# Ermakov systems, exact solution, and geometrical angles and phases

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Ermakov systems are pairs of coupled, time-dependent nonlinear dynamical equations possessing a joint constant of motion. We show how to derive the Ermakov system from nonharmonic oscillators. We present a detailed study of Ermakov systems from a classical and quantum point of view. Finally the nonadiabatic Hannay's angle and Berry's phase for the system are calculated along with its adiabatic limit.

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## I. INTRODUCTION

The study of time-dependent oscillator systems has attracted considerable interest in the literature, both in classical [1–8] and quantum [9–19] mechanics. The origin of this development was no doubt the discovery of an exact invariant by Lewis [2] which was previously known as an approximate adiabatic invariant [1]. The existence of invariants for nonharmonic systems [3–5] was demonstrated by Ray and Reid [6], who derived a family of invariants for a special class of systems with nonlinear equations of motion.

The purpose of this paper is twofold. First we want to derive the invariant and the Ermakov systems described by the Hamiltonian

$$H=\frac{1}{2}\left[Z(t)p^2+2Y(t)pq+X(t)q^2+\frac{Z(t)l^2}{q^2}\right], \tag{1.1}$$

where $q$ and $p$ are the canonical coordinates, $X(t)$, $Y(t)$, and $Z(t)$ are an arbitrary function of time, and $l$ is an arbitrary constant which could be zero. Second, we want to calculate the nonadiabatic Hannay's angle and Berry's phase, and their adiabatic limit [19–26].

This paper is organized as follows. In Sec. II we derive the invariant and the Ermakov systems. In Sec. III, we shall find the exact solution and a nonadiabatic Hannay's angle for classical Ermakov systems. We obtain the exact solution and a nonadiabatic Berry's phase for the Ermakov Schrödinger equation in Sec. IV. In the concluding section we discuss the nature of the superposition law and we end with the adiabatic limit of the results obtained in Secs. III and IV.

## II. DERIVATION OF THE ERMAKOV SYSTEM

For the Hamiltonian

$$H=\sum_n h_n(t)T_n(p,q) \tag{2.1}$$

a dynamical Lie algebra of phase-space functions $T_n$ is constructed which is closed with respect to the Poisson bracket

$$\{T_n,T_m\}=\sum_r C^r_{nm}T_r\ . \tag{2.2}$$

Now the invariant $dI/dt=0$ is written as a member of the dynamical algebra

$$I=\sum_r \lambda_r(t)T_r \tag{2.3}$$

and by means of $\partial I/\partial t=\{I,H\}_{qp}$ and comparison of the coefficients of a system of first-order linear differential equations for the unknown $\lambda_r$ in (2.3) is obtained

$$\dot\lambda_r+\sum_n\left[\sum_n C^r_{nm}h_{m(t)}\right]\lambda_n=0\ , \tag{2.4}$$

where the structure constants $C^r_{nm}$ of the Lie algebra are defined in (2.2).

For the anharmonic oscillator (1.1), this approach leads to a finite algebra containing only $T^l_1=\frac{1}{2}[p^2+l^2/q^2], T_2=pq$, and $T_3=\frac{1}{2}q^2$ with the Poisson brackets $\{T^l_1,T_2\}=-2T^l_1$, $\{T_2,T_3\}-2T_2$, and $\{T_3,T^l_1\}=T_2$, i.e., the algebra $\{T^l_1,T_2,T_3\}$ is identical to the oscillator algebra $\{T^{l=0}_1,T_2,T_3\}$. The differential equation (2.4) reads in this case

$$\begin{aligned}\dot\lambda_1&=2(\lambda_1Y-Z\lambda_2)\ ,\\ \dot\lambda_2&=2(\lambda_1X-Z\lambda_3)\ ,\\ \dot\lambda_3&=2(\lambda_2X-Y\lambda_3)\ ,\end{aligned} \tag{2.5}$$

which can be simplified by setting $\lambda_1=\rho^2$ where $\rho$ is the solution of the auxiliary equation

$$\frac{d}{dt}\dot\rho-\frac{\dot Z}{Z}\dot\rho+\left[(XZ-Y^2)+\frac{\dot Z}{Z}Y-\dot Y\right]\rho=\frac{Z^2}{\rho^3}\ , \tag{2.6}$$

