Causality and quantization of time-delay systems: A two-body model problem

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A model problem consisting of two isolated particles with a mutual interaction depending upon retarded and advanced positions and constrained to one-dimensional motion is analyzed herein. This system has a number of features in common with a previously considered one-particle double-delay model problem. The equations of motion have closed-form solutions and the systems are both deterministic and causal. In both cases, standard low-order approximations to the exact problem, in general, have spurious solutions, which must be recognized and removed in order to extract the physically meaningful parts and to proceed reasonably with quantization. In addition, the two-particle system has specific characteristics. The center-of-momentum motion separates out. The dependence of the solutions on the mass ratio of the two particles can be examined. Further, not only does the relative motion have the ordinary solution, in which the two particles move out of phase, it can also have an extraordinary solution, in which they move in phase with each other. Finally, the total generalized linear momentum and Hamiltonian can be evaluated and seen to be constants of the motion.

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I. INTRODUCTION

In an earlier work [1], hereafter referred to as I, it was noted that, at the present time, there is considerable interest in a diverse range of physical theories involving Lagrangian structures containing entities with higher than first-order derivatives. The reader is referred to I for references. A classic example is the Fokker-Wheeler-Feynman theory of electrodynamics [2] and its generalizations [3–5]. The exact formulation is multitimed, the particles' variables appearing not only at a common central time but also, pairwise, at relatively retarded and advanced times. Now, to obtain a canonical form and to proceed with quantization using conventional procedures, all quantities are power series expanded about the central time, yielding a single timed but, in general, infinite order formulation. Finally, the expansions are truncated at some low order, and either an order reduction technique ([6–10] and references therein) is used, resulting in an approximate first-order formulation, or the truncated form is treated exactly [11–17]. These two approaches can yield different results, in general. Unfortunately, no criteria or tests presently exist to determine the accuracy or validity of these approximations relative to the exact problem. The present work represents a further investigation of the relationship between the exact solutions to multitimed problems and their approximate solutions, as indicated above.

A one-particle model problem, introduced by Feynman and Hibbs [18], involving a central time and relatively retarded and advanced times and constrained to one-dimensional motion, was examined in I. Herein, the model is generalized to consist of two particles, again constrained to one-dimensional motion, and with each particle's variables appearing with the three times. This model problem is defined in Sec. II, which also contains its closed-form solutions. Although some features are similar to the one-particle problem, others are characteristic of a two-particle system with a mass ratio dependence and with specific relative motion relationships. The constants of the motion, being the total generalized linear momentum and the generalized Hamiltonian, are verified in this section also. Some details are relegated to appendices. In Sec. III, a few low-order approximate solutions are given and compared to the exact solutions. The steps required for quantization are summarized in this section, along with some discussion. Again the situation is similar to the one-particle case, with spurious modes being obtained in the approximate problem, in general. The existence of an extraordinary mode produces an added complication. Section IV closes with a summary, an analysis of the results, and further discussion.

II. THE MODEL PROBLEM AND ITS EXACT SOLUTION

The model problem considered herein is a simple extension of a one-particle model problem proposed by Feynman and Hibbs [18] to a two-particle model. The action is taken to be

$$J=\int \{\tfrac{1}{2}m_1\dot{x}_1(t)^2+\tfrac{1}{2}m_2\dot{x}_2(t)^2+\tfrac{1}{16}k[x_1(t)-x_2(t+\tau)-l_0]^2+\tfrac{1}{16}k[x_1(t)-x_2(t-\tau)-l_0]^2 + \tfrac{1}{16}k[x_2(t)-x_1(t+\tau)+l_0]^2+\tfrac{1}{16}k[x_2(t)-x_1(t-\tau)+l_0]^2\}dt. \tag{2.1}$$

TABLE I. ξ as a function of λ for various mass combinations.

