Micromaser with stationary non-Poissonian pumping

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A method is presented for theoretically investigating the properties of a one-atom micromaser with stationary non-Poissonian pumping. The method is based on treating the statistics of the arrival times of the individual pump atoms with the help of the theory of stochastic point processes. Considerable simplification is achieved by assuming the pump statistics to be described by a stationary renewal process. Thus the influence of super- as well as sub-Poissonian pumping with different strengths of correlation between the pump atoms and different correlation decay times may be studied quantitatively. The level-selective statistics of the atoms leaving the cavity is investigated as well as the photon statistics of the cavity field. Moreover, the relation to the other models used in the literature for describing the micromaser pump statistics is discussed. It is found that for sub- (super-)Poissonian pumping, the stationary expectation value of the cavity photon number (which corresponds to its time-averaged value) is larger (smaller) than the conditioned mean photon number that would result from averaging over the number of photons present in the cavity at the time instants immediately before the injection of the individual pump atoms. For Poissonian pumping, both quantities are shown to be equal. The relative standard deviation of the stationary cavity photon number is decreased (increased) for sub- (super-) Poissonian pumping, in comparison to the corresponding values that would result from Poissonian pumping. Moreover, it turns out that for sub- (super-)Poissonian pumping, the normalized coincidence probability density for the detection of the outgoing deexcited atoms is smaller (larger) than the normalized cavity field intensity correlation function at zero time delay. The difference decreases with increasing lifetime of the pump-atom correlations. Finally, the level-selective delayed coincidence probability densities of the outgoing atoms and their waiting-time distributions are found to be affected greatly by the correlation strength as well as the correlation decay time of the incoming pump atoms.

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I. INTRODUCTION

In a one-atom micromaser [1,2] a beam of Rydberg atoms interacts with a single-mode radiation field in a microwave cavity in such a way that with extremely high probability, at most one atom at a time is present in the cavity. The atoms injected into the cavity are excited to the upper level of the microwave transition. Owing to the atom-field interaction, nonclassical properties of the radiation field such as sub-Poissonian photon statistics may occur [1,2]. In order to get information about the microwave field, the energy states of the atoms leaving the cavity are measured in microwave experiments and the level-selective statistics of those outgoing atoms being in the upper or lower energy states, respectively, is determined [3]. In the usual case the injected atoms that pump the micromaser obey Poissonian injection statistics. For a micromaser with Poissonian pumping the relation between the statistical properties of the cavity field and the level-selective statistics of the atoms leaving the cavity has been extensively studied in the past two years [4–10]. However, one may think of other kinds of pump statistics and investigate the problem as to what extent the statistical properties of the pump beam influence the cavity field as well as the level-selective statistics of the outgoing atoms. For this purpose we study the theory of a micromaser with stationary non-Poissonian pumping.

In the literature up to now only the influence of the pump statistics on the cavity field has been considered. The commonly used model is sometimes called pearl-string model because it is based on assuming an incoming beam of ground-state atoms that are equidistant like the pearls on a string. The atoms are supposed to be excited to the upper level of the microwave transition with certain efficiency p ranging from $p \ll 1$ (nearly Poissonian pumping) to $p=1$ (regular pumping) [11]. In order to find the quasi-steady-state photon-number population, a numerical step-by-step microscopic treatment has been adopted that uses a mapping condition for the density matrix of the field [12]. As has been pointed out in Ref. [13] the result depends on the relative location of the time, at which the density matrix is defined, within the interval of periodicity inherent in the pearl-string model [13] since this model is not homogeneous in time, i.e., it does not describe a pumping process that is stationary in the strict sense. Quite recently the pearl-string model has been extended to describe super-Poissonian pumping as well by allowing for the case that more than one atom may arrive simultaneously at the equidistant possible atomic arrival times [14]. Furthermore, transient field properties have also been investigated for the pearl-string model in the sense of cyclic mapping [15].

Another method that has been adopted in order to describe the effect of the pump-atom statistics on lasers and

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micromasers uses the Langevin-equation approach [16]. In contrast to the pearl-string model, this approach refers to strictly stationary pumping. Finally, a stationary quantum-field model of the injected atomic beam in the micromaser has been developed and applied to treat regular pumping as well as a special kind of super-Poissonian pumping [17]. For the latter the steady-state photon statistics of the cavity field has been calculated [17].

In this paper we present a different model for theoretically investigating the properties of a micromaser with stationary non-Poissonian pumping. Our method is based on treating the statistics of the arrival times of the individual pump atoms with the help of the theory of stochastic point processes [18–20], i.e., stochastic processes whose realization is a series of point events occurring in a continuous time. The procedure is somewhat related to the description of sub-Poissonian optical pumping in lasers given in Ref. [21]. To characterize the injection statistics we use the exclusive probability densities $Q_k^{\mathrm{in}}(t_1,t_2,\ldots,t_k)$ $(k=1,2,\ldots)$ for the injection of k pump atoms at the time instants $t_1,t_2,\ldots,t_k$ with no other pump atoms injected in the intervals in between. The whole set of these exclusive probability densities yields a complete description of the pump statistics. In particular it implicitly includes all information about the decay times of pump-atom correlations. In fact, as is well known from the theory of stochastic point processes, an alternative complete characterization of the injection statistics could be provided by the whole set of all coincidence or joint probability densities $P_k^{\mathrm{in}}(t_1,t_2,\ldots,t_k)$ $(k=1,2,\ldots)$ for the injection of k pump atoms at k given instants of time, notwithstanding the possible injection of other pump atoms at other times in between. One set of probability densities being known, the other can be calculated in principle [19].

It is our aim to investigate the photon statistics of the field in the micromaser cavity as well as the level-selective statistics of the outgoing atoms. Starting from the exclusive pump-atom probability densities $Q_k^{\mathrm{in}}(t_1,t_2,\ldots,t_k)$ $(k=1,2,\ldots)$ we therefore derive an evolution equation for the usual unconditioned density operator ρ of the cavity field as well as for an injection-time conditioned field density operator ρ^c referring to the time instants immediately before the injection of the pump atoms. The latter will prove to be the basis for calculating the statistics of the atoms leaving the cavity. It will turn out that for Poissonian pumping, the evolution equations for ρ and for ρ^c are identical and so are the stationary values of these operators. As a consequence, we shall arrive at the interesting result that in the stationary regime of a micromaser with Poissonian pumping, the quantum-mechanical photon-number expectation value is equal to the mean photon number that would be obtained by averaging over the number of photons present in the cavity at the time instants immediately before the transits of the individual pump atoms. For other kinds of pump statistics or for the time instants immediately after the transit of the pump atoms, e.g., this equality does not hold.

For arbitrary pump statistics it proves to be very difficult or even impossible to evaluate the resulting density-matrix equations. Therefore we restrict a quantitative treatment to the case that the statistics of the incoming pump atoms belongs to the class of stationary renewal processes [20]. This yields a considerable simplification of the equations, but, nevertheless, allows us to investigate super- as well as sub-Poissonian pump statistics with different strengths of correlation or anticorrelation, respectively, between the pump atoms and different correlation decay times.

The paper is organized as follows. In Sec. II we introduce the method for describing the pump statistics. The influence of the latter on the properties of the cavity field is investigated in Sec. III by considering the evolution of the field density operator and calculating the steady-state photon statistics. Section IV deals with the level-selective statistics of the atoms leaving the cavity. In particular, we study the dependence of the delayed coincidence probability density for the outgoing deexcited atoms of the strength and of the decay time of the pump-atom correlations and calculate their waiting-time distribution. Finally, in Sec. V we clarify the relation between our approach and the other models used in the literature to describe the pump statistics of a micromaser.

II. DESCRIPTION OF THE PUMP STATISTICS

Since we want to use the theory of stochastic point processes in order to describe the injection statistics of the pump atoms, we start with a brief recapitulation of the properties of these processes (see, e.g., [19,20]), which are fundamental for our treatment of the micromaser. A stochastic point process that is stationary can be completely described by the whole set of exclusive conditional probability densities $(k=1,2,\ldots)$

$$Q_k^c(t_1,t_2,\ldots,t_k|t_0)=Q_k^c(t_1-t_0,\ldots,t_k-t_0|0)\ , \tag{2.1}$$

where $t_0\le t_1\le\cdots\le t_k$. When applied to the incoming beam of pump atoms, the quantities $Q_k^c(t_1,\ldots,t_k|t_0)dt_1dt_2\cdots dt_k$ yield the probabilities that the next k pump atoms arrive in the infinitesimal time intervals (t_i,t_i+dt_i) $(i=1,\ldots,k)$ provided the first pump atom had arrived in the interval (t_0,t_0+dt_0). Here the possibilities that other pump atoms might have arrived in the periods between the time instants $t_0,t_1,\ldots,t_k$ are excluded. The exclusive conditional probability density

$$Q_1^c(\tau|0)\equiv f(\tau) \tag{2.2}$$

determines the waiting-time distribution between consecutive incoming pump atoms and is normalized according to

$$\int_0^\infty d\tau\, f(\tau)=1\ . \tag{2.3}$$

Another way of completely describing the stationary statistics of the incoming pump atoms consists in using the set of conditional coincidence probability densities

$$P_k^c(t_1,t_2,\ldots,t_k|t_0)=P_k^c(t_1-t_0,\ldots,t_k-t_0|0)\ , \tag{2.4}$$

which refer to the case that, provided an incoming atom had arrived at t_0, other atoms will arrive at the time instants $t_1, \ldots, t_k$, notwithstanding that still more pump atoms may be present in the intervals in between these time instants. The stationary injection rate r is given by

$$r = \lim_{\tau \to \infty} P_1^c(\tau|0) \tag{2.5}$$

and the multitime coincidence and exclusive probability densities of the injected pump atoms may be written as

$$P_k^{\text{in}}(t_1, t_2, \ldots, t_k) = r P_{k-1}^c(t_2, \ldots, t_k | t_1) \tag{2.6}$$

and

$$Q_k^{\text{in}}(t_1, t_2, \ldots, t_k) = r Q_{k-1}^c(t_2, \ldots, t_k | t_1) , \tag{2.7}$$

respectively ($k = 2, 3, \ldots$). Provided one set of probability densities P_k^{in} or Q_k^{in} is known, the other can be calculated. In the following we need the relation [19]

$$P_1^c(\tau|0) = Q_1^c(\tau|0) + \sum_{k=1}^{\infty} \int_0^{\tau} dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 Q_{k+1}^c(t_1, t_2, \ldots, t_k, \tau|0) . \tag{2.8}$$

The decay time of the pump-atom correlations is characterized by the time dependence of the normalized delayed coincidence probability density

$$w_2^{\text{in}}(\tau) = \frac{P_2^{\text{in}}(0,\tau)}{r^2} . \tag{2.9}$$

For a sufficiently large time delay τ these correlations vanish, which is expressed by

$$\lim_{\tau \to \infty} w_2^{\text{in}}(\tau) = \frac{1}{r} \lim_{\tau \to \infty} P_1^c(\tau|0) = 1 . \tag{2.10}$$

A special class of stationary stochastic point processes that is most easily accessible to an analytic treatment is formed by the stationary renewal processes [20] having independent and identically distributed interevent times. For these processes the probability densities factorize according to

$$Q_{k+1}^{\text{in}}(t_0, t_1, \ldots, t_k) = r \prod_{i=1}^{k} f(t_i - t_{i-1}) \tag{2.11}$$

and

$$P_{k+1}^{\text{in}}(t_0, t_1, \ldots, t_k) = r^{k+1} \prod_{i=1}^{k} w_2^{\text{in}}(t_i - t_{i-1}) \tag{2.12}$$

where $k = 1, 2, \ldots$. With $\tilde{f}(z)$ and $\tilde{w}_2^{\text{in}}(z)$ denoting the Laplace transforms of the functions $f(\tau)$ and $w_2^{\text{in}}(\tau)$, respectively, the relation

$$\tilde{w}_2^{\text{in}}(z) = \frac{r \tilde{f}(z)}{1 - \tilde{f}(z)} \tag{2.13}$$

holds in the case of a renewal process as follows from Eqs. (2.8) and (2.11) with the help of the convolution theorem. Moreover, in this case the rate r is connected with the mean interevent time $\langle \tau \rangle$ by the relation

$$r^{-1} = \langle \tau \rangle \equiv \int_0^{\infty} d\tau f(\tau) \tau , \tag{2.14}$$

which can be obtained from Eqs. (2.3), (2.5), (2.8), and (2.11).