$\lambda_2=(\rho^2Y-\rho\dot\rho)/Z$, and

$$\lambda_3=\frac{1}{\rho^2}\left[1+\left[\frac{\rho^2Y-\rho\dot\rho}{Z}\right]^2\right].$$

The invariant can be written in the form

$$I=\frac{1}{2}\left\{\rho^2p^2+2\left[\frac{\rho^2Y-\rho\dot\rho}{Z}\right]pq+\frac{1}{\rho^2}\left[1+\left[\frac{\rho^2Y-\rho\dot\rho}{Z}\right]^2\right]q^2+\frac{\rho^2l^2}{q^2}\right\}. \tag{2.7}$$

The equation of motion for $q$ that follows from $H$ (1.1) is

 

$$\frac{d}{dt}\dot{q}-\frac{\dot{Z}}{Z}\dot{q}+\left[(XZ-Y^2)+\frac{\dot{Z}}{Z}Y-\dot{Y}\right]q=\frac{Z^2l^2}{q^3}\ . \tag{2.8}$$

The form of (2.6) and (2.8) is easily recognized as an Ermakov pair with the Ermakov invariant (2.7). The system (2.6)–(2.8) is an Ermakov system.

## III. EXACT SOLUTION OF THE CLASSICAL ERMAKOV SYSTEM

The existence of the invariant $I(p,q,\vec{\lambda}(t))$ $[\vec{\lambda}=(\lambda_1,\lambda_2,\lambda_3)]$ implies that, in principle, through a time-dependent canonical transformation we can choose $I$ to be the new "momentum." This, in fact, is achieved by the time-dependent generating function of the canonical transformation for $q,p$ to $\theta,I$ [20,26]:

$$S(q,I,\vec{\lambda}(t))=\int^q dq'p(q',I,\vec{\lambda}(t)),\ \ p=\frac{\partial S}{\partial q},\ \ \theta=\frac{\partial S}{\partial I}\ . \tag{3.1}$$

Following [25] we write the rate of change of angle of a phase point as the sum of contributions from its motion in phase space and from the changing coordinates $(I,\theta)$:

$$\dot{\theta}=\frac{\partial\mathcal{H}(I,\theta,\vec{\lambda})}{\partial I}+\frac{\partial\theta}{\partial\vec{\lambda}}\cdot\dot{\vec{\lambda}}\ , \tag{3.2}$$

where

$$\mathcal{H}(I,\theta,t)=H(q(I,\theta,\vec{\lambda}(t)),p(I,\theta,\vec{\lambda}(t)),t)$$

and $\partial\theta/\partial\vec{\lambda}$ is the rate at which the angle at fixed $q$ and $p$ changes with the parameters. Integrating (3.2) we obtain $\Delta\theta$, which does not depend on $\theta$; however, each term in the sum does depend on $\theta$. These dependences can be eliminated by averaging over each contour of constant action. We note that this allows (3.2) to be reinterpreted [26] as a Hamilton equation in action angle variables: the changing $\vec{\lambda}$ introduces a time dependence which contributes to the transformed Hamiltonian a term $\partial S/\partial t$, whose I derivative can be shown to equal the extra term $(\partial\theta/\partial\vec{\lambda})\cdot\dot{\vec{\lambda}}$ in (3.2). Thus we obtain

$$\Delta\theta=\int_0^t\left\langle\frac{\partial\mathcal{H}}{\partial I}\right\rangle dt'+\frac{\partial}{\partial I}\int_0^t\left\langle\frac{\partial S}{\partial t'}(I,\theta,\vec{\lambda}(t'))\right\rangle dt'\ , \tag{3.3}$$

where $\langle\ \rangle$ denotes the average over $\theta$ at fixed $\vec{\lambda}$. The first term is the dynamical angle $\Delta\theta^d$ and the second is the classical geometric angle $\Delta\theta^g$.