	Exact solutions			Approximate solutions for $m_1=m_2$	
λ	$m_1=m_2$	$m_1=9m_2$	$m_1=99m_2$	First order	Second order
0.1	0.9988	0.9996	0.9999	0.9988	0.9987 693.69
1.0	0.9004	0.9968	0.9965	0.8944	0.9005 7.6934
2.0	0.7391	0.9551[a]	0.9958[a]	0.7071	0.7420 2.3344
2.3	0.6962 0.7851[a]	0.9696[a]	0.9972[a]	0.6562	0.7008 1.8688 0.7438[a]
3.0	0.6099 0.9972[a]	0.9991[a]	0.9999[a]	0.5547	0.6200 1.2416 0.8607[a]
3.5	0.5588 0.9876[a]	0.9949[a] 0.1768[a]	0.9994[a]	0.4961	0.5744 0.9847 0.8122[a]
6.0	0.3900 0.9794 0.8877 0.7596[a]	0.9958 0.3046	0.9996	0.3162	(Goes complex at $\lambda=$ 5.085) 0.5443[a]
11.0			0.9950[a] 0.0671[a]		

[a]Denotes instantaneously parallel motion; otherwise indicates the usual antiparallel motion.

The symbols have their usual meanings: m_1 and m_2 are the constant masses of the two particles, and k, l_0, and τ are fixed constants. The variational procedure is carried out as indicated in I, with the end-point regions being treated in a like manner. The two equations of motion are

$$m_i\ddot{x}_i=\tfrac{1}{4}k[2x_i(t)-x_j(t+\tau)-x_j(t-\tau)-2S_il_0]. \quad (2.2)$$

Here, $i=1$ or 2 and $j=2$ or 1, respectively; $S_1=+1$ and $S_2=-1$.

There is a net "center-of-momentum" motion for this system with

$$x_i(t)=X_i(t)=l_0\delta_{il}+X_0+Vt \quad (i=1,2), \quad (2.3)$$

with X_0 and V being constants. This suggests that the complete solution can be written as

$$x_i(t)=X_i(t)+S_iA_iy(t), \quad (2.4)$$

with

$$m_iA_i\ddot{y}(t)=\tfrac{1}{4}k\{2A_iy(t)+A_j[y(t+\tau)+y(t-\tau)]\}, \quad (2.5)$$

where $i=1$ or 2 and $j=2$ or 1. As for the previous model problem in I, the physical solutions are taken to be

$$y(t)= \cos(\omega t+\delta). \quad (2.6)$$

This is consistent with the generalized Hamiltonian being a constant of the motion and results in the two coupled equations

$$[m_i\omega^2+k/2]A_i+[(k/2)\cos(\omega\tau)]A_j=0. \quad (2.7)$$

Setting $k=-2\mu\omega_0^2$, with $\mu=m_1m_2/(m_1+m_2)=m_1m_2/M$ being the usual reduced mass, the condition for a nontrivial solution yields

$$\cos^2(\omega\tau)=[1-M\omega^2/m_1\omega_0^2][1-M\omega^2/m_2\omega_0^2]. \quad (2.8)$$

Notice that the relative amplitudes are frequency dependent. Converting to dimensionless variables $\lambda=\omega_0\tau$ and $\xi=\omega/\omega_0$, gives

$$\xi^4-\xi^2+(\mu/M)\sin^2(\lambda\xi)=0. \quad (2.9)$$

Although similar to the previous one-particle model problem in I, the allowed frequency spectrum for the two-body problem has some distinctive features, being dependent not only on λ but also on μ. A sampling of the allowed frequencies is given in Table I. An interesting point is that these solutions imply that the equal mass system is the most likley candidate for extra modes. This is fortuitous, since many expressions simplify for this case. For example, here A_2 becomes equal to $\pm A_1$ and is independent of frequency, apart from the sign. The positive sign is the ordinary solution, in which the two particles move out of phase, occurring at small λ (τ) and continuing for all λ (τ). The minus sign is an extraordinary mode, in which, instantaneously, the two masses move in phase; but because of the time delays, they both undergo oscillatory motion with zero net displacement. This mode is not supported until the delay time becomes sufficiently long.