For later use we make a specific ansatz for the waiting-time distribution $f(\tau)$ by writing

$$f(\tau) = \frac{\lambda_1 \lambda_2}{\lambda_2 + \alpha \lambda_1} (e^{-\lambda_1 \tau} + \alpha e^{-\lambda_2 \tau}) , \tag{2.15}$$

where α, λ_1, and λ_2 are real parameters with $\alpha \geq -1$ and $\lambda_1, \lambda_2 \geq 0$. Assuming the stochastic process to be a stationary renewal process and using Eq. (2.13), we find from this ansatz that the pump-atom correlations are described by a coincidence probability density that takes the form

$$P_2^{\text{in}}(0,\tau) = r^2 w_2^{\text{in}}(\tau) = r^2 (1 + C e^{-\Gamma \tau}) , \tag{2.16}$$

where the injection rate r, the correlation strength C, and the correlation decay rate Γ are given by

$$r = \frac{\lambda_1 \lambda_2 (\lambda_2 + \alpha \lambda_1)}{\lambda_2^2 + \alpha \lambda_1^2} , \tag{2.17}$$

$$C = \alpha \frac{(\lambda_2 - \lambda_1)^2}{(\lambda_2 + \alpha \lambda_1)^2} \geq -1 , \tag{2.18}$$

and

$$\Gamma = \frac{\lambda_2^2 + \alpha \lambda_1^2}{\lambda_2 + \alpha \lambda_1} . \tag{2.19}$$

When $\alpha = C = 0$ Eq. (2.15) or (2.16) describes Poissonian pump statistics, whereas bunching or antibunching of the pump atoms is characterized by $C > 0$ or $C < 0$, respectively.

Starting from Eq. (2.16) for the coincidence probability density of the incoming pump atoms, their waiting-time distribution may be expressed by the parameters r, C, and Γ provided that the injection statistics belongs to the class of stationary renewal processes. Using Eq. (2.13) we find that for antibunched injection with $-1 \leq C < 0$, the expression (2.16) is only compatible with the properties of a renewal process when the inequality

$$\Gamma \geq r(1 + \sqrt{|C|})^2 \quad (-1 \leq C < 0) \tag{2.20}$$

is fulfilled, since otherwise a real-valued positive waiting-time distribution would not result. In contrast to this, for $C \geq 0$ this compatibility is always ensured. The parameters λ_1, λ_2, and α can be found from the equations

$$\lambda_{1/2} = \tfrac{1}{2}[\Gamma + r(1+C)] \pm \tfrac{1}{2}\sqrt{\Gamma^2 + 2r\Gamma(C-1) + r^2(1+C)^2} \tag{2.21}$$

and

$$\alpha=-\frac{\Gamma-\lambda_2(1+C)}{\Gamma-\lambda_1(1+C)} . \tag{2.22}$$

With the help of these results we shall be able to study the micromaser properties for bunched as well as antibunched injection of pump atoms with given correlation strength C and given correlation decay rate Γ, supposing that the injection process is a renewal process and that the inequality (2.20) is valid in the case of antibunching. The strength of the sub- or super-Poissonian character of the pumping process depends on the length of the corresponding counting interval to which it is referred (see Sec. V).

It is interesting to note that pumping according to a stationary renewal process exhibiting complete antibunching ($C=-1$) can be physically realized, at least in principle. Actually, as we have shown in a recent paper [8], the statistics of the deexcited atoms leaving a micromaser with Poissonian pumping that is in the one-photon trapped state is described by such a renewal process where $r=\kappa R\sin^2(q\pi/\sqrt{2})$ with $q=1,2,\ldots$. Here κ and R denote the cavity damping rate and the Poissonian pump rate, respectively. The outgoing deexcited atoms could be used to pump a second micromaser that works on an adjacent lower microwave transition.

III. PROPERTIES OF THE CAVITY FIELD

A. An injection-time conditioned density operator

1. The general evolution equation

Throughout this paper we stick to the conventional assumptions that the damping of the micromaser field can be neglected over the transit time t_{int} of the atoms and that the atoms interact with the field one after the other. Owing to the transit of a single pump atom, the field density operator ρ changes according to

$$\rho(t+t_{\text{int}})=M\rho(t)=(D+E)\rho(t) . \tag{3.1}$$

Here the superoperator M has been formally divided into two parts D and E accounting for the possibilities that the pump atom gets deexcited into the lower level or remains in the upper level, respectively, during the transit. For our purposes we are interested in the diagonal elements $\rho_{nn}\equiv p_n$ of the density matrix in the photon-number representation. Assuming the initial field to be diagonal in this representation, the action of D and E on the field density operator is expressed by the equations [22]

$$(D\rho)_{nn}=\beta_n p_{n-1} \tag{3.2}$$

and

$$(E\rho)_{nn}=(1-\beta_{n+1})p_n . \tag{3.3}$$

Here the abbreviation

$$\beta_n=\sin^2(gt_{\text{int}}\sqrt{n}\,) \tag{3.4}$$

has been introduced with g denoting the atom-field coupling constant. The cavity field is damped over the time between the passage of two consecutive atoms resulting in a change of the field density operator according to [22]

$$\rho(t+\Delta t)=e^{L\Delta t}\rho(t) . \tag{3.5}$$

In the photon-number representation the action of the superoperator L yields the diagonal elements [22]

$$\begin{aligned}(L\rho)_{nm}={}&\gamma(n_b+1)[(n+1)p_{n+1}-np_n]\\&-\gamma n_b[(n+1)p_n-np_{n-1}] ,\end{aligned} \tag{3.6}$$

with γ and n_b being the cavity damping rate and the thermal photon number, respectively.

The change of the cavity field, from an initial time $t=0$ to a final time $t=\tau$, is determined by the combined contributions from the individual atoms that may enter the cavity at certain time instants t_i in the interval $[0,\tau]$ and from cavity damping in between these time instants. (Note that the interaction time t_{int} is assumed to be negligibly short.) For the moment we suppose that an atom has transvered the cavity immediately prior to the initial time $t=0$, whereas the final time $t=\tau$ is located immediately before the injection of another atom (see Fig. 1). With ρ^0 denoting the field density operator at this initial time, we find that the field at the specific final time τ is described by a density operator $\rho^c(\tau)$, which we may call injection-time conditioned and which obeys the evolution equation

$$\rho^c(\tau)=\frac{r^2}{P_2^{\text{in}}(0,\tau)}U_c(\tau)\rho^0=\frac{1}{w_2^{\text{in}}(\tau)}U_c(\tau)\rho^0 , \tag{3.7}$$

where

$$U_c(\tau)=\frac{1}{r^2}\left\{Q_2^{\text{in}}(0,\tau)e^{L\tau}+\sum_{k=1}^{\infty}\int_0^{\tau}dt_k\int_0^{t_k}dt_{k-1}\cdots\int_0^{t_2}dt_1 Q_{k+2}^{\text{in}}(0,t_1,\ldots,t_k,\tau)e^{L(\tau-t_k)}Me^{L(t_k-t_{k-1})}\cdots Me^{Lt_1}\right\} . \tag{3.8}$$

On the right-hand side of the above equation we took into account, with proper probability, all possibilities that exactly k atoms ($k=0,1,2,\ldots$) may have entered the cavity in the interval $[0,\tau]$, where (for $k\geq 1$) this has occurred at the time instants $t_1,t_2,\ldots,t_k$, which are randomly distributed according to the corresponding exclusive probability densities $Q_{k+2}^{\text{in}}(0,t_1,\ldots,t_k,\tau)$ of the injected pump atoms. Here the ordering of the noncommutative operators e^{Lt} and M results in a natural way from considering the alternating action of cavity damp-

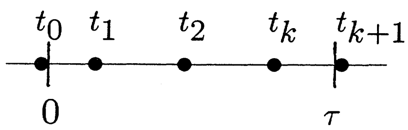

FIG. 1. Relative positions of the pump-atom injection times t_i, with respect to the considered time interval $[0,\tau]$.

ing and atomic transit. The coincidence probability density $P_2^{\rm in}(0,\tau)$ of having indeed two atoms injected at the initial time $t=0$ and the final time $t=\tau$ appears in the denominator of Eq. (3.7) in order to ensure the correct normalization. In fact, since the operators L and M are trace conserving, we find with the help of Eq. (2.8) that $\mathrm{Tr}[\rho^c(\tau)]=1$. When, after interaction with the field, the pump atoms are not selected with respect to their energy levels, the initial density operator ρ^0 can be written as

$$\rho^0=M\rho^c(0)\ , \tag{3.9}$$

with $\rho^c(0)$ referring to the time instant immediately before $t=0$.

For arbitrary pump statistics the evaluation of $\rho^c(\tau)$ with the help of Eqs. (3.7) and (3.8) is practically impossible since one would have to know all exclusive injection probability densities up to an infinite high order and perform an infinite series of integrations. However, the task is greatly facilitated when the pump statistics is described by a renewal process. In this case, because of the factorization condition (2.11), the convolution theorem can be applied to the Laplace transform of Eq. (3.8), yielding the transformed equation

$$\tilde{U}_c(z)=\frac{1}{r}\sum_{k=0}^{\infty}\tilde{F}(z)[M\tilde{F}(z)]^k=\frac{1}{r}\tilde{F}(z)[1-M\tilde{F}(z)]^{-1}\ , \tag{3.10}$$

where the operator

$$\tilde{F}(z)=\int_0^\infty d\tau\, e^{-z\tau}f(\tau)e^{L\tau}=\tilde{f}(z-L) \tag{3.11}$$

has been introduced. Equation (3.10) is equivalent to

$$\tilde{U}_c(z)=\frac{1}{r}\sum_{k=0}^{\infty}[\tilde{F}(z)M]^k\tilde{F}(z)=\frac{1}{r}[1-\tilde{F}(z)M]^{-1}\tilde{F}(z)\ . \tag{3.12}$$

After multiplying both sides of the above expression from the left by $1-\tilde{F}(z)M$ and performing the inverse transformation, we obtain the implicit solution

$$U_c(\tau)=\frac{1}{r}f(\tau)e^{L\tau}+\int_0^\tau dt\, f(\tau-t)e^{L(\tau-t)}MU_c(t)\ . \tag{3.13}$$

Equation (3.13), together with Eqs. (3.7) and (2.13), determines the evolution of the injection-time conditioned field density operator ρ^c of a micromaser whose pump statistics is described by a renewal process characterized by the waiting-time distribution $f(t)$. Explicitly, the solution can be written with the help of Eqs. (3.10) and (3.12) as

$$U_c(\tau)=\frac{1}{r}\sum_{\nu}\mathrm{Res}\{\tilde{F}(z)[1-M\tilde{F}(z)]^{-1}\}_{z=z_\nu} \tag{3.14}$$

$$=\frac{1}{r}\sum_{\nu}\mathrm{Res}\{[1-\tilde{F}(z)M]^{-1}\tilde{F}(z)\}_{z=z_\nu}\ . \tag{3.15}$$

2. *The steady-state solution*

When τ is sufficiently large, the operator $\rho^c(\tau)$ takes on the stationary value

$$\bar{\rho}^c=\lim_{\tau\to\infty}\rho^c(\tau)\ . \tag{3.16}$$

Making use of the facts that $\mathrm{Res}\tilde{F}(z)|_{z=0}$ since $\lim_{\tau\to\infty}f(\tau)e^{L\tau}=0$ and taking into account Eqs. (3.7), (2.10), (3.15), and (3.12), we may write

$$\bar{\rho}^c=\frac{1}{r}\mathrm{Res}\{[1-\tilde{F}(z)M]^{-1}-1\}\tilde{F}(z)\rho^0|_{z=0}$$

$$=\frac{1}{r}\mathrm{Res}\left\{\tilde{F}(z)M\sum_{k=0}^{\infty}[\tilde{F}(z)M]^k\tilde{F}(z)\rho^0\right\}_{z=0}=\tilde{F}(0)M\bar{\rho}^c. \tag{3.17}$$

Because of Eq. (3.11) this expression is equivalent to the mapping condition

$$\bar{\rho}^c=\int_0^\infty dt\, f(t)e^{Lt}M\bar{\rho}^c\ , \tag{3.18}$$

which determines the steady-state operator $\bar{\rho}^c$ from the requirement that it is kept unchanged by the action of an atomic transit and a subsequent mean loss event. Obviously the above mapping condition is justified only for pumping according to a renewal process. It yields a specific stationary injection-time conditioned density operator $\bar{\rho}^c$ that, apart from the special case of Poissonian pumping, is different from the stationary unconditioned density operator $\bar{\rho}$, as will be shown in the next subsection. By using $\bar{\rho}^c$ one may calculate the mean values of the field variables that would be found in the stationary micromaser regime by averaging over the results of measurements that are performed always immediately before the transit of an atom. (If one were interested in the average field values referring to the time instants immediately after the transit of an atom one would have to use the operator $M\bar{\rho}^c$.) As it must be, we infer from Eq. (3.18) that the operator $\bar{\rho}^c$ does not depend on ρ^0. Therefore the explicit solution may be written with the help of Eqs. (3.7) and (3.14) as

$$\bar{\rho}^c=\bar{U}_c=\frac{1}{r}\tilde{F}(0)\mathrm{Res}[1-M\tilde{F}(z)]^{-1}|_{z=0}\ , \tag{3.19}$$

where ρ^0 has been replaced by the unity operator.