Expressing $p$ in terms of $I$, $q$, and $\vec{\lambda}$, we obtain

$$p=\frac{1}{\rho^2}\frac{(-q^4+2I\rho^2q^2-\rho^4l^2)^{1/2}}{q}-\frac{Y-\dot{\rho}\rho^{-1}}{Z}q \tag{3.4}$$

and evaluating the indefinite integral of Eq. (3.1) we find

$$S(q,I,\vec{\lambda}(t))=-\frac{1}{2}\left[\frac{Y-\dot{\rho}\rho^{-1}}{Z}\right]q^2-I\tan^{-1}\left[\frac{I\rho^2-q^2}{(-q^4+2I\rho^2q^2-\rho^4l^2)^{1/2}}\right]$$
$$-l\tan^{-1}\left[\frac{Iq^2-l^2\rho^2}{l(-q^4+2I\rho^2q^2-\rho^4l^2)^{1/2}}\right]+\frac{(-q^4+2I\rho^2q^2-\rho^4l^2)^{1/2}}{2\rho^2}+\text{const}\ , \tag{3.5}$$

which gives

$$\theta=\frac{\partial S}{\partial I}=-\frac{1}{2}\tan^{-1}\left[\frac{I\rho^2-q^2}{(-q^4+2I\rho^2q^2-\rho^4l^2)^{1/2}}\right]\ . \tag{3.6}$$

From Eqs. (3.4) and (3.6), it follows that

$$q=\rho^2(\sqrt{I^2-l^2}\sin2\theta+I)^{1/2} \tag{3.7}$$

and

$$p=\frac{1}{\rho(\sqrt{I^2-l^2}\sin2\theta+I)^{1/2}}\times\left\{\sqrt{I^2-l^2}\left[\cos2\theta-\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\sin2\theta\right]-\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]I\right\}\ . \tag{3.8}$$

From (3.3), the dynamical angle shift is

$$\Delta\theta^d=\int_0^t\left\langle\frac{\partial\mathcal{H}}{\partial I}\right\rangle dt'=\int_0^t\left\{\frac{Z}{\rho^2}+\frac{1}{2}\left[\frac{d}{dt'}\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]-2\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right]\right\}dt'\ , \tag{3.9}$$

the geometrical angle shift is

$$\Delta\theta^g=\frac{\partial}{\partial I}\int_0^t\left\langle\frac{\partial S}{\partial t'}(I,\theta,\vec{\lambda}(t'))\right\rangle dt'=\int_0^t\left[-\frac{1}{2}\frac{d}{dt'}\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]+\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right]dt'\ , \tag{3.10}$$

and the total angle is

$$\Delta\theta=\int_0^t\frac{Z}{\rho^2}dt'\ . \tag{3.11}$$

Now we use the exact solution to discuss the nonadiabatic Hannay's angle. If the parameters $\vec{\lambda}(t)$ are periodic functions of time with the same period $T$, i.e., $\vec{\lambda}(t+T)=\vec{\lambda}(t)$, the cyclic evolution of the system is

$$\Delta\theta^d=\int_0^T\left[\frac{Z}{\rho^2}-\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right]dt \tag{3.12}$$

and

$$\Delta\theta^g=\oint_C\left[\frac{Y\rho-\dot{\rho}}{Z}\right]d\rho \ . \tag{3.13}$$

Equation (3.13) is the geometric angle (nonadiabatic Hannay angle [20]) in a cyclic evolution over a circuit $C$ in parameter space, irrespective of whether the evolution is adiabatic or not.