This problem has the standard constants of the motion, which can be identified as follows [7,8,12–17,19]. The total generalized linear momentum is (see Appendix A)

$$P=p_1^1+p_2^1, \tag{2.10}$$

where

$$p_i^p=\sum_{m=p}^{\infty}(-1)^{m-p}\frac{d^{m-p}}{dt^{m-p}}\frac{\partial L}{\partial x_i^{(m)}} \quad (p\geq 1). \tag{2.11}$$

Here $x_i^{(m)}$ denotes the mth time derivative of x_i. Notice that this form assumes that the solutions belong to C^∞, consistent with (2.3) and (2.6). To evaluate these quantities the Lagrangian, being the integrand of (2.1), must be repesented by its power series about t. After some straightforward algebra, one obtains (A9)

$$p_i^1=m_i x_i^{(1)}+\frac{k}{8}\sum_{n=1}^{\infty}\frac{\tau^{n+1}}{(n+1)!}[1-(-1)^n][x_i^{(n)}+x_j^{(n)}]. \tag{2.12}$$

Hence (A10),

$$P=\sum_{i=1}^{2}\left\{m_i x_i^{(1)}+\frac{k}{2}\sum_{n=0}^{\infty}\frac{\tau^{2n+2}}{(2n+2)!}x_i^{(2n+1)}\right\}. \tag{2.13}$$

For the solutions given by (2.3), (2.4), and (2.6) this reduces to

$$\begin{aligned}P&=[m_1+m_2+\tfrac{1}{2}k\tau^2]V-(1/\omega)\{[(m_1\omega^2+k/2)A_1+(k/2)\cos(\omega\tau)A_2]\\&\quad-[(m_2\omega^2+k/2)A_2+(k/2)\cos(\omega\tau)A_1]\}[\sin(\omega t+\delta)]\\&=[m_1+m_2+\tfrac{1}{2}k\tau^2]V.\end{aligned} \tag{2.14}$$

The last step follows from (2.7). Thus, the total generalized linear momentum is a constant depending solely upon the center-of-momentum (CM) motion. Since the equations of motion are linear, this result is valid for any linear combination of solutions for different allowed frequencies.

We remark that, if the usual classical result is used for the total linear momentum, one has

$$\begin{aligned}P(\text{classical})&=m_1x_1^{(1)}(t)+m_2x_2^{(1)}(t)\\&=(m_1+m_2)V-\omega(m_1A_1-m_2A_2)[\sin(\omega t+\delta)].\end{aligned} \tag{2.15}$$

Now, although it is true that P(classical) is not instantaneously a constant of the motion, its time average is, and hence the internal time delays do not generate a macroscopic motion of the system in the absence of external forces.

The generalized Hamiltonian is given by [7,8,12–17,19]

$$H=\sum_{j=1}^{2}\sum_{p=1}^{\infty}x_j^{(p)}p_j^n-L \tag{2.16}$$

and is a constant of the motion by definition. As for the total linear momentum, it becomes (see Appendix B)

$$H=H_{\rm CM}+H_{\rm rel}, \tag{2.17}$$

where

$$H_{\rm CM}=\tfrac{1}{2}(m_1+m_2+\tfrac{1}{2}k\tau^2)V^2 \tag{2.18}$$

and

$$H_{\rm rel}=\tfrac{1}{2}m_1\omega^2A_1^2+\tfrac{1}{2}m_2\omega^2A_2^2-\tfrac{1}{4}k\omega\tau\,\sin(\omega\tau)A_1A_2. \tag{2.19}$$

We conclude this section by noting that this two-particle system is just as deterministic and causal as the one-particle system considered in I. This is consistent with the results found earlier [20] for the Fokker-Wheeler-Feynman model [2] and the Weiss model [4]. A center-of-momentum motion exists, which separates out from the relative motion. It is interesting that, along with the ordinary relative motion in which the two particles move out of phase, in some circumstances an extraordinary relative motion can exist in which the two particles move in phase. Further, constants of the motion exist, being generalizations of the no-time-delay problem.