3. *Specialization to the specific pumping model*

For later application we now model the pump-atom waiting-time distribution $f(\tau)$ as the sum of two exponentially decaying functions according to Eq. (2.15). The problem of solving the integral operator equation (3.13) is then reduced to the solution of a system of two coupled linear differential operator equations. In fact, by differentiating Eq. (3.13) and taking into account Eq. (2.15), we find that

$$U_c(\tau)=U_1(\tau)+U_2(\tau)\ , \tag{3.20}$$

where

$$\dot{U}_1=LU_1+\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}M(U_1+U_2)-\lambda_1U_1\ , \tag{3.21}$$

$$\dot{U}_2=LU_2+\alpha\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}M(U_1+U_2)-\lambda_2U_2\ , \tag{3.22}$$

with

$$U_1(0)=\frac{\lambda_2^2+\alpha\lambda_1^2}{(\lambda_2+\alpha\lambda_1)^2}\underline{1} \tag{3.23}$$

and

$$U_2(0)=\alpha U_1(0)\ , \tag{3.24}$$

where $\underline{1}$ is the unitary operator. Introducing the abbreviations

$$q_n=(U_1)_{nn},\quad \hat{q}_n=(U_2)_{nn} \tag{3.25}$$

for the diagonal elements of the operators U_1 and U_2 in the photon-number representation and making use of Eqs. (3.1)–(3.3) and (3.6), we obtain from Eqs. (3.21) and (3.22)

$$\begin{aligned}\dot{q}_n=&\gamma(n_b+1)(n+1)q_{n+1}\\&-[\lambda_1+\gamma(n+n_b+2nn_b)]q_n+\gamma nn_bq_{n-1}\\&+\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}[(1-\beta_{n+1})(q_n+\hat{q}_n)\\&\qquad+\beta_n(q_{n-1}+\hat{q}_{n-1})]\ ,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\dot{\hat{q}}_n=&\gamma(n_b+1)(n+1)\hat{q}_{n+1}\\&-[\lambda_2+\gamma(n+n_b+2nn_b)]\hat{q}_n+\gamma nn_b\hat{q}_{n-1}\\&+\alpha\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}[(1-\beta_{n+1})(q_n+\hat{q}_n)\\&\qquad+\beta_n(q_{n-1}+\hat{q}_{n-1})]\end{aligned} \tag{3.27}$$

($n=0,1,2,\ldots$). According to Eqs. (3.7), (3.20), and (3.25), the injection-time conditioned photon-number probability is given by

$$p_n^c(\tau)\equiv\rho_{nn}^c(\tau)=\frac{1}{w_2^{\rm in}(\tau)}[q_n(\tau)+\hat{q}_n(\tau)]\rho_{nn}^0\ , \tag{3.28}$$

where we take into account that the initial density operator ρ^0 is supposed to be diagonal in the photon-number representation. When we assume that the photon-number probability distribution can be truncated at a certain maximum photon number $n_{\max}$, the time-dependent solutions $q_n(\tau)$ and $\hat{q}_n(\tau)$ are found by solving the system of $2(n_{\max}+1)$ coupled differential equations established by Eqs. (3.26) and (3.27) where the initial conditions follow from Eqs. (3.23)–(3.25).

In order to determine the steady-state injection-time conditioned density operator $\bar{\rho}^c\equiv\bar{U}_c$ one may either find the stationary solution of Eqs. (3.21) and (3.22) or solve the equation

$$\bar{\rho}^c=\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}\int_0^\infty dt(e^{-\lambda_1t}+\alpha e^{-\lambda_2t})e^{Lt}M\bar{\rho}^c\ , \tag{3.29}$$

which results from the mapping condition (3.18). Upon integration the above equation yields

$$\begin{aligned}(L-\lambda_1)&\bar{\rho}^c(M\bar{\rho}^c)^{-1}(L-\lambda_2)\\&=\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}[\lambda_2-L+\alpha(\lambda_1-L)]\ .\end{aligned} \tag{3.30}$$

For the special case that the pump atoms are completely antibunched, i.e., for $\alpha=-1$, the right-hand side is equal to a c number and we obtain the condition

$$(L-\lambda_2)(L-\lambda_1)\bar{\rho}^c=\lambda_1\lambda_2M\bar{\rho}^c\ , \tag{3.31}$$

which, in the photon-number representation, with the help of Eqs. (3.1)–(3.3) and (3.6) yields a three-term recursion relation for the injection-time conditioned photon-number probabilities $\bar{p}_n^c\equiv\bar{\rho}_{nn}^c$. When the thermal photon number n_b can be neglected, this relation takes the simple form

$$\lambda_1\lambda_2\beta_n\bar{p}_{n-1}^c=n\gamma(n\gamma+\lambda_1+\lambda_2)\bar{p}_n^c-n(n+1)\gamma^2\bar{p}_{n+1}^c\ , \tag{3.32}$$

valid for $\alpha=-1$ and $n_b=0$. For arbitrary values of α we resort to the decomposition

$$\bar{p}_n^c=[\bar{U}_c]_{nn}=\bar{q}_n+\bar{\hat{q}}_n\ , \tag{3.33}$$

with $\bar{q}_n$ and $\bar{\hat{q}}_n$ denoting the stationary solution of the system of differential equations given by Eqs. (3.26) and (3.27), which can be easily obtained numerically. With some effort it is even possible to derive a coupled set of recursion relations by setting the derivatives in Eqs. (3.26) and (3.27) equal to zero. When, for simplicity, the thermal photon number is assumed to be negligible, we obtain the relations

$$\begin{aligned}\beta_n\bar{p}_{n-1}^c=&\left[\frac{\lambda_2+\alpha\lambda_1}{\lambda_1\lambda_2}\frac{(n\gamma+\lambda_1)(n\gamma+\lambda_2)}{n\gamma(1+\alpha)+\lambda_2+\alpha\lambda_1}\right.\\&\left.-(1-\beta_{n+1})\right]\bar{p}_n^c+(n+1)\gamma\frac{\lambda_2+\alpha\lambda_1}{\lambda_1\lambda_2}\\&\times\frac{(\lambda_1-\lambda_2)\bar{q}_{n+1}-(n\gamma+\lambda_1)\bar{p}_{n+1}^c}{n\gamma(1+\alpha)+\lambda_2+\alpha\lambda_1}\end{aligned} \tag{3.34}$$

and

$$\bar{q}_n=\frac{(n\gamma+\lambda_2)\bar{p}_n^c+(n+1)\gamma[(1+\alpha)\bar{q}_{n+1}-\bar{p}_{n+1}^c]}{n\gamma(1+\alpha)+\lambda_2+\alpha\lambda_1}\ . \tag{3.35}$$

On alternately applying Eqs. (3.35) and (3.34) the steady-state distribution $\bar{p}_n^c$ can be determined by starting from the assumption that $\bar{p}_{n_{\max}+1}^c=\bar{q}_{n_{\max}+1}=0$ for a sufficiently large value of $n_{\max}$ and by setting $\bar{p}_{n_{\max}}^c=C_0$, where the constant C_0 has to be determined at the end of the calculation from the normalization condition $\sum_n \bar{p}_n^c = 1$. We note that for $\alpha=0$, i.e., for a micromaser with Poissonian pumping, Eq. (3.35) is equivalent to the equality $\bar{q}_n = \bar{p}_n^c$, which, after substitution into Eq. (3.34), yields the recursion relation $\lambda_1 \beta_n \bar{p}_{n-1}^c = n\gamma \bar{p}_n^c$.

B. The unconditioned density operator

Let us now consider the evolution of the cavity-field density operator ρ from an initial time $t=0$ to a final time $t=\tau$, which are both located arbitrarily with respect to the injection times of the pump atoms. In reference to Fig. 1, this means that the injection time t_{k+1} is now located anywhere between $t=\tau$ and ∞, whereas the time instant t_0 lies anywhere between $t=-\infty$ and 0. Following the same arguments that led to Eqs. (3.7) and (3.8), we find that

$$\rho(\tau) = U(\tau)\rho(0) \,, \tag{3.36}$$

where

$$U(\tau) = \int_\tau^\infty dt_1 \int_{-\infty}^0 dt_0 Q_2^{\mathrm{in}}(t_0,t_1) e^{L\tau} + \sum_{k=1}^{\infty} \int_\tau^\infty dt_{k+1} \int_0^\tau dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 \int_{-\infty}^0 dt_0 Q_{k+2}^{\mathrm{in}}(t_0,t_1,\ldots,t_{k+1}) e^{L(\tau-t_k)} M e^{L(t_k-t_{k-1})} \cdots M e^{Lt_1} \,. \tag{3.37}$$

The quantities

$$W_k(\tau) = \int_\tau^\infty dt_{k+1} \int_0^\tau dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 \int_{-\infty}^0 dt_0 Q_{k+2}^{\mathrm{in}}(t_0,t_1,\ldots,t_{k+1}) \tag{3.38}$$

yield the probabilities that k pump atoms are injected into the micromaser over an arbitrary time interval of length τ.

As in Sec. III A, we again specialize to pumping according to a renewal process in order to treat the above equations. Replacing in Eq. (3.38) the exclusive probability densities Q_k^{in} by the expressions following from Eq. (2.11), performing the Laplace transformation, and making use of the convolution theorem we obtain the atom-number probabilities ($k \geq 1$) of the pumping process

$$W_k(\tau) = r \int_\tau^\infty dt' \int_0^\tau dt'' f(t'-t'') \times \sum_\nu \mathrm{Res}\{[\tilde{f}(z)]^{k-1} \tilde{\varphi}(z)\}_{z=z_\nu} \,, \tag{3.39}$$

where $\tilde{f}(z)$ and $\tilde{\varphi}(z)$ denote the Laplace transforms of the waiting-time distribution $f(t)$ and of the function

$$\varphi(t) = \int_{-\infty}^0 dt' f(t-t') = \int_t^\infty dt' f(t') \,. \tag{3.40}$$

The probability of finding no pump atom at all in the interval of length τ reads

$$W_0(\tau) = r \int_\tau^\infty dt_1 \int_{-\infty}^0 dt_0 f(t_1-t_0) = r \int_\tau^\infty dt\, \varphi(t) \,. \tag{3.41}$$

With the help of the identity

$$z\tilde{\varphi}(z) = 1 - \tilde{f}(z) \,, \tag{3.42}$$

following from Eqs. (3.40) and (2.3) it can be verified that the normalization condition $\sum_{k=0}^\infty W_k(\tau) = 1$ is indeed fulfilled (see the Appendix). Now we substitute the factorization condition (2.11) into Eq. (3.37). Using again the convolution theorem of the Laplace transformation and summing up all contributions we arrive at the equation

$$U(\tau) = W_0(\tau) e^{L\tau} + r \sum_\nu \mathrm{Res}\{\tilde{\varphi}(z-L)[1 - M\tilde{f}(z-L)]^{-1} \times M\tilde{\varphi}(z-L) e^{\tau z}\}_{z=z_\nu} \,, \tag{3.43}$$

which, together with Eq. (3.36), describes the evolution of the unconditioned field density operator ρ of a micromaser whose pump statistics is described by a renewal process. As has been shown in Ref. [17] by a completely different approach, the evolution of this density operator can be expressed by an integro-differential equation, the expanded version of which [see Eq. (5.40) of Ref. [17]] is equal to the equation that results from substituting Eq. (2.11) into Eq. (3.37).