## IV. EXACT SOLUTION OF THE ERMAKOV SYSTEM: QUANTUM POINT OF VIEW

In the quantum theory of (1.1), $q$ and $p$ become quantum-mechanical operators $p=-\iota\hbar\partial/\partial q$; the auxiliary function $\rho$ remains a $c$ number. The invariant $I$, (2.7), is a constant Hermitian operator

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{1}{\iota\hbar}[I,H]=0 \ . \tag{4.1}$$

Lewis [2] and Lewis and Riesenfeld [9] first used constant operators to solve time-dependent quantum-mechanical problems.

$I$ satisfying (4.1) has constant eigenvalues, which we write as $\lambda_n$:

$$I\psi_n(q,t)=\lambda_n\psi_n(q,t) \ . \tag{4.2}$$

Here, $\psi_n(q,t)$ denote eigenfunctions of $I$ which will, in general, be time dependent. Lewis and Riesenfeld [9] showed that the general solution to the Schrödinger equation for (1.1),

$$\frac{1}{2}\left[-\hbar^2 Z\frac{\partial^2}{\partial q^2}-\iota\hbar Y-2\iota\hbar Yq\frac{\partial}{\partial q}+Xq^2+\frac{Zl^2}{q^2}\right]\psi=\iota\hbar\frac{\partial\psi}{\partial t} \ , \tag{4.3}$$

can be written in the form

$$\psi(q,t)=\sum_n C_n e^{\iota\alpha_n(t)}\psi_n(q,t) \ , \tag{4.4}$$

where $C_n$ are constants, $\psi_n(q,t)$ are the eigenfunctions of $I$ defined by (4.2), and the phase functions $\alpha_n(t)$ are found from the equation

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle\psi_n\left|\iota\hbar\frac{\partial}{\partial t}-H\right|\psi_n\right\rangle \ , \tag{4.5}$$

where the states $\psi_n(q,t)$ are orthonormal, $\langle\psi_n|\psi_{n}\rangle=\delta_{nn'}$. For simplicity we assume $I$ to have a discrete spectrum.

The key point of our analysis is to perform the unitary transformation

$$\psi_n'(q,t)=\exp\left[\frac{\iota}{\hbar}\left[\frac{Y\rho-\dot{\rho}}{Z\rho}\right]\frac{q^2}{2}\right]\psi_n(q,t)=U\psi_n(q,t) \ . \tag{4.6}$$

The operator $I$ changes into $I'$:

$$I'=UIU^\dagger \ . \tag{4.7}$$

The operator eigenvalue equation (4.2) is mapped into

$$I'\psi_n'(q,t)=\lambda_n\psi_n'(q,t) \ , \tag{4.8}$$

where we find by straightforward calculation that

$$I'=\frac{1}{2}\left[-\hbar^2\rho^2\frac{\partial^2}{\partial q^2}+\frac{q^2}{\rho^2}+\frac{\rho^2 l^2}{q^2}\right] \ . \tag{4.9}$$

If we now define a new independent variable $\sigma=q/\rho$, we can write the eigenvalue equation in the form

$$\frac{1}{2}\left[-\hbar^2\frac{\partial^2}{\partial\sigma^2}+\sigma^2+\frac{l^2}{\sigma^2}\right]\phi_n(\sigma)=\lambda_n\phi_n(\sigma) \tag{4.10}$$

or $I'\phi_n(\sigma)=\lambda_n\phi_n(\sigma)$, where

$$\psi_n'(q,t)=\frac{1}{\rho^{1/2}}\phi_n(\sigma)=\frac{1}{\rho^{1/2}}\phi_n\left[\frac{q}{\rho}\right] \ . \tag{4.11}$$

The factor $1/\rho^{1/2}$ is introduced into (4.11) so that the normalization condition

$$\int\psi_n'^*(q,t)\psi_n'(q,t)dq=\int\phi_n^*(\sigma)\phi_n(\sigma)d\sigma=1 \tag{4.12}$$

holds. The important point is that the transformed eigenvalue problem (4.10) is an ordinary one-dimensional time-independent Schrödinger equation with potential $V(\sigma)=\frac{1}{2}[\sigma^2+(l/\sigma)^2]$.

