 \text{Im} \int \xi \cdot (w^* \vec{\nabla} w + v^* \vec{\nabla} v) d^D \xi. \quad (8)$$

Besides, we can determine the ζ derivative of both the contributions on the right-hand side of Eq. (8) by making use of the following: on the one hand, one starts with Eq. (3) and multiplies it by $(\xi \cdot \vec{\nabla} w^*)$, which leads, after a straightforward integration over space, to

$$\begin{aligned} & \frac{\partial}{\partial \zeta} \text{Im} \int (\xi \cdot \vec{\nabla} w) w^* d^D \xi \\ &= \int \{ 2|\vec{\nabla} w|^2 + \text{Re}[(\xi \cdot \vec{\nabla} v^*) w^2], \} d^D \xi. \end{aligned} \quad (9)$$

On the other hand, we multiply (4) by $(\xi \cdot \vec{\nabla} v^*)$, which yields

$$\begin{aligned} & \frac{\partial}{\partial \zeta} \text{Im} \int (\xi \cdot \vec{\nabla} v) v^* d^D \xi \\ &= \int \left\{ |\vec{\nabla} v|^2 + \text{Re}[(\xi \cdot \vec{\nabla} w^2) v^*] + \frac{D}{2} \text{Re}(w^2 v^*) \right\} d^D \xi. \end{aligned} \quad (10)$$

We finally differentiate Eq. (8) once more with respect to ζ , and use the former relations (9) and (10) to obtain the virial identity

$$\frac{\partial^2 I}{\partial \zeta^2} = (4-D) \int (|\vec{\nabla} w|^2 + \frac{1}{2} |\vec{\nabla} v|^2) d^D \xi + D \left(H - \frac{\beta}{2} \int |v|^2 d^D \xi \right). \quad (11)$$

From this relation, it can easily be concluded that no collapse [in the sense that $I(\zeta) \rightarrow 0$ at a finite ζ with, e.g., $I_{\zeta\zeta} < 0$] of solutions $w(\xi, \zeta)$ and $v(\xi, \zeta)$ can occur for $D < 4$. Indeed, let us *a priori* suppose the contrary, i.e., that $I(\zeta) \rightarrow 0$ at a finite $\zeta = \zeta_c$; then necessarily both the integrals $\int \xi^2 |v|^2 d^D \xi$ and $\int \xi^2 |w|^2 d^D \xi$ have to vanish separately and simultaneously as $\zeta \rightarrow \zeta_c$. Employing the Schwarz inequality after a simple integration by parts, we obtain $(\int |g|^2 d^D \xi)^2 \leq (4/D^2) \int \xi^2 |g|^2 d^D \xi \int |\vec{\nabla} g|^2 d^D \xi$ applied to any L^2 -integrable function g , so that the previous assumption should imply that both the gradient norms $\int |\vec{\nabla} w|^2 d^D \xi$ and $\int |\vec{\nabla} v|^2 d^D \xi$ diverge as $\zeta \rightarrow \zeta_c$. By virtue of the constancy of H and since the finiteness of N ensures that the two masses $N_w \equiv \int |w|^2 d^D \xi$ and $N_v \equiv \int |v|^2 d^D \xi$ remain bounded, the quantity $I_{\zeta\zeta}$ should thus diverge in the vicinity of the collapse focus ζ_c , hence predicting the spreading of both wave forms, which contradicts the starting hypothesis. Therefore, ζ -dependent solutions w and v can never collapse at any finite distance ζ , and will be expected to exist globally for every ζ by keeping a bounded gradient norm.

The absence of a finite-distance collapse seems in contrast with solutions to the nonlinear Schrödinger equation, which is recovered in the limit of large $|\beta| \rightarrow +\infty$ as displayed in [4–6], since for space dimensions $D \geq 2$, the resulting cubic NSE admits collapsing solutions when $\beta > 0$. However, performing in Eqs. (3) and (4) the high-order perturbative expansion of v ,

$$v = v_0/\beta + v_1/\beta^2 + \dots, \quad v_0 = w^2, \quad (12)$$

in the limit $\beta \rightarrow \infty$, we obtain a modified NSE for w ,

$$i \frac{\partial w}{\partial \zeta} + \Delta w + \frac{1}{\beta} \left(|w|^2 - \frac{4}{\beta^2} |w|^4 \right) w + \frac{2}{\beta^2} [w^* (\vec{\nabla} w)^2 - |w|^2 \Delta w] = 0, \quad (13)$$

from which the cubic NSE is simply restored by retaining the dominant contributions including the first-order term in $1/\beta$. From Eq. (13), it is then observed that, whereas the correction of the dispersive part may be neglected for large β , the salient modification carried into the equation for w by the perturbation (12) consists in a higher-order nonlinearity that always exhibits a sign opposite that of the cubic term. Consequently, the whole nonlinear potential plays the role of a saturating nonlinearity that ultimately arrests the collapse as $|w|^2$ grows up along the longitudinal axis. These arguments display further evidence of the absence of collapse in $\chi^{(2)}$ materials.

