Coulomb potential from a particle in uniform ultrarelativistic motion

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The Coulomb potential produced by an ultrarelativistic particle (such as a heavy ion) in uniform motion is shown in the appropriate gauge to factorize into a longitudinal $\delta(z-t)$ dependence times a simple two-dimensional potential solution in the transverse direction. This form makes manifest the source of the energy independence of the interaction.

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In the consideration of the electromagnetic interaction of ultrarelativistic heavy ions, to produce, for example, electron-positron pairs, one may assume uniform straight-line trajectories for the moving ions and then express the time-dependent classical effect as the Liénard-Wiechert potential

$$V(\boldsymbol{\rho},z,t)=\frac{\alpha Z(1-v\alpha_z)}{\sqrt{[(\mathbf{b}-\boldsymbol{\rho})/\gamma]^2+(z-vt)^2}}. \tag{1}$$

$\mathbf{b}$ is the impact parameter, perpendicular to the z axis along which the ion travels; $\boldsymbol{\rho}$, z, and t are the coordinates of the potential relative to a fixed target (or ion); α_z is the Dirac matrix; and Z,v, and γ are the charge, velocity, and γ factor of the moving ion ($\gamma=1/\sqrt{1-v^2}$), respectively. It has previously been shown that in the large-γ limit a gauge transformation on this potential can remove both the large positive and negative time contributions of the potential as well as the γ dependence [1]. If one makes the gauge transformation on the wave function

$$\psi=e^{-i\chi(\mathbf{r},t)}\psi', \tag{2}$$

where

$$\chi(\mathbf{r},t)=\frac{\alpha Z}{v}\ln[\gamma(z-vt)+\sqrt{b^2+\gamma^2(z-vt)^2}], \tag{3}$$

the interaction $V(\boldsymbol{\rho},z,t)$ is gauge transformed to

$$V(\boldsymbol{\rho},z,t)=\frac{\alpha Z(1-v\alpha_z)}{\sqrt{[(\mathbf{b}-\boldsymbol{\rho})/\gamma]^2+(z-vt)^2}}-\frac{\alpha Z[1-(1/v)\alpha_z]}{\sqrt{b^2/\gamma^2+(z-vt)^2}}. \tag{4}$$

The work of Toshima and Eichler [2] previouly utilized a similar transform to solve the difficulty in calculations with interactions that drop off slowly as $1/t$.

In the limit of large γ with an impact parameter not too large, $v=1$ and Eq. (4) can be expressed in a simple multipole decomposition [1]

$$\begin{aligned}
V(\boldsymbol{\rho},z,t)=\alpha Z(1-\alpha_z)\frac{2\pi}{r}
&\times\Bigg\{-\ln\left(\frac{r^2-t^2}{b^2}\right)\sum_l Y_l^0(\theta_t,0)Y_l^0(\theta,0),\quad r>\sqrt{b^2+t^2}\\
&+\sum_{m>0}\frac{2\cos m\phi}{m}\sum_l Y_l^m(\theta_t,0)Y_l^m(\theta,0)\\
&\times\left[\left(\frac{r^2-t^2}{b^2}\right)^{m/2},\quad |t|<r<\sqrt{b^2+t^2}\right.\\
&\left.+\left(\frac{b^2}{r^2-t^2}\right)^{m/2}\right]\Bigg\},\quad r>\sqrt{b^2+t^2},
\end{aligned} \tag{5}$$

where $\cos\theta_t = t/r$ and $V(\boldsymbol{\rho},z,t)=0$ in the region $r<|t|$. If one notices that for any m

$$\sum_l Y_l^m(\theta_t,0)Y_l^m(\theta,0)=\frac{1}{2\pi}\delta(\cos\theta-\cos\theta_t), \tag{6}$$

then the expression becomes

$$V(\boldsymbol{\rho},z,t)=\alpha Z(1-\alpha_z)\delta(z-t)\times\left\{-\ln\frac{\rho^2}{b^2},\ \rho>b\right.+\sum_{m>0}\frac{2\cos m\phi}{m}\times\left[\left(\frac{\rho}{b}\right)^m,\ \rho<b\right.\left.\left.+\left(\frac{b}{\rho}\right)^m\right]\right\},\quad \rho>b. \tag{7}$$

The expression now has cylindrical symmetry, with a δ function in $(z-t)$ and a Fourier series in ϕ, which involves only ρ and b in the coefficients.