III. APPROXIMATE SOLUTIONS

In this section we consider the consequences of truncating the expansion, given by (A3), for the Lagrangian at various low orders of τ. Equation (A3) can be rewritten as

$$L=\tfrac{1}{2}m_1x_1^{(1)^2}+\tfrac{1}{2}m_2x_2^{(1)^2}+\tfrac{1}{4}k(x_1-x_2-l_0)^2+\tfrac{1}{8}k\tau^2\left\{2x_1^{(1)}x_2^{(1)}+\frac{d}{dt}[(x_1-x_2-l_0)(x_1^{(1)}-x_2^{(1)})]\right\}$$
$$-\tfrac{1}{48}k\tau^4\left\{x_1^{(2)}x_2^{(2)}-\frac{1}{2}\frac{d}{dt}[3x_1^{(1)}x_1^{(2)}+x_1^{(1)}x_2^{(2)}+x_1^{(2)}x_2^{(1)}+3x_2^{(1)}x_2^{(2)}]\right\}+\cdots . \quad (3.1)$$

Here the Lagrangian has been put into minimal form, as discussed in earlier papers [1,15–17]. The equations of motion are the same as those given by (2.2) on expanding in τ and truncating at the same order. Thus,

$$m_ix_i^{(2)}=\tfrac{1}{2}k(x_i-x_j-l_0S_i)-\tfrac{1}{4}k\tau^2x_j^{(2)}-\tfrac{1}{48}k\tau^4x_j^{(4)}+\cdots . \quad (3.2)$$

The zero-order approximation, that is, the first three terms of (3.1), gives the standard two-particle linear coupled oscillator problem, the only solution being

$$A_2=(m_1/m_2)A_1, \quad \xi=\omega/\omega_0=1. \quad (3.3)$$

This solution is valid only for $\lambda=\omega_0\tau\simeq0$, as one should expect. For comparison with higher-order approximations, the quantization process for this case is briefly reviewed. Rewrite (2.4) as

$$x_i=l_0\delta_{il}+X(t)+S_iy_i, \quad (3.4)$$

with

$$X(t)=X_0+Vt, \quad y_i=A_iy. \quad (3.5)$$

The conversion to center-of-momentum and relative coordinate requires the use of the definition of the relative coordinate

$$x_1-x_2-l_0=y_1+y_2=Y, \quad (3.6)$$

as well as the condition of conservation of total linear momentum, the latter being

$$m_1y_1-m_2y_2=0. \quad (3.7)$$

Hence, as usual,

$$y_i=(m_j/M)Y. \quad (3.8)$$

Thus,

$$L_0=\tfrac{1}{2}MX^{(1)2}+\tfrac{1}{2}\mu Y^{(1)2}+\tfrac{1}{4}kY^2. \quad (3.9)$$

The canonical coordinates are the center-of-momentum and relative variables X and Y, respectively. The canonical momenta, Hamiltonian, and quantization are standard.

The second-order solution, that is, using the first four terms in (3.1), gives

$$A_2/A_1\geq0, \quad \xi^2=[1-\mu\lambda^2/M]/[1-\mu\lambda^4/4M]. \quad (3.10)$$

For $m_2=m_1$, $\xi^2=1/[1+\lambda^2/4]$, which is defined for the entire range of λ and is tabulated in Table I. This gives a reasonable approximation to ξ for the entire range of λ considered. No indication of the extraordinary mode is seen. For $m_2\neq m_1$, ξ^2 goes negative for $\lambda^2>M/\mu$, occurring before the zero in the denominator and giving cutoffs for λ at 3.3 and 10.1 for $m_1=9m_2$ and $m_1=99m_2$, respectively. From Table I, this indicates that this approximation is actually only valid for $\lambda\lesssim1$. The reduction to canonical form is similar to that for the zero-order approximation. Equations (3.4), (3.5), and (3.6) are the same. In (3.7), (3.8), and (3.9), m_i, M, and μ are replaced by m_i', M_1, and μ', where

$$m_i'=m_i+\tfrac{1}{4}k\tau^2 \text{ and } M'=m_1'+m_2', \quad (3.11)$$

$$\mu'=(m_1'm_2')/M'-\tfrac{1}{4}k\tau^2. \quad (3.12)$$

The internal mode frequency (3.10) is included exactly. Thus, conversion to canonical form and quantization are again standard for the region of validity; i.e., $\lambda\lesssim1$.

