Stabilization and the zero-range models

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It is shown that the high-frequency approximation describes the lack of stabilization for the zero-range models in dimensions higher than 1. The same conclusion has been drawn from the exact solution of this model in the three-dimensional space. Hence, there are no contradictions between the predictions of the exactly soluble models and the high-frequency approximation. On the other hand both the exact numerical analysis and this approximation leads to the stabilization for the one-dimensional δ potential. This means that the high-frequency approximation does correctly predict both the presence and the absence of stabilization effects.

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The quantum processes that arise from the interaction of atoms, molecules, or solids with strong laser fields have recently attracted a lot of attention [1]. The zero-range potentials have proved to be the very convenient models for a qualitative analysis of such processes. In particular, they have been used for the investigation of multiphoton processes in atomic physics in the presence of a strong laser field. It appears that the zero-range potentials qualitatively well describe such processes as the low-frequency free-free transitions [2], the higher harmonic generation [3], the rescattering effects in the above threshold ionization [4], or the electron photodetachment [5]. On the other hand, we observe recently a very energetic discussion of what is known as the stabilization of ionization in the presence of a superintense laser field [6]. Zero-range potentials have also been used for the analysis of this phenomenon with sometimes contradicting conclusions. For instance, it was claimed that the one-dimensional zero-range model describes both the stabilization effect [7] and the lack of it [8]. The recently published accurate numerical analysis of this model [9] shows that the stabilization does occur, although it is still not known whether the trajectories of the poles of the multichannel scattering amplitude in the complex energy plane can continuously be connected to the bound-state energy of the radiationless model.

For the three-dimensional problem the analysis of the stabilization effect could even be made more transparent, because there exists the exactly soluble case of a circularly polarized laser field [2,10], that allows us to write down an exact transcendental equation to determine the complex-valued energy. The asymptotic analysis of this equation shows [11] that there is no stabilization for this model, which was also confirmed to some extent by the numerical analysis [12]. On the basis of this finding the conclusion has been drawn [11] that there is every reason to believe that this statement remains true for the ground states of atoms, for instance the hydrogen atom. However, before making such generalizations we should be sure that the zero-range models for dimensions higher than 1 really contradict the high-frequency mechanism of the stabilization [13,14]. The aim of this Brief Report is to show that the high-frequency approximation does describe the lack of stabilization for zero-range interactions in dimensions higher than 1, therefore there is no contradiction between this approximation and the results that can be deduced from the exactly soluble models with local interactions.

The convenient starting point for the derivation of what is called the high-frequency approximation is the Schrödinger equation in the form (the units in which $\hbar=c=m=1$ are used here) [14]

$$i\partial_t \psi(\mathbf{r},t)=[-\tfrac{1}{2}\Delta+V(\mathbf{r}+\boldsymbol{\alpha}(t))y]\psi(\mathbf{r},t). \tag{1}$$

Let us stress, however, that by a proper analysis of the wave equation in both the position and the velocity gauges one will also end up with the same conclusions. Namely, that in order to discuss quantum processes governed by the exact time-dependent Schrödinger equation (1) it suffices in the high-frequency limit to consider the truncated system of the time-independent equations [13,14]

$$[E+\tfrac{1}{2}\Delta-V_0(\mathbf{r})]\psi_0(\mathbf{r})=0 \tag{2}$$

and

$$[E+n\omega+\tfrac{1}{2}\Delta-V_0(\mathbf{r})]\psi_n(\mathbf{r})=V_n(\mathbf{r})\psi_0(\mathbf{r}), \quad n=\pm 1,\pm 2,\ldots \tag{3}$$

for the Fourier components of the exact time-dependent wave function

$$\psi(\mathbf{r},t)=\sum_{n\in\mathbb{Z}} e^{-i(E+n\omega)t}\psi_n(\mathbf{r}), \tag{4}$$

in which $\mathbb{Z}$ is the set of all integers, E is a quasienergy, and $V_n(\mathbf{r})$ are the Fourier components of the time-dependent potential $V(\mathbf{r}+\boldsymbol{\alpha}(t))$. For quasibound states considered in this paper the quasienergy E is a complex quantity with a negative imaginary part. The analysis of corrections shows [14] that such a truncation is justified provided that the absolute value of E obtained from the basic equation (2) is smaller than the laser-photon energy ω. Let us emphasize however that it does not necessarily mean that the frequency has to be really large, because for very high intensities of the laser field all quasienergies of quasibound states collapse into the Rydberg states [15], for which the high-frequency condition can be fulfilled by optical fields.