It is interesting to note that for the special case of Poissonian pumping, the evolution operator $U(\tau)$ is equal to the injection-time conditioned operator $U_c(\tau)$ given by the inverse Laplace transform of Eq. (3.10). This can be easily verified by substituting the Poissonian waiting-time distribution $f(t) = r\exp(-rt)$ into the preceding equations, which, in particular, results in $W_0(\tau) = \varphi(\tau) = r^{-1} f(\tau)$. As a consequence, for a micromaser with Poissonian pumping, the steady-state density operators $\bar{\rho}^c$ and $\bar{\rho}$ are identical. A simple relation between these operators can be established also for non-Poissonian pumping. For this purpose some further transformations are necessary. Since $\lim_{\tau\to\infty} W_0(\tau) = 0$, the steady-state result can be written with the help of Eqs. (3.36) and (3.43) as

$$\lim_{\tau\to\infty} \rho(\tau) = \bar{\rho} = \tilde{\varphi}(-L) \mathrm{Res}\{[1 - M\tilde{f}(z-L)]^{-1}\}_{z=0} \times M r \tilde{\varphi}(-L) \rho(0) \,. \tag{3.44}$$

When the operator defined by $\mathrm{Res}\{[1-M\tilde{f}(z-L)]^{-1}\}_{z=0}$ acts on any operator A whose trace is unity, the operator A can be replaced by the unity operator as we conclude from Eqs. (3.19), (3.11), and (3.7). Since the operators M and L are trace conserving and since $\mathrm{Tr}[\rho(0)]=1$ we find that

$$\mathrm{Tr}[Mr\tilde{\varphi}(-L)\rho(0)] = r\,\mathrm{Tr}\left[M\int_0^\infty dt\, e^{Lt}\int_t^\infty dt' f(t')\rho(0)\right]=1 , \quad (3.45)$$

where use has been made of Eq. (A4) of the Appendix. Therefore the product $Mr\tilde{\varphi}(-L)\rho(0)$ may be dropped in Eq. (3.44). Moreover, making use of Eq. (3.42), we replace $\tilde{\varphi}(-L)$ by the expression

$$\tilde{\varphi}(-L)=-L^{-1}[1-\tilde{f}(-L)] , \quad (3.46)$$

thus arriving at

$$\mathrm{Res}\sum_{k=0}^{\infty}[M\tilde{f}(z-L)]^k|_{z=0} - \tilde{f}(-L)\,\mathrm{Res}[1-M\tilde{f}(z-L)]^{-1}|_{z=0}+L\bar{\rho}=0 . \quad (3.47)$$

Because of Eqs. (3.19), (3.11),and (3.12) this is equivalent to the relation

$$r(M-1)\bar{\rho}^{\,c}+L\bar{\rho}=0 , \quad (3.48)$$

which for Poissonian pumping, where $\bar{\rho}^{\,c}=\bar{\rho}$, reduces to the well-known condition for the steady state of the micromaser. For pumping according to a more general renewal process, the above equation still has a simple physical interpretation that becomes apparent when we switch over to the photon-number representation. Taking into account Eqs. (3.1)–(3.3) and (3.6), we find that Eq. (3.48) requires the condition

$$r\beta_n\bar{p}^{\,c}_{n-1}=(1+n_b)\gamma n\bar{p}_n \quad (3.49)$$

to be valid. Performing the sum over the photon-number probabilities we obtain

$$rW_1^d=\gamma(\bar{n}-n_b) , \quad (3.50)$$

where

$$\bar{n}=\sum_n n\bar{p}_n \quad (3.51)$$

is the mean cavity photon number in the stationary micromaser regime and

$$W_1^d=\sum_n \beta_n\bar{p}^{\,c}_{n-1} \quad (3.52)$$

has the meaning of the average transition probability into the deexcited state for each pump atom crossing the cavity (see Sec. IV). According to Eq. (3.50), for $n_b=0$ the production rate of deexcited atoms is equal to the loss rate of cavity photons in the stationary regime of any micromaser whose pumping is described by a renewal process. When the pumping process is different from a renewal process, Eq. (2.14) is no longer valid, the mean time interval $\langle\tau\rangle$ between two consecutive pump atoms thus being different from the inverse of the injection rate r. Since the average number of pump atoms arriving over the cavity damping time $T_c=\gamma^{-1}$ is given by $T_c/\langle\tau\rangle$, one would expect that in the general case for $n_b=0$ the relation $\bar{n}=(\gamma\langle\tau\rangle)^{-1}W_1^d$ holds.

C. Photon statistics in the cavity

Applying the previous equations, we now investigate the photon statistics of the cavity field in the stationary micromaser regime dependent on the correlation strength C and on the correlation decay rate Γ of the incoming pump atoms, which are assumed to be distributed according to a renewal process. For this purpose we first have to determine the parameters λ_1,λ_2, and α with the help of Eqs. (2.21) and (2.22) and then find the steady-state solution $\bar{p}_n^c=\bar{\rho}_{nn}^c$ for the injection-time conditioned photon-number probabilities ($n=0,1,\ldots$). By using Eq. (3.49) the unconditioned probability densities $\bar{p}_n$ can be easily obtained in a further step with the help of a recursive procedure. Besides the steady-state mean value $\bar{n}$ and standard deviation $\Delta n^2=\sum_n n^2\bar{p}_n-\bar{n}^2$ of the cavity photon number, we also calculate, for the purpose of comparison, the injection-time conditioned mean photon number

$$\bar{n}^c\equiv\sum_n n\bar{p}_n^c , \quad (3.53)$$

which is the average value that would be obtained by measuring the photon number always immediately before the injection of a pump atom. (Instead of this quantity, one might also discuss the average value $\sum_n n(M\bar{\rho}^c)_{nn} =\sum_n n[\beta_n\bar{p}_{n-1}^c+(1-\beta_{n+1})\bar{p}_n^c]$, which refers to the time instants immediately after the transits of the pump atoms, or the corresponding level-selective average values $\sum_n n(D\bar{\rho}^c)_{nn}$ and $\sum_n n(E\bar{\rho}^c)_{nn}$; cf. Eqs. (3.2) and (3.3).)

For the special case that the micromaser is in the one-photon trapped state arising from the conditions $gt_{\mathrm{int}}=q\pi$ ($q=1,2,\ldots$) and $n_b=0$, all results can be obtained analytically since the photon-number probabilities p_n and p_n^c vanish for $n\geq 2$. In particular we find

$$\bar{n}^c=\frac{\left[1+C\dfrac{\gamma}{\Gamma+\gamma}\right]N_{\mathrm{ex}}\beta_1}{1+\left[1+C\dfrac{\gamma}{\Gamma+\gamma}\right]N_{\mathrm{ex}}\beta_1} \quad (3.54)$$

and

$$\bar{n}=\frac{N_{\mathrm{ex}}\beta_1}{1+\left[1+C\dfrac{\gamma}{\Gamma+\gamma}\right]N_{\mathrm{ex}}\beta_1} , \quad (3.55)$$

where $N_{\mathrm{ex}}=r/\gamma$ and $\beta_1=\sin^2(\pi/\sqrt{2})$. These equations reflect the fact that for Poissonian pump statistics (i.e., for $C=0$) the averaged photon number $\bar{n}^c$ referring to the time instants immediately before the injections of the individual pump atoms is exactly equal to the stationary mean photon number $\bar{n}$, which, due to ergodicity, corresponds to the time-averaged cavity photon number. For antibunched or bunched injection, i.e., for $C<0$ or $C>0$, the inequalities $\bar{n}^c<\bar{n}$ and $\bar{n}^c>\bar{n}$, respectively, are fulfilled.

The relation $\bar{n}^c=\bar{n}$, valid for Poissonian pumping, is not restricted to the one-photon trapped state, but holds for arbitrary values of the micromaser parameters since in this case the density operators $\bar{\rho}^c$ and $\bar{\rho}$ are identical, as derived in the preceding paragraph. Likewise, for sub- and super-Poissonian pump statistics arising from antibunched or bunched injection of pump atoms, the relations $\bar{n}^c \le \bar{n}$ and $\bar{n}^c \ge \bar{n}$, respectively, are confirmed for arbitrary values of the micromaser pump parameter $\Theta=\sqrt{N_{\mathrm{ex}}}gt_{\mathrm{int}}$ by numerical calculations, as illustrated in Fig. 2. It is found that in all cases the difference between $\bar{n}$ and $\bar{n}^c$ depends linearly on the correlation strength C, which takes on negative or positive values for sub- and super-Poissonian pumping, respectively. A rough explanation of the described behavior is based on the following argument, which refers to the time dependence of the microscopic field in the cavity. Obviously all minima of the instantaneous cavity photon number must be located immediately before the transit of an atom since only the latter may increase the number of photons. On the

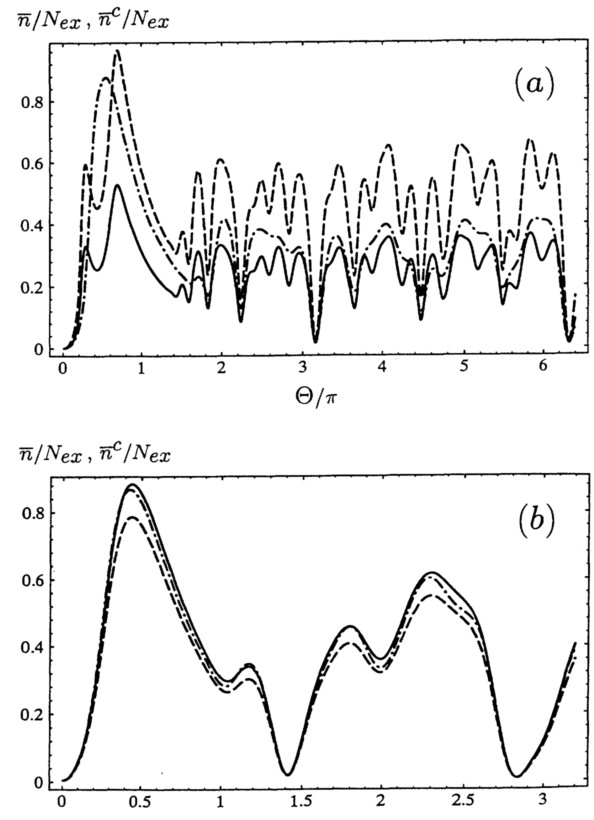

FIG. 2. Mean photon number $\bar{n}$ (full line) and conditioned photon-number expectation value $\bar{n}^c$ referring to the time instants immediately before the injection of the pump atoms (dashed line) (a) for super-Poissonian pumping with $N_{\mathrm{ex}}=10$, $C=1$, and $\Gamma=0.2\gamma$ and (b) for sub-Poissonian pumping with $N_{\mathrm{ex}}=2$, $C=-1$, and $\Gamma=8\gamma$. The thermal photon number is $n_b=0.01$ and for comparison the mean photon number $\bar{n}$ for Poissonian pumping at the corresponding values of N_{ex} is also plotted (dash-dotted line).

assumption that the pump atoms are equidistant in time, which corresponds to a special kind of sub-Poissonian statistics, it is therefore clear that the time-averaged photon number $\bar{n}$ is larger than the average value $\bar{n}^c$. In the general case, however, the relation between $\bar{n}$ and $\bar{n}^c$ crucially depends on the distribution of the distances between neighboring atoms. These distances affect the time average $\bar{n}$, but do not enter into the determination of $\bar{n}^c$ since the latter quantity is simply given by the arithmetic mean of the photon numbers being present immediately before the injections of the individual pump atoms. Thus it can be qualitatively understood that for pump atoms arriving in bunches, the time-averaged photon number $\bar{n}$ is smaller than $\bar{n}^c$ owing to the comparatively long periods without atoms between the bunches.