The fourth-order equations, that is, using all terms in (3.1) except the total time derivatives, corresponding to (2.7), are

$$[(m_i/\mu)\xi^2-1]A_i=[1-\tfrac{1}{2}\lambda^2\xi^2+\tfrac{1}{24}\lambda^4\xi^4]A_j. \quad (3.13)$$

For $m_2=m_1$, as for the exact problem, one has solutions for $A_2=A_1$, the ordinary case, and for $A_2=-A_1$, the extraordinary case. In the first case, the solutions are

$$\xi_\pm^2=\frac{24}{\lambda^4}\{(1+\lambda^2/4)\pm[1+\lambda^2/2-\lambda^4/48]^{1/2}\}. \quad (3.14)$$

$\xi_\pm$ are given in Table I, one sees that ξ_- is a reasonable approximation for one exact root for $\lambda\lesssim5$, whereas ξ_+ always represents a spurious root. The approximation breaks down for $\lambda\gtrsim5$. In the second case, there is a single nonzero root

$$\xi_e^2=\frac{12}{\lambda^4}[\lambda^2-4]. \quad (3.15)$$

ξ_e becomes real at $\lambda=2$ and by $\lambda=2.3$ is a moderate approximation to the extraordinary exact root for the remaining values of λ. For $m_2<m_1$, the secular equation from (3.13) is a fourth-order polynomial in ξ^2, with one root being $\xi^2=0$. For $m_1=99m_2$, for example, the three remaining roots, for $\lambda\lesssim10$, consist of one approximating the true root and of two spurious roots, one giving ξ real and large and the other giving ξ pure imaginary and large. In this instance, some order reduction technique [6–10] would be appropriate, although care would be required to get the phases correct.

To proceed to a canonical form, the Lagrangian must be put into a suitable form. Equations (3.4), (3.5), and (3.6) apply unchanged. The corresponding conserved total generalized linear momentum [see (2.10) and (2.11)] now gives

$$m_1' x_1^{(2)} + m_2' x_2^{(2)} + \tfrac{1}{48} k \tau^4 [x_1^{(4)} + x_2^{(4)}] = 0 \quad (3.16)$$

or

$$m_1' y_1 - m_2' y_2 + \tfrac{1}{48} k \tau^4 [y_1^{(2)} - y_2^{(2)}] = 0 . \quad (3.17)$$

Although (3.6) and (3.17) can be solved for y_1 and y_2 as power series in τ^4 for unequal masses, the algebra becomes lengthy and it is no more informative than the equal mass case. Thus, for the remainder of the disussion for this approximation, only equal masses are considered. In this case, the solution is

$$y_i = Y/2 . \quad (3.18)$$

Now, one finds ($k = -2\mu\omega_0^2$)

$$L_4 = \tfrac{1}{2} M' X^{(1)2} + \tfrac{1}{2} \mu' Y^{(1)2} - \tfrac{1}{2} \mu \omega_0^2 Y^2 - \tfrac{1}{96} \mu \omega_0^2 \tau^4 Y^{(2)2} . \quad (3.19)$$

Notice that, to arrive at this form, one must explicitly invoke the condition $X^{(2)} = 0$. This is necessary to maintain consistency with the equations of motion. It is also consistent with the translational invariance of the Lagrangian [21]. The relative part is similar to a model problem discussed previously [17], with the most straightforward method being a conversion to a problem of Lagrange. This is achieved by defining the constraint

$$Y^{(1)} - \omega_0 Z = 0 \quad (3.20)$$

and the effective Lagrangian

$$\Gamma_r = \tfrac{1}{2} \mu' \omega_0^2 Z^2 - \tfrac{1}{2} \mu \omega_0^2 Y^2 - \tfrac{1}{96} \mu \lambda^4 Z^{(1)2} + \gamma (Y^{(1)} - \omega_0 Z) , \quad (3.21)$$

where γ is a Lagrange multiplier. Formally, Γ_r is a first-order Lagrangian, and all of the standard procedures apply. The canonical coordinates are Y and Z, and the canonical momenta are

$$P_Y = \gamma , \quad P_Z = -\tfrac{1}{48} \mu \lambda^4 Z^{(1)} . \quad (3.22)$$