The expression of Eq. (5) was obtained by ignoring terms of order $(\ln\gamma)/\gamma^2$ [1]. This means that to the same accuracy in $1/\gamma$ the $z-t$ dependence of the potential can be adequately represented by $\delta(z-t)$ in Eq. (7). We would like to find the function of the transverse variable $V(\rho)$ whose Fourier series is expressed in Eq. (7)

$$V(\boldsymbol{\rho},z,t)=V(\boldsymbol{\rho})\delta(z-t). \tag{8}$$

$V(\boldsymbol{\rho})$ may be evaluated from the form of Eq. (4):

$$\begin{aligned} V(\boldsymbol{\rho}) &= \int_{-\infty}^{\infty} V(\boldsymbol{\rho},z,t)dz \\ &= \alpha Z(1-\alpha_z)\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{[(\mathbf{b}-\boldsymbol{\rho})/\gamma]^2+(z-t)^2}} - \frac{1}{\sqrt{b^2/\gamma^2+(z-t)^2}}\right]dz \\ &= \alpha Z(1-\alpha_z)\ln\frac{z'+\sqrt{z'^2+[(\mathbf{b}-\boldsymbol{\rho})/\gamma]^2}}{z'+\sqrt{z'^2+b^2/\gamma^2}}\Bigg|_{-\infty}^{\infty}. \end{aligned} \tag{9}$$

The gauge transformed form allows the removal of the infinities and factors of γ in the evaluation of the integral. The result is

$$V(\boldsymbol{\rho},z,t)=-\delta(z-t)\alpha Z(1-\alpha_z)\ln\frac{(\mathbf{b}-\boldsymbol{\rho})^2}{b^2}. \tag{10}$$

One can construct the Fourier series for Eq. (10) and verify that the result is identical to Eq. (7).

The b^2 in the denominator of the logarithm is removable by a gauge transformation. If one wished to have a potential with the same gauge for all impact parameters one would remove it. The b^2 has no effect on the $m>0$ terms in Eq. (7), but just adds the constant seen in the $m=0$ term.

Of course, once the existence of the effective two-dimensional potential is established, one really does not have to carry out the integral of Eq. (9); the solution

$$V(\boldsymbol{\rho})=-\alpha Z(1-\alpha_z)\ln(\mathbf{b}-\boldsymbol{\rho})^2 \tag{11}$$

immediately follows from Maxwell's equations and symmetry.

This Brief Report has pointed out that implicit in the previously presented multipole decomposition formula Eq. (5) is the extremely simple cylindrically symmetrical expression Eq. (10). These expressions are valid from the smallest relevant impact parameters just outside nuclear touching and throughout the region where b is not too large and the interaction is strong enough to be nonperturbative. At large enough b/γ, the $\delta(z-t)$ form becomes a poor approximation, but then alternative perturbation theory expressions can be utilized for calculating processes such as electron-positron pair production [3].

Unfortunately, the result of Eq. (10) has no obvious application to coupled channels calculations utilizing the conventional angular momentum algebra [4–6], since radial multipole forms such as those of Eq. (5) are necessary when angular integration is done analytically. However, the reduction of the interaction from three dimensions to the two of Eq. (10) might make direct solution of the time-dependent Dirac equation, without using coupled channels, a viable alternative for the calculation of pair production induced by ultrarelativistic heavy ions.

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[1] A. J. Baltz, M. J. Rhoades-Brown, and J. Weneser, Phys. Rev. A **44**, 5569 (1991).
[2] N. Toshima and J. Eichler, Phys. Rev. A **42**, 3896 (1990).
[3] A. J. Baltz, M. J. Rhoades-Brown, and J. Weneser, Phys. Rev. A **48**, 2002 (1993).
[4] Klaus Rumrich, Klaus Momberger, Gerhard Soff, Walter Greiner, Norbert Grün, and Werner Scheid, Phys. Rev. Lett. **66**, 2613 (1991).
[5] A. J. Baltz, M. J. Rhoades-Brown, and J. Weneser, Phys. Rev. A **47**, 3444 (1993).
[6] A. J. Baltz, M. J. Rhoades-Brown, and J. Weneser, Phys. Rev. A **50**, 4842 (1994).