Another interesting conclusion that can be drawn from inspection of Eqs. (3.54) and (3.55) refers to the influence of the pump-atom correlation-decay constant Γ. When the latter is much larger than the cavity damping rate γ, the quantities $\bar{n}$ and $\bar{n}^c$ become equal and do not differ from the quantities $\bar{n}=\bar{n}^c$, which would result from Poissonian pumping (i.e., for $C=0$) at the same value of N_{ex}. On the other hand, for $\Gamma \ll \gamma$ we find that $\bar{n}^c \approx (1+C)\bar{n}$. At reasonable values of N_{ex}, however, this limiting case is restricted to super-Poissonian pumping ($C>0$) because of the condition (2.20). Again the described dependence on Γ does not only hold for the one-photon trapped state but is also valid for arbitrary values of the pump parameter Θ, as can be confirmed by numerical calculations (cf. Fig. 3). Moreover, when $n_b=0$ the above limiting cases may be also discussed analytically for arbitrary values of Θ. To do so we consider the coupled set of recursion relations (3.35) and (3.36) and take into account Eqs. (2.21) and (2.22), which determine the parameters λ_1, λ_2, and α from r, C, and Γ. When $\Gamma \gg r$ we find that $\lambda_1 \approx r$, $\lambda_2 \approx \Gamma$, and $\alpha \approx C$. On the additional assumption $\Gamma \gg \gamma$, Eqs. (3.34) and (3.35) yield, after some simplification, the recursion relation

$$n\gamma \bar{p}_n^c = r\beta_n \bar{p}_{n-1}^c \quad (\Gamma \gg r, \gamma) \tag{3.56}$$

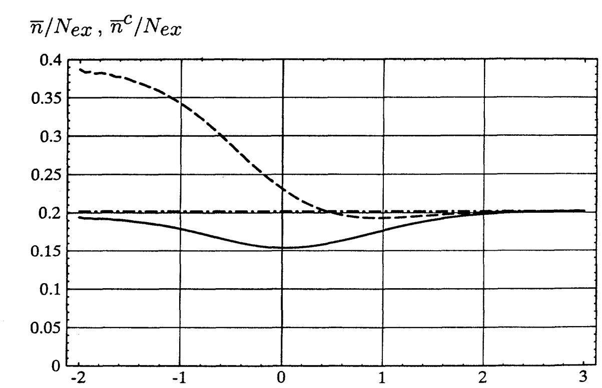

FIG. 3. Same as Fig. 2 for super-Poissonian pumping with $N_{\mathrm{ex}}=10$ and $C=1$ dependent on the pump-atom correlation-decay constant Γ at the pump parameter $\Theta=3.6\pi$ for the thermal photon number $n_b=0$.

for the conditioned photon-number probabilities referring to the time instants immediately before the injection of the individual pump atoms. Equation (3.56) corresponds to the well-known recursion relation for Poissonian pumping. In the opposite case $\Gamma \ll r$ [which is restricted to $C>0$ because of the condition (2.20)] we obtain from Eqs. (2.21) and (2.22) the values $\lambda_1 \approx \Gamma/(1+C)$, $\lambda_2 \approx r(1+C)$, and $\alpha \approx 4r^2/\Gamma^2 \gg 1$. By substituting these values into Eqs. (3.34) and (3.35) and assuming in addition that $\Gamma \ll \gamma$, we arrive, after a little algebra, at the recursion relation

$$n\gamma \bar{p}_n^c = (1+C) r\beta_n \bar{p}_{n-1}^c \quad (\Gamma \ll r,\gamma,\ C \geq 0) . \tag{3.57}$$

Since, for $n_b=0$, according to Eq. (3.49) the unconditioned photon-number probability is connected to $\bar{p}_n^c$ by the relation

$$\gamma n \bar{p}_n = r\beta_n \bar{p}_{n-1}^c , \tag{3.58}$$

we obtain with the help of Eqs. (3.51), (3.53), and (3.56) at once the desired results $\bar{n}^c \approx \bar{n}$ for $\Gamma \gg r,\gamma$ and $\bar{n}^c \approx (1+C)\bar{n}$ for $\Gamma \ll r,\gamma$ and $C \geq 0$.

For super-Poissonian pumping with the correlation strength $C=1$ and the correlation decay rate $\Gamma=\gamma$, the distribution of the stationary photon-number probabilities is shown in Fig. 4 for two different values of the pump parameter Θ that correspond to the single- and the double-peaked (bistable) case. The distinct increase of $\bar{p}_n$ for small photon numbers n in comparison to the values that would arise from Poissonian pumping has already been found numerically in Ref. [17]. From Fig. 5 it becomes obvious that for super- or sub-Poissonian pumping the relative standard deviation of the stationary photon-number distribution is increased or decreased, respectively, in comparison to the Poissonian case. Again, with growing pump-atom correlation decay rate Γ, the difference from the Poissonian result is diminished. The influence of Γ will be discussed in more detail in Sec. IV in connection with the normalized intensity correlation function at zero time delay

$$g_2(0) = \frac{\overline{n(n-1)}}{\bar{n}^2} = 1 + \frac{1}{\bar{n}} \left[\frac{\Delta n^2}{\bar{n}} - 1 \right] . \tag{3.59}$$

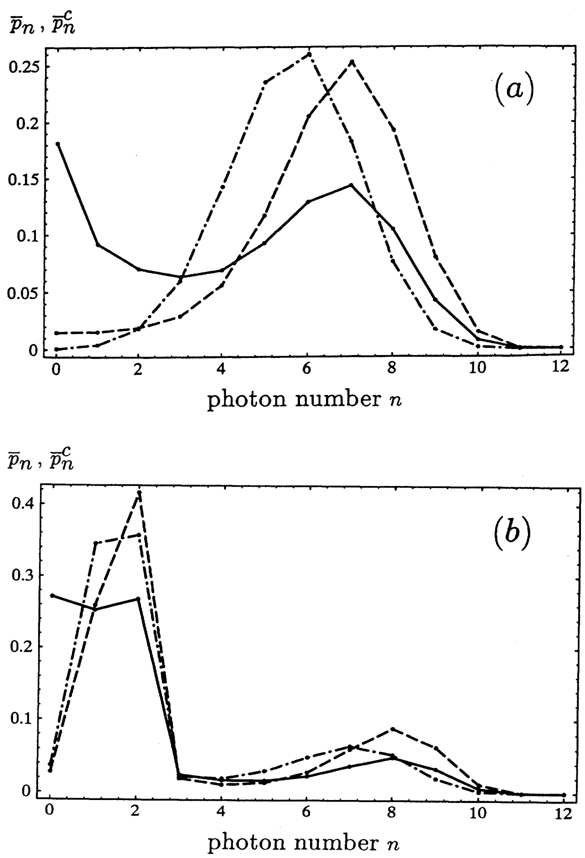

FIG. 4. Photon-number probability $\bar{p}_n$ (full line) and injection-time conditioned photon-number probability $\bar{p}_n^c$ (dashed line) for $N_{ex}=10$, $C=1$, $\Gamma=\gamma$, and $n_b=0.01$ at the pump parameters (a) $\Theta=0.9\pi$ and (b) $\Theta=1.9\pi$ in comparison to the photon-number distribution for Poissonian pumping with $N_{ex}=10$ (dash-dotted line).

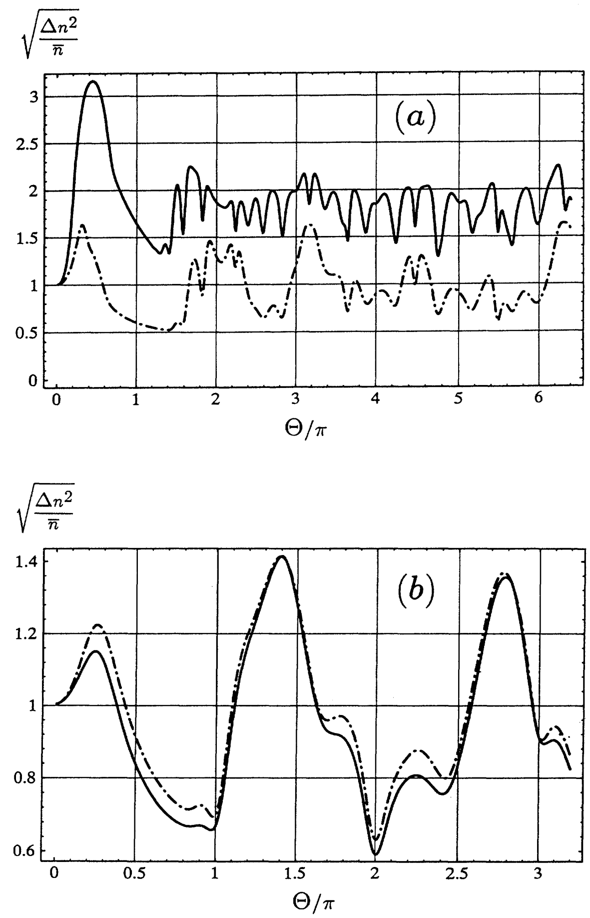

FIG. 5. Relative standard deviation $\sqrt{\Delta n^2/\bar{n}}$ of the cavity photon number (full line) (a) for super-Poissonian pumping and (b) for sub-Poissonian pumping. The parameters are the same as in Fig. 2 and the dash-dotted line again refers to Poissonian pumping.

IV. STATISTICS OF THE OUTGOING ATOMS

A. General expressions

Now we want to investigate the level-selective statistics of the atoms leaving the cavity, which is directly accessible to experimental observations through measuring the waiting-time distributions, coincidence probabilities, or counting statistics over a definite time interval [4–10]. Since the second factorial moment of any counting statistics can be easily determined from the two-time coincidence probability density [cf. Eq. (5.2) in Sec. V], we focus only on the latter quantity and on the waiting-time distribution.