As is easily verified, by use of the orthogonality relation

$$\int_0^\infty dx\, e^{-x}x^\alpha L_k^\alpha(x)L_{k'}^\alpha(x)=\delta_{kk'}\frac{\Gamma(\alpha+k+1)}{k!} \quad (\alpha>-1) \ , \tag{4.13}$$

the normalized solution $\psi_n'(q,t)$ can be written as

$$\psi_n'(q,t)=\frac{(-1)^k}{\hbar^{1/4}\rho^{1/2}}\left[\frac{k!}{\Gamma(k+c+1)}\right]^{1/2}\sigma^c e^{-\sigma^2/2}\times L_k^{c-1/2}\left[\frac{\sigma^2}{\hbar}\right]\quad\text{for } n=2k \ , \tag{4.14}$$

where the $L_k^2(\sigma^2)$'s denote generalized Laguerre polynomials satisfying

$$\left[\sigma\frac{d^2}{d\sigma^2}+(c+\tfrac{1}{2}-\sigma)\frac{d}{d\sigma}+k\right]L_k^{c-1/2}(\sigma^2)=0 \tag{4.15}$$

and the constant eigenvalue $\lambda_n$ is exactly given by $\lambda_n=(n+c+\frac{1}{2})\hbar$ where $\hbar^2c(c-1)=l^2$. The complete normalized state is thus given by

$$\psi_n(q,t)=\frac{1}{\rho^{1/2}}\exp\left[-\iota\left[\frac{Y\rho-\dot{\rho}}{Z\rho}\right]\frac{q^2}{2\hbar}\right]\phi_n\left[\frac{q}{\rho}\right]. \tag{4.16}$$

There remains the problem of finding the phases $\alpha_n(t)$ which satisfy

$$\hbar\dot{\alpha}_n(t)=\left\langle\psi_n\left|\iota\hbar\frac{\partial}{\partial t}-H\right|\psi_n\right\rangle, \tag{4.17}$$

where

$$\dot{\alpha}_n^g(t)=\left\langle\psi_n\left|\iota\frac{\partial}{\partial t}\right|\psi_n\right\rangle \tag{4.18}$$

is the geometrical phase and

$$\dot{\alpha}_n^d(t)=-\left\langle\psi_n\left|\frac{H}{\hbar}\right|\psi_n\right\rangle \tag{4.19}$$

is the dynamical phase. Carrying out the unitary transformation $U$ the right-hand side of Eq. (4.17) becomes

$$\hbar\dot{\alpha}_n(t)=\left\langle\psi_n'\left|\iota\hbar\frac{\partial}{\partial t}+\iota\hbar\frac{\dot{\rho}}{\rho}q\frac{\partial}{\partial q}+\iota\hbar\frac{\dot{\rho}}{2\rho}-\frac{Z}{\rho^2}I'\right|\psi_n'\right\rangle, \tag{4.20}$$

where we have used the auxiliary Eq. (2.6) to eliminate $X(t)$ from $H$. Next substituting $\psi_n'(q,t)=(1/\rho^{1/2})\phi_n(q/\rho)$ into (4.20) we find

$$\hbar\dot{\alpha}_n(t)=\left\langle\phi_n\left|-\frac{Z}{\rho^2}I'\right|\phi_n\right\rangle. \tag{4.21}$$