Thus, if we want to observe the stabilization effect, we have to be sure that there are quasibound states where electrons can remain, i.e., that the effective potential V_0 supports at least one bound state. For potentials that are negative and vanish at infinity this is always true in the one-dimensional space, but it is not so for two- or three-dimensional cases [16]. In particular, as it will follow shortly, this statement is not valid for two- or three-dimensional zero-range medals, hence one should not expect stabilization for them. However, before discussing this subject we have to provide the mathematically rigorous definition of the zero-range interaction. It has been proven [17] that such an interaction can be considered as the limiting case of a separable potential. To be more specific, let us consider the time-independent Schrödinger equation with the nonlocal separable potential

$$(E+\tfrac{1}{2}\Delta)\psi(\mathbf{r})=-\lambda S(\mathbf{r})\int d^dr' S^*(\mathbf{r}')\psi(\mathbf{r}'), \qquad (5)$$

in which d is the space dimension, $\lambda>0$ is a coupling constant, and

$$S(\mathbf{r})=(\pi\sigma^2)^{-d/2}\exp\left(-\frac{\mathbf{r}^2}{\sigma^2}\right), \qquad (6)$$

with an arbitrary real and positive parameter σ. The local zero-range potential is attained in the limit $\sigma\to 0$, in which $S(\mathbf{r})\to\delta(\mathbf{r})$. However, in order to get the finite expression for the scattering amplitude we have to assume that the coupling constant λ in this limit behaves like

$$\lambda^{-1}=\begin{cases}(2|E_B|)^{-1/2}, & d=1\\ -[C+\ln(|E_B|\sigma^2)]/2\pi, & d=2\\ \dfrac{1}{\pi\sqrt{2\pi}}\left(\dfrac{1}{\sigma}-\sqrt{\pi|E_B|}\right), & d=3,\end{cases} \qquad (7)$$

where E_B is the energy of the bound state, whereas C is the Euler constant. It is clear from this expression that only for $d=1$ the inverse of the coupling constant λ^{-1} stays finite. This means that the two- and three-dimensional δ potentials are too strong to support the bound state of a finite energy, and in order to prevent the collapse of this bound state to $-\infty$ we have somehow to weaken such potentials. This goal can be achieved either by an infinitesimally small coupling constant, or by some extra operations that accompany the δ function [18] (it is not the aim of this Brief Report to prove the equivalence of these two approaches; let us only note that they lead to the same expressions for cross sections and ionization rates). Since the procedure described above is more convenient for the discussion of the high-frequency limit, therefore we shall use it in our further analysis.

The dynamics of a charged particle interacting with such a nonlocal potential and an electromagnetic plane wave in the dipole approximation is governed by the gauge-covariant Schrödinger equation [19]

$$\begin{aligned}(i\partial_t+\tfrac{1}{2}\Delta)\psi(\mathbf{r},t)=&-\lambda\exp[ie\mathbf{A}(t)\cdot\mathbf{r}]S(\mathbf{r}+\boldsymbol{\alpha}(t))\\ &\times\int d^dr'\exp[-ie\mathbf{A}(t)\cdot\mathbf{r}']\\ &\times S^*(\mathbf{r}'+\boldsymbol{\alpha}(t))\psi(\mathbf{r}',t),\end{aligned} \qquad (8)$$

in which $\ddot{\boldsymbol{\alpha}}(t)=e\mathscr{E}(t)$ and $\mathscr{E}(t)$ is a time-dependent oscillating electric field which is supposed to describe a monochromatic laser field. We assume that the laser field is circularly polarized,