When, in a stationary micromaser regime, an atom leaving the cavity is found to be in the deexcited state, the (non-normalized) reduced density operator describing the state of the field immediately after the transit of this atom is given by $D\bar{\rho}^c$ according to Eq. (3.1) and to the definition of the operator ρ^c in Sec. III. Provided the deexcited atoms are detected with efficiency η, the probability density for the detection of deexcited atoms, i.e., the detection rate, is given by [4–10]

$$P_1^d = \eta r W_1^d = \eta r \,\mathrm{Tr}(D\bar{\rho}^c) = \eta r \sum_n \beta_n \bar{p}_{n-1}^c \tag{4.1}$$

with W_1^d being the average transition probability into the deexcited state for a single atom. In order to calculate the joint probability $W_2^{d,d}(\tau)$ that two atoms, which are known to be injected with time difference τ, both leave the cavity in the deexcited state, we have to apply the evolution equation (3.7) of the density operator $\rho^c(\tau)$. In contrast to Eq. (3.9), the initial density operator of the evolution in this case has to be identified with $D\bar{\rho}^c$. Thus we arrive at

$$W_2^{d,d}(\tau) = \frac{r^2}{P_2^{\mathrm{in}}(0,\tau)} \mathrm{Tr}[DU_c(\tau)D\bar{\rho}^c] \; . \tag{4.2}$$

The coincidence probability density for the detection of two deexcited atoms with the time difference τ using a detector with efficiency η is therefore given by

$$P_2^{d,d}(\tau) = \eta^2 P_2^{\mathrm{in}}(0,\tau) W_2^{d,d}(\tau) = \eta^2 r^2 \,\mathrm{Tr}[DU_c(\tau)D\bar{\rho}^c] \; . \tag{4.3}$$

Now we turn to the waiting-time distribution $P_w(\tau)$, defined as the conditional probability density of detecting the next atom at time $t+\tau$ provided the first atom had been detected at time t. With $Q_2^{d,d}(\tau)$ denoting the exclusive probability density for detecting two deexcited atoms with a time difference τ on condition that no other deexcited atom has been detected in between we have

$$P_w(\tau) = \frac{1}{r} Q_2^{d,d}(\tau) \; . \tag{4.4}$$

In analogy to Eq. (4.3) we may write

$$Q_2^{d,d}(\tau) = \eta^2 r^2 \,\mathrm{Tr}[DV_c(\tau)D\bar{\rho}^c] \; , \tag{4.5}$$

where the evolution of the operator $V_c(\tau)$ is determined by the contributions of all atoms traversing the cavity and not being detected in the deexcited state. Following the arguments used to treat a micromaser with Poissonian pumping [6–9], we find the evolution equation of V_c by replacing the operator $M = D + E$ on the right-hand side of Eq. (3.8) by the operator $M' = (1-\eta)D + E = M - \eta D$. The latter refers to the transit of a single atom and takes into account only the possibilities that this atom either leaves the cavity in the excited state or it is deexcited but not detected. Thus for pumping according to a renewal process with the waiting-time distribution (2.15) we obtain, along the same lines used to derive Eqs. (3.20)–(3.22), the representation

$$V_c(\tau) = V_1(\tau) + V_2(\tau) \; , \tag{4.6}$$

where

$$\begin{aligned} \dot{V}_1 &= LV_1 + \frac{\lambda_1\lambda_2}{\lambda_2 + \alpha\lambda_1}(M - \eta D)(V_1 + V_2) - \lambda_1 V_1 \; , \\ \dot{V}_2 &= LV_2 + \alpha\frac{\lambda_1\lambda_2}{\lambda_2 + \alpha\lambda_1}(M - \eta D)(V_1 + V_2) - \lambda_2 V_2 \; , \end{aligned} \tag{4.7}$$

with $V_{1/2}(0) = U_{1/2}(0)$ [cf. Eqs. (3.23) and (3.24)].

Up to now we only considered the statistics of the

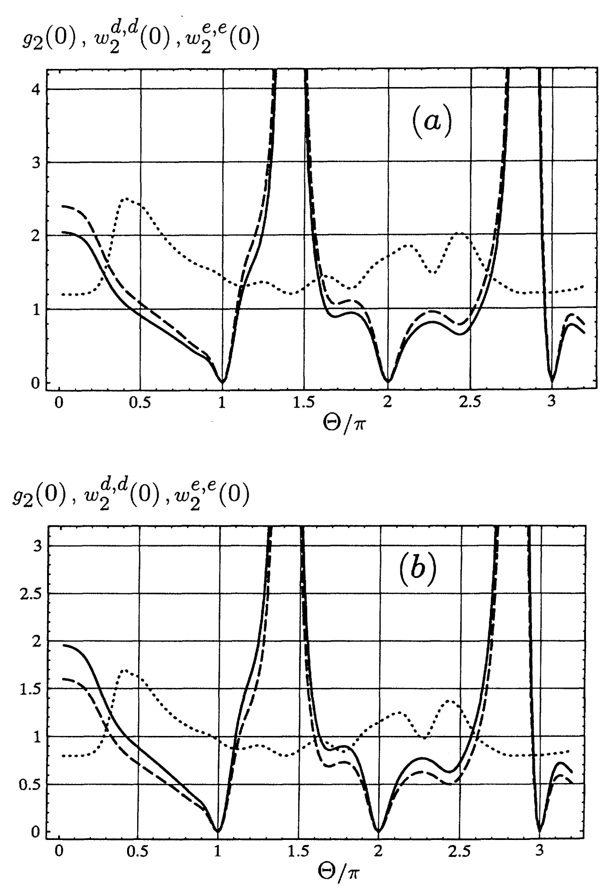

FIG. 6. Normalized coincidence probability densities $w_2^{d,d}(0)$ for the outgoing deexcited atoms (dashed line) and $w_2^{e,e}(0)$ for the outgoing excited atoms (dotted line) in comparison to the normalized intensity correlation function $g_2(0)$ of the cavity field (full line) at $n_b = 10^{-6}$ for pumping with $N_{\mathrm{ex}} = 10$ and $\Gamma = 8\gamma$ at the correlation strengths (a) $C = 0.2$ and (b) $C = -0.2$.

atoms leaving the cavity in the deexcited state. When one is interested in the statistics of the outgoing excited atoms or in cross correlations between both kinds of atoms, one simply has to replace the operator D at the appropriate positions in Eqs. (4.6) and (4.7) by the operator E introduced in Eqs. (3.1) and (3.3).

B. Numerical results

According to Eqs. (4.1) and (4.3), the normalized delayed coincidence probability density

$$w_2^{d,d}(\tau)=\frac{P_2^{d,d}(\tau)}{(P_1^d)^2} \tag{4.8}$$

for the detection of two deexcited atoms with time difference τ at the exit of the micromaser does not depend on the detector efficiency η; nor does the corresponding coincidence probability density for the detection of excited atoms

$$w_2^{e,e}(\tau)=\frac{P_2^{e,e}(\tau)}{(P_1^e)^2}=\frac{\mathrm{Tr}[EU_c(\tau)E\bar{\rho}^c]}{[\mathrm{Tr}(E\bar{\rho}^c)]^2} . \tag{4.9}$$

In Fig. 6 the zero-time-delay values of the above quantities are depicted as being dependent on the micromaser pump parameter for two special cases of super- and sub-Poissonian pumping. For comparison, the normalized zero-time-delay intensity correlation function $g_2(0)$ of the cavity field given by Eq. (3.59) is also plotted. Obviously the behavior of $g_2(0)$ and $w_2^{d,d}(0)$ is very similar: for super-Poissonian pumping the latter quantity is only slightly larger than the former, whereas for sub-Poissonian pumping the opposite case is true. This close resemblance is not surprising since for Poissonian pumping and negligible thermal photon number it has been proved that both quantities are exactly equal for any value of the pump parameter Θ [5,7,10] and that such an equality holds even for the normalized values of the delayed coincidence probability density of deexcited atoms and of the corresponding delayed intensity correlation function [7,8,10]. For a fixed value of Θ the dependence

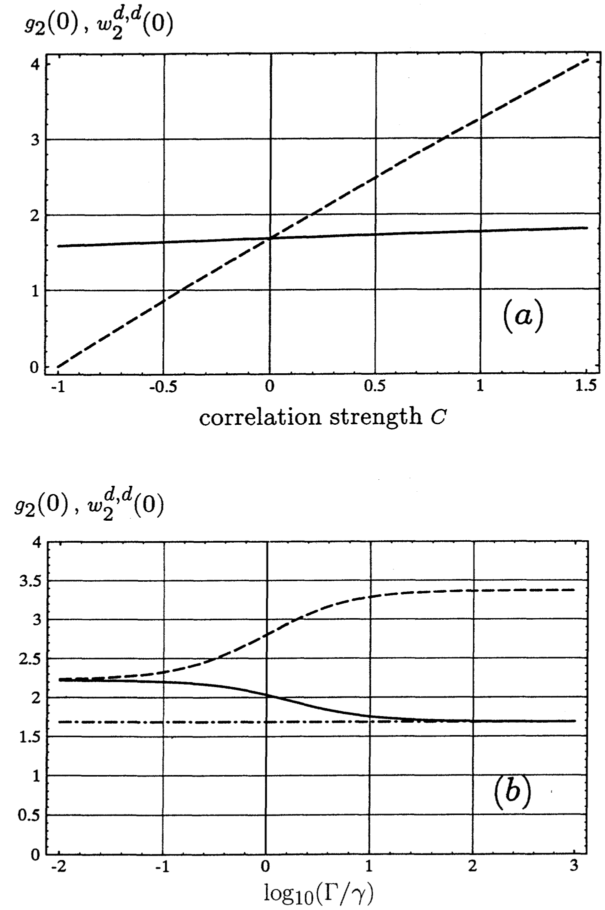

FIG. 7. Normalized coincidence probability density $w_2^{d,d}(0)$ (dashed line) and intensity correlation function $g_2(0)$ (full line) for pumping with $N_{\mathrm{ex}}=2$ at $\Theta=1.2\pi$ and $n_b=0$. (a) The dependence on the correlation strength C is depicted for $\Gamma=8\gamma$; (b) the dependence on the correlation decay constant Γ is shown for $C=1$. The dash-dotted line in (b) refers to Poissonian pumping where $w_2^{d,d}(0)=g_2(0)$ for $n_b=0$.

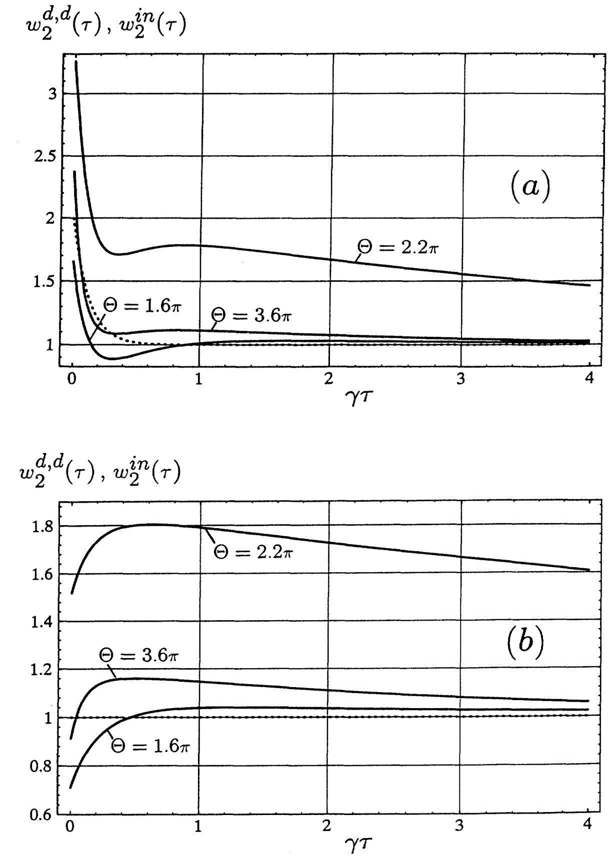

FIG. 8. (a) Normalized delayed coincidence probability density $w_2^{d,d}(\tau)$ for deexcited atoms (full lines) at $n_b=0.01$ for super-Poissonian pumping with $N_{\mathrm{ex}}=10$, $C=1$, and $\Gamma=8\gamma$. The dotted line shows the normalized delayed coincidence probability density $w_2^{\mathrm{in}}(\tau)$ of the incoming pump atoms. (b) For comparison, the corresponding curves are also plotted for Poissonian pumping with the same value of N_{ex}.

of $g_2(0)$ and $w_2^{d,d}(0)$ on the correlation strength C of the incoming pump atoms and on their correlation decay rate Γ is shown in Fig. 7. Making use of the approximations (3.56) and (3.57), which hold in the limit of large and small values of Γ, respectively, we find, with the help of Eq. (4.1), (4.3), and (3.58), that for negligible thermal photon number the approximate relations

$$w_2^{d,d}(0)=(1+C)\sum_n \beta_n\beta_{n-1}\bar{p}^c_{n-2} \Big/ \left[\sum_n \beta_n\bar{p}^c_{n-1}\right]^2$$

$$\approx(1+C)g_2(0) \quad \text{for } \Gamma \gg r,\gamma \tag{4.10}$$

and

$$w_2^{d,d}(0)\approx g_2(0) \quad \text{for } \Gamma \ll r,\gamma;\ C\geq 0 \tag{4.11}$$

are valid. To obtain Eqs. (4.10) and (4.11) the expression $U_c(0)=(1+C)\underline{1}$ following from Eqs. (3.7) and (2.16) has been used. In general, it can be stated that because of Eq. (3.56) and because of the equivalence of $\bar{p}^c$ and $\bar{p}$ for a micromaser with Poissonian pumping [see the remarks following Eq. (3.43)] for rapidly decaying pump-atom correlations (i.e., for $\Gamma \gg \gamma, r$), the photon statistics of the cavity field is equal to that which would arise from Poissonian pumping with the same pumping rate [cf. Figs. 3 and 7(b)]. However, the presence of the pump-atom correlations is nevertheless reflected in the statistics of the atoms leaving the cavity [see Eq. (4.10) and Fig. 7(b)].