As no catastrophic singularity develops in either of the coupled waves, we now search to identify the regular shapes of solutions w and v . Therefore, we look for stationary (ζ -independent) spatially localized solutions to Eqs. (3) and (4) of the form [3–6] $w(\xi, \zeta) = w'(\xi) \exp(i\lambda \zeta)$, $v(\xi, \zeta) = v'(\xi) \exp(2i\lambda \zeta)$, where the new space-dependent complex-valued functions w' and v' satisfy the coupled set of ordinary differential equations

$$-\lambda w' + \Delta w' + w'^* v' = 0, \quad (14)$$

$$-(\beta + 4\lambda) v' + \Delta v' + w'^2 = 0. \quad (15)$$

We first notice that both stationary states (w', v') reach an extremum at the center $\xi = 0$, at least for $D > 1$. In addition, the differential equations (14) and (15) do not admit a simple rescaling permitting us to reduce the eigenvalue λ to unity without loss of generality, except in the extreme cases of a

resonant wave mixing $\beta=0$ and of the NSE limit $|\beta| \rightarrow +\infty$ [this can be seen by performing the substitutions $w'(\xi) \rightarrow \lambda w'(\sqrt{\lambda}\xi)$ and $v'(\xi) \rightarrow \lambda v'(\sqrt{\lambda}\xi)$ in Eqs. (14) and (15), leading to a modifying of $\beta \rightarrow \beta/\lambda$]. Furthermore, localized solitonlike solutions may exist under the sufficient requirement $\lambda > \max\{0, -\beta/4\}$. Following this condition, we conclude that bright solitons, corresponding to the NSE limit with a positive β , are always ensured to exist as soon as $\lambda > 0$, whereas those corresponding to the opposite case $\beta < 0$ may exhibit a localized shape, provided that the eigenvalue λ belongs to the restricted spectrum $\lambda > -\beta/4$. Unlike the NSE limit, localized bright solitons can thus develop even for a negative β , as discussed by, e.g., Torner [6]. In what follows, we will assume that the previous condition for having localized stationary solutions is fulfilled.

Detailing the properties of the ground states w' and v' , we multiply Eq. (14) by (w'^*) , on the one hand, and by $(\vec{\nabla} w'^*)$, on the other hand, then integrate in space both of the resulting equations. We next repeat the previous operations in Eq. (15) by formally replacing $w' \rightarrow v'$. A simple combination of the space-integrated results yields the double identity

$$\begin{aligned} \int \operatorname{Re}(w'^2 v'^*) d^D \xi &= \frac{4}{D} \int (|\vec{\nabla} w'|^2 + \frac{1}{2} |\vec{\nabla} v'|^2) d^D \xi \\ &= \frac{2}{6-D} \left(4\lambda N_{GS} + \beta \int |v'|^2 d^D \xi \right), \end{aligned} \quad (16)$$

showing that the nonlinear potential $\int \operatorname{Re}(w'^2 v'^*) d^D \xi$ remains positive when it is expressed in terms of the soliton solutions. We can then express the Hamiltonian integral (6) as a function of the ground-state solutions: using the right-hand side of identity (16), we obtain

$$\begin{aligned} H_{GS} &= \left(1 - \frac{4}{D} \right) \int (|\vec{\nabla} w'|^2 + \frac{1}{2} |\vec{\nabla} v'|^2) d^D \xi + \frac{\beta}{2} \int |v'|^2 d^D \xi \\ &= \frac{D-4}{6-D} 2\lambda N_{GS} + \frac{\beta}{6-D} \int |v'|^2 d^D \xi, \end{aligned} \quad (17)$$

where the index GS henceforth refers to the ground states (w', v') . Thus, H_{GS} is ensured negative as $\beta < 0$ and $D < 4$. We now prove the stability of the coupled-soliton solution by arguing that the functional

$$S = H - \frac{\beta}{2} \int |v|^2 d^D \xi \quad (18)$$

can be viewed as a Lyapunov function that remains bounded from below for any space dimension $D < 4$, and whose bound admits for a fixed invariant N a global minimum reached on the coupled-soliton family. With this aim, we begin estimating the nonlinear potential of (6) by making use of the Schwarz and Sobolev inequalities,