Using (4.10) and the normalization of $\phi_n$ we have

$$\alpha_n(t)=-(n+c+\tfrac{1}{2})\int_0^t\frac{Z(t')}{\rho^2}dt'. \tag{4.22}$$

Substituting for $\phi_n$ in the expression for $\dot{\alpha}_n^g$ (4.19), the geometrical phase is hence found to be

$$\alpha_n^g=-(n+c+\tfrac{1}{2})\int_0^t\left[-\frac{1}{2}\frac{d}{dt'}\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]+\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right]dt' \tag{4.23}$$

and the dynamical phase is

$$\alpha_n^d=-(n+c+\tfrac{1}{2})\int_0^t\left\{\frac{Z}{\rho^2}+\frac{1}{2}\left[\frac{d}{dt'}\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]-2\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right]\right\}dt'. \tag{4.24}$$

Finally the exact solution of the Schrödinger equation (4.3) is

$$\psi(q,t)=\sum_n C_n\exp\left[\iota\frac{-\lambda_n}{\hbar}\int_0^t\frac{Z(t')}{\rho^2}dt'\right]\psi_n(q,t). \tag{4.25}$$

Now we use the exact solution to discuss the nonadiabatic Berry's phase [22]. If the parameters $\vec{\lambda}(t)$ are periodic functions of time with the same period $T$, i.e., $\vec{\lambda}(t+T)=\vec{\lambda}(t)$, the cyclic evolution of the system is

$$\alpha_n^d=-(n+c+\tfrac{1}{2})\int_0^t\left[\frac{Z}{\rho^2}-\left[\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right]dt' \tag{4.26}$$

and

$$\alpha_n^g=-(n+c+\tfrac{1}{2})\oint_C\left[\frac{Y\rho-\dot{\rho}}{Z}\right]d\rho. \tag{4.27}$$

## V. CONCLUSION

We have proved in this paper that the solution to the classical Ermakov system reduces to solving one of the equations (2.6) or (2.8) in terms of solutions of the other equation. These results are sometimes referred to as the nonlinear superposition law in the literature [27].

In the quantum case, the solution to the Ermakov Schrödinger equation (4.3) reduces to solving the one-dimensional Schrodinger equation (4.10). In fact, Eq. (4.25) together with (4.16) and (4.23) is a quantum-mechanical superposition law.

We conclude with the adiabatic limit of our, so far, exact treatment. Let us assume that the external parameters $(X,Y,Z)$ perform an adiabatic excursion during the time $T$ in the parameter space so that $(X,Y,Z)(0)=(X,Y,Z)(T)$. In the adiabatic limit, the $\dot{\rho}$ term in the auxiliary equation (2.6) may be ignored; then we obtain

$$\frac{Z}{\rho^2}=w_D\left[1-\frac{Z}{w_D^2}\frac{d}{dt}\left[\frac{Y}{Z}\right]\right]^{1/2},$$

where $w_D^2=(XZ-Y^2)$, $XZ>Y^2$. Furthermore, expanding with respect to

$$\frac{Z}{w_D^2}\frac{d}{dt}\left[\frac{Y}{Z}\right]\ll 1,$$

we obtain

$$\frac{Z}{\rho^2}=w_D-\frac{Z}{2w_D}\frac{d}{dt}\left[\frac{Y}{Z}\right].$$

When this adiabatic expression is substituted into (3.11) or (4.22), we may obtain the total angle (or phase) accumulated in a cyclic adiabatic evolution:

$$\Delta\theta=\int_0^T dt\, w_D-\oint_C d\vec{R}\frac{Z}{2w_D}\nabla_{\vec{R}}\left[\frac{Y}{Z}\right],\quad \vec{R}=(X,Y,Z),$$

and

$$\alpha_n(t)=-(n+c+\tfrac{1}{2})\hbar\int_0^T dt\, w_D+(n+c+\tfrac{1}{2})\hbar\oint_C d\vec{R}\frac{Z}{2w_D}\nabla_{\vec{R}}\left[\frac{Y}{Z}\right],$$

where the first term is the dynamical angle (or phase) and the second is the geometrical Hannay's angle (or Berry's phase) [24,25].

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