On the other hand, when the lifetime of the pump-atom correlations is rather large in comparison to the cavity lifetime and to the average time interval between the injection of consecutive pump atoms, it should be possible, because of Eq. (4.11), to determine the field quantity $g_2(0)$ directly by measuring the coincidence rate of deexcited atoms [cf. Fig. 7(b)]. [Note that because of the restriction (2.20), this limiting case can be treated with the model of pumping according to a renewal process only for super-Poissonian pump statistics where $C>0$.]

The dependence of the normalized two-time coincidence probability density $w_2^{d,d}(\tau)$ of deexcited atoms on the delay time τ is greatly affected by the properties of the pump-atom correlations as becomes obvious from Figs. 8 and 9. The same holds true for the τ dependence of the waiting-time distribution $P_w(\tau)$, which is plotted in Fig. 10 as a result of numerically evaluating Eqs. (4.4)–(4.7).

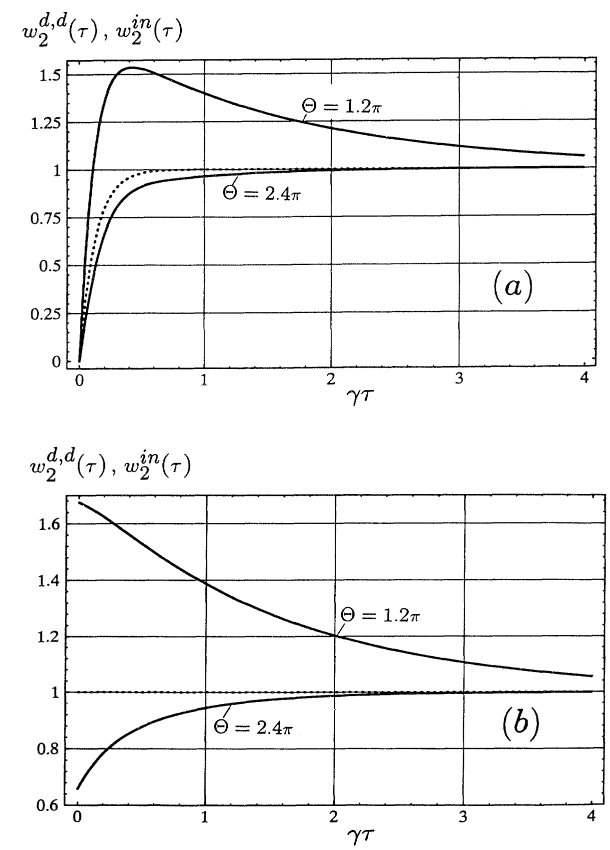

FIG. 9. Same as Fig. 8, but for sub-Poissonian pumping with $N_{ex}=2$, $C=-1$, and $\Gamma=8\gamma$.

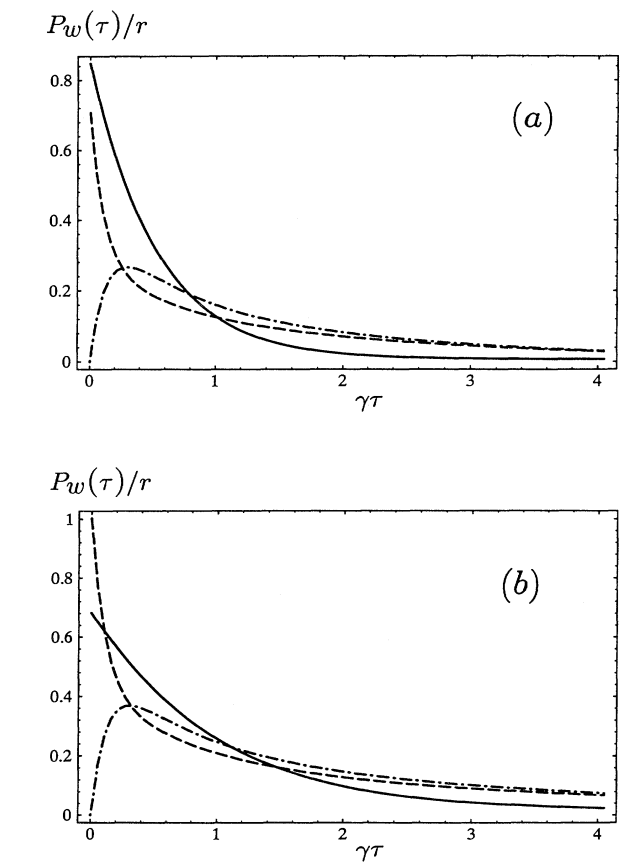

FIG. 10. Normalized waiting-time distribution $Pw(\tau)/r$ of the outgoing deexcited atoms measured with detector efficiency (a) $\eta=1$ and (b) $\eta=0.5$ for pumping with $N_{ex}=2$ at the pump parameter $\Theta=1.2\pi$ and the thermal photon number $n_b=0.01$. The full lines correspond to Poissonian pumping ($C=0$), the dashed lines to super-Poissonian pumping with $C=1$ and $\Gamma=8\gamma$, and the dash-dotted lines to sub-Poissonian pumping with $C=-1$ and $\Gamma=8\gamma$.

V. CONNECTION BETWEEN THE DIFFERENT TREATMENTS OF THE PUMP STATISTICS

A. Pump-atom counting statistics in our model

In order to compare our results with those arising from other treatments of the micromaser pump statistics it is necessary to discuss in some detail the mutual relationship and the ranges of validity of the different pumping models used in the literature. For this purpose let us first consider the counting statistics of the incoming pump atoms, which is obtained from our description of the pump statistics. As is well known from the theory of stochastic point processes, the kth-order factorial moments of the counting statistics referring to a counting interval of length T can be calculated by performing a k-fold time integration over the k-time coincidence probability density [19]. Denoting the mean number of pump atoms arriving over a period of length T by $\langle N\rangle_T$, we may write, for stationary pumping,

$$\langle N\rangle_T = TP_1^{\rm in} = rT \ . \tag{5.1}$$

The corresponding equations for the higher-order factorial moments read [19]

$$\langle N(N-1)\cdots(N-k+1)\rangle_T = \int_0^T dt_1 \int_0^T dt_2 \cdots \int_0^T dt_k P_k^{\rm in}(t_1,t_2,\ldots,t_k) \quad (k=1,2,\ldots) \ , \tag{5.2}$$

where now the former restriction $t_1 \le t_2 \le \cdots \le t_k$ has been dropped. When the special ansatz

$$P_2^{\rm in}(t_1,t_2) = r^2[1+C\exp(-\Gamma|t_2-t_1|)] \tag{5.3}$$

[cf. Eq. (2.16)] is used for the two-time coincidence probability density we find

$$\langle N(N-1)\rangle_T = \langle N\rangle_T^2 + \frac{2Cr}{\Gamma}[1-\varepsilon(T)]\langle N\rangle_T \ , \tag{5.4}$$

where

$$\varepsilon(T) = \frac{1-e^{-\Gamma T}}{\Gamma T} \ . \tag{5.5}$$

In general, in the case of stationary non-Poissonian pumping the counting statistics depends on the length T of the counting interval. For the limiting cases that T is either very short or very large in comparison to the decay time Γ^{-1} of the pump atom correlations, we obtain from Eqs. (5.4) and (5.5)

$$\frac{\langle \Delta N^2\rangle_T}{\langle N\rangle_T} = \begin{cases} 1+CrT & \text{for } \Gamma T \ll 1 & (5.6)\\ 1+2\dfrac{Cr}{\Gamma} & \text{for } \Gamma T \gg 1 \ , & (5.7)\end{cases}$$

where the atom-number variance $\langle \Delta N^2\rangle_T = \langle N^2\rangle_T - \langle N\rangle_T^2$ has been introduced and Eq. (5.1) has been taken into consideration. It is clear that $\langle \Delta N^2\rangle_T$ cannot be smaller than zero. For this reason, in the case of a negative correlation strength C, Eq. (5.7) imposes a restriction on the parameters C, r, and Γ occurring in the ansatz (5.3). In fact, we have to suppose that for $-1 \le C < 0$ the condition $\Gamma \ge 2r|C|$ is fulfilled, which is less restrictive than the inequality (2.20) resulting from the additional assumption of the pumping process being a renewal process. Because of this condition Eq. (5.6) yields

$$\frac{\langle \Delta N^2\rangle_T}{\langle N\rangle_T} \ge 1-\frac{1}{2}\Gamma T \approx 1 \quad \text{for } \Gamma T \ll 1 \ . \tag{5.8}$$

Obviously with respect to a counting interval, which is small in comparison to the lifetime of the pump-atom correlation, $\langle \Delta N^2\rangle_T$ cannot be noticeably smaller than $\langle N\rangle_T$, i.e., sub-Poissonian counting statistics practically does not occur in this case.

For later use we now calculate the third-order factorial moment of the counting statistics of the pump atoms. To do so we assume that the pumping process is a stationary renewal process fulfilling the factorization condition [20]

$$P_3^{\rm in}(t_1,t_2,t_3) = \frac{1}{r}P_2^{\rm in}(t_3-t_2)P_2^{\rm in}(t_2-t_1) \ , \tag{5.9}$$

which is valid for $t_3 \ge t_2 \ge t_1$. From Eq. (5.2) we then obtain

$$\begin{aligned}\langle N(N-1)(N-2)\rangle_T = \frac{1}{r}\Bigg\{ &\int_0^T dt_1 \int_0^{t_1} dt_2 \Bigg[\int_0^{t_2} dt_3 P_2^{\rm in}(t_1-t_2)P_2^{\rm in}(t_2-t_3) \\ &\quad + \int_{t_2}^{t_1} dt_3 P_2^{\rm in}(t_1-t_3)P_2^{\rm in}(t_3-t_2) + \int_{t_1}^{T} dt_3 P_2^{\rm in}(t_3-t_1)P_2^{\rm in}(t_1-t_2)\Bigg] \\ &+ \int_0^T dt_1 \int_{t_1}^{T} dt_2 \Bigg[\int_0^{t_1} dt_3 P_2^{\rm in}(t_2-t_1)P_2^{\rm in}(t_1-t_3) \\ &\quad + \int_{t_1}^{t_2} dt_3 P_2^{\rm in}(t_2-t_3)P_2^{\rm in}(t_3-t_1) + \int_{t_2}^{T} dt_3 P_2^{\rm in}(t_3-t_2)P_2^{\rm in}(t_2-t_1)\Bigg]\Bigg\} \ . \end{aligned} \tag{5.10}$$

Making use of the ansatz (5.3) and performing the integrations, we find, after a little algebra,

$$\langle N(N-1)(N-2)\rangle_T = \langle N\rangle_T^3 + \frac{2rC}{\Gamma}[3-\varepsilon(T)]\langle N\rangle_T^2 + \left[\frac{2rC}{\Gamma}\right]^2 \left\{2+e^{-\Gamma T} - 3\varepsilon(T) - \frac{C}{2}[5+e^{-\Gamma T}-6\varepsilon(T)]\right\}\langle N\rangle_T \ , \tag{5.11}$$

with $\varepsilon(T)$ being given by Eq. (5.5). It should be kept in mind that the condition (2.20) has to be fulfilled in order to make the ansatz (5.3) compatible with the requirement that the statistical process is a renewal process (compare Sec. II). When the counting interval T is much larger than the lifetime of the pump-atom correlations, Eq. (5.11) becomes independent of T and reads, for $\Gamma T \gg 1$,

$$\langle N(N-1)(N-2)\rangle_T = \langle N\rangle_T^3 + 3\frac{2rC}{\Gamma}\langle N\rangle_T^2 + \left[2-\frac{5}{2}C\right]\left[\frac{2rC}{\Gamma}\right]^2\langle N\rangle_T . \tag{5.12}$$