$$\mathscr{E}(t)=\mathscr{E}_0[\mathbf{e}_x\cos(\omega t+\delta)+\mathbf{e}_y\sin(\omega t+\delta)], \qquad (9)$$

where $\mathscr{E}_0$ is the field amplitude, ω is the frequency, whereas δ is an arbitrary phase. Since σ is supposed to be infinitesimally small, hence $S(\mathbf{r})$ is very close to the δ function, therefore we shall neglect in our further discussions the exponentials $\exp[ie\mathbf{A}(t)\cdot\mathbf{r}]$ and $\exp[-ie\mathbf{A}(t)\cdot\mathbf{r}']$ in this equation. In other words, this means that in the limit $\sigma\to 0$ our nonlocal separable potential becomes the local one. Let us emphasize however that it does not mean we have made some approximations. The analysis that will follow can be performed with the full equation (8) and we have checked that all physical quantities in the zero-range limit are independent of these exponentials and that their role consists only in longer intermediate expressions.

Hence, Fourier analyzing the time-dependent Schrödinger equation we arrive at

$$(E+n\omega+\tfrac{1}{2}\Delta)\psi_n(\mathbf{r})=-\lambda\sum_{n'\in\mathbb{Z}}\int d^dr' V_{n-n'}(\mathbf{r},\mathbf{r}')\psi_{n'}(\mathbf{r}'), \qquad (10)$$

where

$$V_n(\mathbf{r},\mathbf{r}')=\sum_{k\in\mathbb{Z}}S_{n+k}(\mathbf{r})S_k^*(\mathbf{r}'),$$

$$S(\mathbf{r}+\boldsymbol{\alpha}(t))=\sum_{k\in\mathbb{Z}}S_k(\mathbf{r})e^{-ik\omega t}. \qquad (11)$$

The high-frequency approximation says that the infinite system of time-independent equations (10) can be truncated in the first step to the single equation for the elastic component ψ_0. Let us however not make such a drastic simplification and assume that the infinite system (10) can be truncated to any finite system of equations of the form

$$\begin{aligned}(E+n\omega+\tfrac{1}{2}\Delta)\psi_n(\mathbf{r})&\\ =-\lambda\sum_{n'\in\mathbb{Z}_0}\int d^dr' V_{n-n'}(\mathbf{r},\mathbf{r}')\psi_{n'}(\mathbf{r}'),&\\ n\in\mathbb{Z}_0,&\end{aligned} \qquad (12)$$

in which $\mathbb{Z}_0$ is an arbitrary finite subset of integers that contains 0. Our goal is to show that such a truncated system in the zero-range limit cannot support any bound states (provided of course that the laser field does not vanish), thus cannot describe the stabilization effect. To this end let us replace the system of differential equations (12) by the equivalent system of integral equations that already incorporates boundary conditions proper for quasibound states [19],

$$\psi_n(\mathbf{r})=-\lambda\sum_{n'\in\mathbb{Z}_0}\int d^dr'd^dr''G_0^-(\mathbf{r},\mathbf{r}'';E+n\omega)\times V_{n-n'}(\mathbf{r}'',\mathbf{r}')\psi_{n'}(\mathbf{r}'),\quad n\in\mathbb{Z}_0, \tag{13}$$

in which the Green function G_0^- is equal to

$$G_0^-(\mathbf{r},\mathbf{r}';E)=\mathrm{i}^{1+d/2}\int_0^\infty ds(2\pi s)^{-d/2}\times\exp\left(-\mathrm{i}Es-\mathrm{i}\frac{(\mathbf{r}-\mathbf{r}')^2}{2s}\right),\quad \mathrm{Im}E<0. \tag{14}$$