$$\begin{aligned} \int \operatorname{Re}(w^2 v^*) d^D \xi \\ \leq C \left(\int |\vec{\nabla} w|^2 d^D \xi \right)^{\frac{D}{4}} \left(\int |w|^2 d^D \xi \right)^{1-\frac{D}{4}} \sqrt{\int |v|^2 d^D \xi}, \end{aligned} \quad (19)$$

where C denotes a positive constant. By means of this inequality, H is bounded from below as follows for $D < 4$:

$$\begin{aligned} H - \frac{\beta}{2} \int |v|^2 d^D \xi \\ \geq \int |\vec{\nabla} w|^2 d^D \xi - 2^{1-\frac{D}{4}} C N^{\frac{3}{2}-\frac{D}{4}} \left(\int |\vec{\nabla} w|^2 d^D \xi \right)^{\frac{D}{4}}, \end{aligned} \quad (20)$$

where the total mass N has been used to bound the norms N_w and N_v . Note that as the latter norms always remain bounded separately, the functional (18) can really be regarded as a proper Lyapunov functional, even though the second integral is not a true constant of motion: at fixed N and up to some additional positive contributions in λN assuring the positiveness of the Lyapunov function, this previous choice applying to both cases $\beta \geq 0$ could be checked to restore the forthcoming stability results in a way similar to the ones deduced from a more general prescription as, e.g., $S = H + \lambda N$. Like this one, the integral (18) is indeed bounded from below by a functional of $\int |\vec{\nabla} w|^2 d^D \xi$ reaching a global minimum, as seen from the right-hand side of the inequality (20). The latter estimate indicates that Eqs. (3) and (4) admit some fixed-point (stationary) solutions that are stable. In fact, the important integral, from which the stability of the stationary solutions follows, is the Hamiltonian H that exhibits a strict minimum. To examine which kind of fixed-point solutions can realize this minimum of H , we use the property according to which functional S admits a single minimum: we introduce

$$T \equiv \int (|\vec{\nabla} w|^2 + \frac{1}{2} |\vec{\nabla} v|^2) d^D \xi, \quad I_0 \equiv \operatorname{Re} \int (w^2 v^*) d^D \xi, \quad (21)$$

such that $S = T - I_0$. Following the standard procedure reviewed in [1] and [9], the minimum of S is identified by using the scale transformations

$$w(\vec{\xi}, \zeta) \rightarrow a^{-\frac{D}{2}} w(\vec{\xi}/a, \zeta), \quad v(\vec{\xi}, \zeta) \rightarrow a^{-\frac{D}{2}} v(\vec{\xi}/a, \zeta) \quad (22)$$

that preserve the L^2 norms attached to each wave w and v in expression (5). Here, a denotes a constant parameter playing the role of a Lagrange multiplier that only affects the energy integral when one inserts (22) into S , leading to $S_a \equiv T/a^2 - I_0/a^{D/2}$. Differentiating S_a with respect to the parameter a constrained on the value $a=1$ then yields the minimum of S . By doing so, one deduces that the latter functional admits a single minimum reached when the solutions satisfy the relation $I_0 = 4T/D$, which is nothing else but the relation (16) realized by the ground-state solutions. Hence, as inferred from the previous variational problem $\delta S = 0$, S contains a stable fixed point, on which its minimum is reached and which corresponds to the coupled-soliton solutions. This minimum also corresponds to the minimum of H , so that we now dispose the inequality

$$H \geq H_{GS}. \quad (23)$$

We have thus shown that not only are localized solutions to (3) and (4) stable in the Lyapunov sense, but also that the stable stationary solutions realizing a minimum of the energy integral exactly consist of the soliton solutions w' and v' . Once created, and provided that their respective initial data guarantee a nonlinearity level sufficiently strong to form a self-trapping attractor, the two waves w and v may therefore converge along ζ towards some stable stationary shapes whose spatial distribution in the transverse plane fits with w' and v' , respectively.

We now investigate the different types of dynamics of the fundamental and second-harmonic waves when they propagate in a dispersive quadratic medium. To do this, we again make use of the virial identity and deduce that the integral S (18) in the last contribution of the right-hand side of (11)

plays a major role in the behavior of the coupled solitons. Indeed, when S is assumed to be positive, $I_{\zeta\zeta}$ is larger than a strictly positive constant, so that $I(\zeta)$ must necessarily diverge as $\zeta \rightarrow +\infty$ for initial data without a space-dependent phase. Thus, the mean square radius of the two solitonlike waves centered on the origin increases with ζ , which means that both of them tend to spread out in the transverse plane as follows: either the fundamental and second harmonic waves can spread out on the center when their respective dispersions strongly dominate the attracting $\chi^{(2)}$ nonlinearity, or in the case of a moderate dispersion, they can form solitons, but the latter must separate and then move away from each other symmetrically with respect to the center. In this latter situation, both modes spread out as $\zeta \rightarrow +\infty$ since the nonlinear coupling linking w to v vanishes in this same limit. From a physical viewpoint, the basic condition $S \geq 0$ will be definitively ensured whenever initially one has