B. Connection to the pearl-string model

We now investigate the second and the third factorial moment of the pump-atom counting statistics for the other models used in the literature to describe the statistical nature of the pumping process. In the original pearl-string model the pumping is provided by a regular flux of ground-state atoms out of which some atoms are excited to the upper level of the maser transition [11]. When the excitation probability is denoted by p, the probability to find N excited atoms out of M equidistant incoming atoms is given by the binomial distribution

$$p_N(M) = \begin{bmatrix} M \\ N \end{bmatrix} p^N(1-p)^{M-N} , \tag{5.13}$$

which yields the mean value

$$\langle N\rangle_M = \sum_N N p_N(M) = Mp \tag{5.14}$$

[11,14]. The factorial moments are easily calculated to be

$$\langle N(N-1)\cdots(N-k+1)\rangle_M = M(M-1)\cdots(M-k+1)p^k \quad (k=1,2,\ldots) . \tag{5.15}$$

In particular, one finds [11,14]

$$\frac{\langle \Delta N^2\rangle_M}{\langle N\rangle_M} = 1-p . \tag{5.16}$$

In this model the relative variance does not depend on the number M of incoming equidistant atoms or on the length of the considered time interval, respectively, in contrast to the result for a strictly stationary pumping model [cf. Eq. (5.4)–(5.7)]. The third-order factorial moment can be written as

$$\langle N(N-1)(N-2)\rangle_M = \langle N\rangle_M^3 - 3p\langle N\rangle_M^2 + 2\langle N\rangle_M \tag{5.17}$$

and in a similar way the higher-order factorial moments may be expressed as a power series in $\langle N\rangle_M$ with coefficients that only depend on p. Therefore, in the pearl-string model, which is not stationary in the strict sense inasmuch as there exists only a discrete set of possible atomic arrival times, the counting statistics of excited pump atoms does not depend on the length of the counting interval.

When the distance Δt between two consecutive atoms in the incoming regular beam of ground-state atoms is assumed to be small in comparison to all relevant time intervals T, the pearl-string model is nearly homogeneous in time and approximately describes stationary pumping. Inspection of Eqs. (5.7) and (5.16) suggests that the equation

$$\frac{2rC}{\Gamma} = -p \quad (C \le 0) \tag{5.18}$$

connects our description of the pump statistics with this homogeneous limit of the pearl-string model provided $\Gamma T \gg 1$. However, for $C \neq 0$ the connection established by Eq. (5.18) is valid with respect to higher-order moments of the number of pump atoms only when for any relevant time interval T the quantities $\langle N\rangle_T$ or $\langle N\rangle_M$, respectively, are large in comparison to unity, as can be seen from Eqs. (5.12) and (5.17). In fact, when in both equations the last term on the right-hand side is neglected, they can be transformed with the help of Eqs. (5.7) and (5.16), respectively, to yield the identical expressions

$$\langle (N-\langle N\rangle_T)^3\rangle_T = \langle (N-\langle N\rangle_M)^3\rangle_M = 0 \tag{5.19}$$

for $\langle N\rangle_T \gg 1$, $\Gamma T \gg 1$, and $\langle N\rangle_M = pM = pT/\Delta t \gg 1$. The binomial distribution (5.13) in this limit is approximately equal to a Gaussian one; the statistics therefore are completely determined by the knowledge of the mean value and of the variance. With respect to the micromaser, the physically relevant time interval T is the cavity lifetime γ^{-1}. In our treatment, which is based on the ansatz (5.3), the relations $\langle N\rangle_T \gg 1$ and $\Gamma T \gg 1$ therefore are equivalent to $N_{\text{ex}} = r/\gamma \gg 1$ and $\Gamma \gg \gamma$, respectively. From the preceding considerations we conclude that, for $-1 \le C < 0$ and in the limit of large values of N_{ex} and Γ/γ, our description of the pump statistics corresponds to a pearl-string model with $p = -2rC/\Gamma$ and $\Delta t = p/r$. The restriction (2.20) yields $p \le 2|C|/(1+\sqrt{|C|})^2 \le \frac{1}{2}$.

C. Connection to the Langevin-equation approach

Now we turn to the Langevin-equation description of the pump statistics [16]. In this description the pump fluctuations are taken into account by adding a fluctuating force $F(t)$ to the mean injection rate r and by assuming that this fluctuating force is δ correlated, obeying the equations

$$\langle F(t)\rangle = 0 \tag{5.20}$$

and

$$\langle F(t)F(t')\rangle = (1-p)\delta(t-t') , \tag{5.21}$$

where $0 \le p \le 1$. Hence the Langevin-equation approach refers to pump-atom correlations that practically decay infinitely fast. Therefore it can be approximately applied when the pump-atom correlation time Γ^{-1} is small in comparison to the cavity lifetime γ^{-1}. However, it is worth noting that, in general, the above approach does not completely describe the pump statistics as long as the higher-order correlation functions of $F(t)$ are not specified. This becomes evident by considering the moments of the number of pump atoms arriving over a time interval of length T, which can be calculated from

$$\langle N^m \rangle = \int_0^T dt_1 \cdots \int_0^T dt_m \left\langle \prod_{i=1}^{m} [r+F(t_i)] \right\rangle , \quad (5.22)$$

where $m=1,2.\ldots$ With the help of Eqs. (5.20) and (5.21) we obtain $\langle N \rangle_T = rT$ and

$$\langle (N-\langle N \rangle_T)^2 \rangle_T \equiv \langle \Delta N^2 \rangle_T = (1-p)\langle N \rangle_T . \quad (5.23)$$

Now we assume that the higher-order correlation functions obey the Gaussian factorization conditions, i.e., that for $k=1,2\ldots,$

$$\langle F(t_1)F(t_2) \cdots F(t_{2k-1}) \rangle = 0 \quad (5.24)$$

and

$$\langle F(t_1)F(t_2) \cdots F(t_{2k}) \rangle = \sum \langle F(t_{i_1})F(t_{i_2}) \rangle \cdots \langle F(t_{i_{2k-1}})F(t_{i_{2k}}) \rangle , \quad (5.25)$$

where the summation has to be performed over all $1\times 3 \times \cdots \times (2k-1)$ possibilities of decomposing the left-hand side into a product of two-time correlation functions with the original time ordering being preserved. Using Eq. (5.22) and (5.23) and applying the binomial theorem, we then find after some algebra that

$$\langle (N-\langle N \rangle_T)^{2k-1} \rangle_T = 0 \quad (5.26)$$

and

$$\langle (N-\langle N \rangle_T)^{2k} \rangle_T = 1\times 3 \times \cdots \times (2k-1) \langle \Delta N^2 \rangle_T^k . \quad (5.27)$$

We mention that the above equations could be obtained more easily by integration from the Gaussian probability distribution

$$p_N(T) = \frac{1}{\sqrt{2\pi\langle \Delta N^2 \rangle_T}} \exp\left[-\frac{(N-\langle N \rangle_T)^2}{2\langle \Delta N^2 \rangle_T} \right] . \quad (5.28)$$

When the parameter p is set equal to zero, the variance corresponds to that of a Poissonian distribution, as can be seen from Eq. (5.23). The third-order moment, however, is different from its Poissonian value and also from the limiting value that would follow from a binomial distribution. For $\langle N \rangle_T \gg 1$, however, this difference can be neglected. Because of the equivalence of Eqs. (5.23) and (5.16) we may conclude that in the case of $\langle N \rangle_T$ being large for all relevant time intervals T, i.e., for $N_{ex} = r/\gamma \gg 1$ in the micromaser, the pearl-string model and the Langevin-equation approach correspond to one another and are connected to our description of the pump statistics by Eq. (5.18) as long as we assume that the pump-atom correlations are decaying rapidly enough, i.e., for $\Gamma \gg \gamma$.

D. Connection to the quantum-field pump model

Finally, we discuss the relation between our treatment and the quantum-field model of the atomic injection introduced in Ref. [17]. While our approach is more evident from heuristic physical considerations, the latter starts from first principles, but refers only to the unconditioned density matrix ρ of the cavity field without discussing the level-selective statistics of the atoms leaving the cavity. In order to yield equations that are practically tractable, both approaches need the assumption that the injection statistics is described by a stationary renewal process. As it must be, the resulting evolution equations for the density matrix ρ are identical. This becomes obvious by comparing the expanded version of the integro-differential equation derived in Ref. [17] with our result [see Eq. (5.40) of Ref. [17] and Eqs. (3.36), (3.37), and (2.11) of this paper]. However, the ansatz used in Ref. [17] is only suited to treat super-Poissonian pumping with fixed correlation strength.

VI. CONCLUSIONS

We treated the photon statistics of the cavity field and the level-selective statistics of the outgoing atoms for a one-atom micromaser with stationary non-Poissonian pumping with the help of the theory of stochastic point processes. On the assumption that the pump statistics can be described by a renewal process, we investigated the effect of sub- as well as super-Poissonian pumping, both analytically and numerically, dependent on the correlation strength (being negative in case of pump-atom antibunching) and on the correlation decay time of the incoming pump atoms.

An interesting result refers to the case that the lifetime of the pump-atom correlations is rather large in comparison to the cavity damping time and to the average time interval between the injection of consecutive pump atoms. In this case the normalized coincidence probability density of the outgoing deexcited atoms and the normalized cavity-field intensity correlation function for zero time delay are shown to be equal for negligible thermal photon number. For rapidly decaying pump-atom correlations, however, the statistics of the atoms leaving the cavity is strongly affected by the presence of the pump-atom correlations, whereas the photon statistics of the cavity field is found to be the same as that which would arise from Poissonian pumping, with the same pumping rate.

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APPENDIX

We want to calculate the expression

$$\sum_{k=0}^{\infty} W_k(\tau) = W_0(\tau) + r\int_\tau^\infty dt' \int_0^\tau dt'' f(t'-t'') \times \sum_\nu \operatorname{Res} \frac{\tilde{\varphi}(z)}{1-\tilde{f}(z)} \bigg|_{z=z_\nu} \quad (A1)$$

which follows from Eq. (3.40). Inserting Eqs. (3.42) and (3.43) we arrive at

$$\sum_{k=0}^{\infty} W_k(\tau)=r\int_{\tau}^{\infty}dt''\int_{-\infty}^{\tau}dt'f(t''-t') . \qquad (A2)$$

On the other hand, because of Eqs. (2.5), (2.8), (2.11), and (3.43) we may write

$$r=\mathrm{Res}\frac{\tilde{f}(z)}{1-\tilde{f}(z)}\bigg|_{z=0}=\mathrm{Res}\frac{\tilde{f}(z)}{z\tilde{\varphi}(z)}\bigg|_{z=0}=\frac{\tilde{f}(0)}{\tilde{\varphi}(0)}=\frac{1}{\tilde{\varphi}(0)} , \qquad (A3)$$

where we made use of the facts that the functions $f(t)$ and $\varphi(t)$ vanish for $t\rightarrow\infty$, therefore possessing no pole at $z=0$, and that the function $f(t)$ is normalized to unity. Therefore we obtain, with the help of Eq. (3.41), the relation

$$\frac{1}{r}=\tilde{\varphi}(0)=\int_{0}^{\infty}d\tau''\int_{\tau''}^{\infty}d\tau' f(\tau') , \qquad (A4)$$

which, after appropriate transformation, yield the desired result

$$r\int_{\tau}^{\infty}dt''\int_{-\infty}^{\tau}dt'f(t''-t')=\sum_{k=0}^{\infty}W_k(\tau)=1 . \qquad (A5)$$

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