We explicitly assume that quasienergies for quasibound states are complex with negative imaginary parts. With such a choice of the Green function all integrals that appear further are well defined and convergent. Hence our conclusions are not affected by the fact that quasienergies are analytically continued into the lower half of the complex plane. Let us define further

$$F_{n,k}=\int d^drS^*_{n+k}(\mathbf{r})\psi_n(\mathbf{r}), \tag{15}$$

where the first index n belongs to $\mathbb{Z}_0$, whereas the second index k runs over the whole set of integers. One can easily prove that the system of integral equations (13) is now reduced to the system of algebraic equations

$$F_{n,k}=-\lambda\sum_{n'\in\mathbb{Z}_0}\sum_{k'\in\mathbb{Z}}\int d^drd^dr'S^*_{n+k}(\mathbf{r})G_0^-(\mathbf{r},\mathbf{r}';E+n\omega)\times S_{n+k'}(\mathbf{r}')F_{n',k'},\quad n\in\mathbb{Z}_0,\ k\in\mathbb{Z}. \tag{16}$$

The space integration is carried out using the well-known expressions for the Gauss integrals, and finally we end up with the finite for any k system of algebraic equations, which we write as (k plays the role of a parameter, which can only happen for the circularly polarized field)

$$\sum_{n'\in\mathbb{Z}_0}M^{(k)}_{nn'}F_{n',k}=0,\quad n\in\mathbb{Z}_0,\ k\in\mathbb{Z}, \tag{17}$$

where

$$M^{(k)}_{nn'}=\lambda^{-1}\delta_{nn'}+\mathrm{i}^{1+n+k+d/2}(2\pi)^{-d/2}\int_0^\infty ds(s+\mathrm{i}\sigma^2)^{-d/2}\times J_{n+k}\left(\frac{\alpha_0^2}{s+\mathrm{i}\sigma^2}\right)\exp\left(-\mathrm{i}(E+n\omega)s-\mathrm{i}\frac{\alpha_0^2}{s+\mathrm{i}\sigma^2}\right) \tag{18}$$

in which J_n is the Bessel function and $\alpha_0=-e\mathscr{E}_0/\omega^2$. Let us note that the second term of $M^{(k)}_{nn'}$ is independent of n', which again can only take place for the circularly polarized field.

We can now make the zero-range limit $\sigma\to 0$. In this limit the integral is divergent for small s only for $d=3$, whereas the convergence for large s is guaranteed by a supposed negative imaginary part of E. In order to calculate the exact form of the divergent term we split the integral into two parts, from 0 to ε and from ε to ∞, where ε is such that $\varepsilon\gg\sigma^2$ and $\varepsilon\ll\alpha_0^2$. In the second part we can put $\sigma=0$ in the intergand, whereas in the first part we expand the Bessel function in the asymptotic series and perform the remaining integrations term by term. Finally, we obtain the following expression for the matrix $M^{(k)}_{nn'}$,

$$M^{(k)}_{nn'}=-\frac{1}{\pi}\ln\sigma\delta_{nn'}+\mathscr{F},\quad d=2 \tag{19}$$

and

$$M^{(k)}_{nn'}=-\frac{1}{\pi\sigma\sqrt{2\pi}}\delta_{nn'}+\frac{1}{2\pi^2\alpha_0}\ln\sigma+\mathscr{F},\quad d=3, \tag{20}$$

in which $\mathscr{F}$ means "finite terms in the limit $\sigma\to 0$." It is clear that the determinants of these matrices cannot be equal to zero for any finite and physically accepted (i.e., for instance with the negative imaginary part) quasienergy E; let us remind the reader that $M^{(k)}_{nn'}$ are the matrices of a finite rank and in the strict high-frequency approximation they are just complex numbers. This means that the truncated system (12) does not possess quasibound states of finite quasienergies, hence one cannot expect the stabilization for this system. On the other hand, one can show that the infinite system (10) does support the quasibound state of the quasienergy that fulfill the well-known transcendental equation, and which leads to the lack of stabilization for asymptotically large intensities of radiation. Hence, both the high-frequency approximation and the exact solution of the zero-range models predict the same behavior of quantum systems irradiated by intense laser fields, and there is no reason to claim that the high-frequency approximation incorrectly describes the stabilization effect for more complicated systems such as the hydrogen atom.

To recapitulate, we have proved that there are two- and three-dimensional systems for which the stabilization does not occur; this fact does not contradict the high-frequency mechanism of this phenomenon. Moreover, as the one-dimensional model shows [9], the excited states are not necessarily indispensable for the stabilization, although they may be helpful.

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