The corresponding Hamiltonian is

$$H_r = \omega_0 P_Y Z - (24/\mu\lambda^4) P_Z^2 - \tfrac{1}{2} \mu' \omega_0^2 Z^2 + \tfrac{1}{2} \mu \omega_0^2 Y^2 . \quad (3.23)$$

The four canonical equations of motion contain the constraint, the second part of (3.22), and the original equation of motion. Thus, this problem has exactly the same information as the original problem. In principle, (3.23) can be quantized directly. However, H_r is not at all informative in its present form. Thus, we first apply a canonical transformation that diagonalizes it. This is now known to be possible [17] and is done by setting

$$Y = \alpha_1 q_1 + (\alpha_1/\mu\omega_0\xi_+) p_2 ,$$

$$Z = \alpha_1 \xi_+ q_2 + (\alpha_1/\mu\omega_0) p_1 ,$$

$$P_Y = \mu\omega_0 (\alpha_1/\xi_+) q_2 + (\alpha_1/\xi_-^2) p_1 ,$$

$$P_Z = \mu\omega_0 (\alpha_1/\xi_+^2) q_1 + (\alpha_1/\xi_+\xi_-^2) p_2 , \quad (3.24)$$

with, for equal masses,

$$\mu' = \mu(1 + \lambda^2/4) , \quad (3.25)$$

$$\alpha_1^2 = \xi_+^2 \xi_-^2 / (\xi_+^2 - \xi_-^2) , \quad (3.26)$$

$\xi_\pm^2$ being given by (3.14). H_r reduces to

$$H_r = [(1/2\mu) p_1^2 + \tfrac{1}{2} \mu \omega_0^2 \xi_-^2 q_1^2] - [(1/2\mu) p_2^2 + \tfrac{1}{2} \mu \omega_0^2 \xi_+^2 q_2^2] . \quad (3.27)$$

As for the exact problem, the Hamiltonian is a constant of the motion and separates for the two allowed solutions. It is not the total energy of the system; however, there is no guarantee that if one starts from a particular Lagrangian, this will be the case. Nevertheless, the total energy can now be identified and quantization achieved by standard procedures.

At this point, it is useful to discuss the above derivations in terms of two scenarios. The first scenario, which is the usual situation for physical problems, is that no essential information is available about the exact solutions. In this case, no value judgment can be made about the physical reality of either mode and no basis exists for the rejection of the one mode. Some additional information must be invoked, such as comparison with a physical system. Furthermore, no indication of the extraordinary mode appears and we would have no basis to include such a mode. The second scenario, being the present situation, is that considerable information is known about the exact problem. In the present case, it is known that the ξ_+ mode is spurious and must be rejected. To include the extraordinary mode, (3.18) needs to be replaced by

$$y_i = Y/2 + S_i Y_e , \quad (3.28)$$

where Y_e gives the extraordinary contribution. Now (3.6) is still satisfied; however, (3.17) cannot be satisfied. Also, the direct substitution of (3.28) into (3.4) and then into (3.1) gives

$$L_4' = \tfrac{1}{2} M' [X^{(1)} + Y_e^{(1)}]^2 + \tfrac{1}{24} \mu \omega_0^2 \tau^4 Y_e^{(2)2} + \tfrac{1}{2} \mu' Y^{(1)2} - \tfrac{1}{2} \mu \omega_0^2 Y^2 - \tfrac{1}{98} \mu \omega_0^2 \tau^2 Y^{(2)2} . \quad (3.29)$$