$$H \geq (\beta/2)N \quad \text{if } \beta > 0, \quad (24)$$

$$\text{or } H \geq 0 \quad \text{if } \beta \leq 0. \quad (25)$$

Established from the virial (11), these inequalities correspond to some conditions that are *sufficient* to initiate the spreading process.

When the above constraints are not satisfied initially, then the solitons may merge and, due to their stable nature, propagate without any deformation inside the medium as ζ increases. More precisely, starting with some initial waves, such as $I_{\zeta\zeta}(0) < 0$, solitons mutually trap by first self-contracting. As no collapse occurs, they afterwards converge towards stable solitonlike shapes satisfying $I_{\zeta\zeta}(\zeta) = I_{\zeta\zeta GS} = 0$ [the vanishing of $I_{\zeta\zeta}(\zeta)$ on the GS solutions simply results from inserting (17) into the virial identity (11)] and propagate as an undeformed self-trapped structure in the nonlinear medium. Note that assuring $I_{\zeta\zeta} < 0$ amounts to imposing the *necessary* conditions inferred from $S < 0$:

$$H < (\beta/2)N \quad \text{if } \beta > 0, \quad (26)$$

$$\text{or } H < 0 \quad \text{if } \beta \leq 0. \quad (27)$$

Keeping in mind the boundedness of H (23), condition (27) moreover consists in assuring a sufficiently high initial power $N > 2H_{GS}/\beta$ in the case of a positive phase mismatch $\beta > 0$. These results are in agreement with Torner *et al.*'s numerical simulations where (1+1)- [5] and (2+1)- [10] dimensional bright solitonlike waves have been observed to spread out and to "walk" away from each other *below* an intensity threshold, while they stick together and develop a

trapped coupled-soliton state *above* the same threshold. In accordance with the requirements (25) and (26), Refs. [5,10] exhibit coupled waves with a positive H [with, e.g., $v(\xi, 0) = 0$], which may remain self-trapped for $\beta > 0$, but spread out for $\beta < 0$.

In conclusion, we have shown that the coupled-soliton solution constitutes stable stationary solutions towards which the fundamental and second-harmonic waves in materials with a second-order nonlinearity can converge along the propagation axis. This is a simple consequence of the robustness of the Hamiltonian that has been proven to remain bounded from below for any space dimension number $D < 4$. This result applies to any value of the phase mismatch, which generalizes the recent estimates obtained for $\beta = 0$ in [10, 11]. Accordingly with this result, no collapse has been demonstrated to occur. The proof of the absence of collapse has been established from the virial relation, which enabled us to distinguish two types of motions depending on the incident beam power N : either N exceeds a critical threshold and both waves can coexist in a mutual trapping and propagate with a stable shape given by the soliton solutions, or N lies *below* this threshold and the solitonlike states continue to exist but decouple and ultimately spread out in the transverse plane. In the former situation, the nonlinear effects appear to be initially strong enough to maintain a mutual trapping for both waves, whereas in the latter one, the dispersion effects dominate the nonlinearity.

Finally we point out that all the above-discussed results may immediately be applied to structures "moving" with a velocity \vec{c} , since Eqs. (3) and (4) are invariant with respect to the transformations

$$w(\vec{\xi}, \zeta) \rightarrow \tilde{w}(\vec{\xi} - \vec{c}\zeta, \zeta) \exp(i(\vec{c} \cdot \vec{\xi} / 2 - c^2 \zeta / 4)),$$

$$v(\vec{\xi}, \zeta) \rightarrow \tilde{v}(\vec{\xi} - \vec{c}\zeta, \zeta) \exp(2i(\vec{c} \cdot \vec{\xi} / 2 - c^2 \zeta / 4)).$$

Concerning the case of a spatial soliton, it is worth mentioning that the velocity \vec{c} can induce a bending of the beam through the medium; for instance, if the incoming beam exhibits a spatially varying phase (as in, e.g., $\vec{c} \cdot \vec{\xi}$), the spatial solitary structure will be bent with an angle $\theta = \tan^{-1}|\vec{c}|$ with respect to the incident beam. Since this moving soliton is stable, this previous property has obvious applications in connection with steering of optical beams and with all-optical switching.

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