Thus, the center-of-momentum motion and the extraordinary motion do not separate, contrary to the exact problem. Therefore, to include the extraordinary mode, it is necessary to invoke some *ad hoc* procedures. For example, if one states that Y_e is purely an internal coordinate and that it must be considered in the center-of-momentum frame, $X^{(1)}$ would be set to zero in (3.29). This gives the additional Lagrangian for the extraordinary mode

$$L_{4e} = \tfrac{1}{2} M' Y_e^{(1)2} + \tfrac{1}{24} \mu \omega_0^2 \tau^4 Y_e^{(2)2} . \quad (3.30)$$

The equation of motion yields (3.15), and quantization can be achieved in the standard way by setting

$$Y_e^{(1)} = \frac{(12)^{1/2}}{\lambda^2} \omega_0 Z_e , \quad M' = -\mu(\lambda^2 - 4) . \quad (3.31)$$

The fact that M' goes negative with the onset of the extraordinary mode also suggests that, at this point, this approximation is no longer valid for a description of the center-of-momentum motion. Now

$$L_{4e}=\frac{1}{2}\mu Z_e^{(1)2}-\frac{1}{2}\mu\omega_0^2\xi_e^2Z_e^2. \quad (3.32)$$

Thus, quantization proceeds in the standard way, giving a reasonably valid result. Again one sees that some knowledge of the exact solutions is essential in properly extracting the correct information from such approximations, allowing the identification of spurious modes and the inclusion of omitted modes.

We close this section by noting that (2.14) and (2.18) also suggest that this model inadequately describes a center-of-momentum motion with equal masses for $\lambda^2>2$. Alternately, this could be interpreted as a physical limit on λ. Thus, ultimately a physial constraint will be required to fix this parameter.

IV. DISCUSSION

An analysis of the model problem introduced by Feynman and Hibbs [18], consisting of a single particle constrained to move in one dimension with an interaction dependent upon both a central time and relatively retarded and advanced times, was given in I. Herein, this model is generalized to the case of two particles constrained to move in one dimension and with a mutual interaction having a similar time dependence in both particle's variables. There are a number of similarities between the two models. Both have exact closed-form solutions. Thus, both the deterministic and causal nature of the equations of motion can be clearly demonstrated [20,22,23]. Also, in the usual approximations, where power-series expansions are made about the central time and truncated at some order, in general spurious solutions are generated. These must be identified and removed at some suitable point in the analysis. Of course, the two-particle system introduces a number of new characteristics. In this case, a center-of-momentum motion and the relative motion can be identified and their respective solutions obtained independently. The total generalized momentum is a constant of the motion and is a function solely of the center-of-momentum motion. The generalized Hamiltonian separates into the center-of-momentum and relative contributions, each of which is a constant of the motion. The dependence of the solutions on the ratio of the two masses has been determined (Table I). This model indicates that the most likely candidate for additional modes to exist, apart from the ordinary modes, is the equal mass case. In this instance there can be an extraordinary mode. It is hoped that the analyses of these model problems will assist in the study of more complex realistic models.

Perhaps at this point a few more general comments are appropriate. Our objectives have been much more modest than those of Hoyle and Narliker [24] (and references therein). Their objective has been to consider the possibility that the Fokker-Wheeler-Feynman–like models [2–5,23] provide complete cosmological theories. We, on the other hand, have considered that at this time the lack of an accepted Lagrangian-based relativistic classical dynamics theory on a par with the nonrelativistic case is anomalous and represents a gap in our understanding of physical reality. Our first objective has been to remove this gap. Thus, the questions asked and the answers sought are, in general, quite different for our different objectives. For our purposes, the only available candidates are the Fokker-Wheeler-Feynman–like models. These models, by construction, satisfy time-reversal symmetry, the equations of motion for each particle can be written in proper covariant form, and the theory can be seen to have a particle symmetric structure [2–5,25]. In addition, one requires the equations of motion for a finite number of particles in a local region and without external influences to be both deterministic and causal, just as in the nonrelativistic case. We have accomplished this in a series of preceding works [5,15–17,20,25]. Of course, the simple Newtonian interpretation of the equations of motion must be abandoned. However, the variational principle does not guarantee any particular interpretation. The next step we have considered is the quantization of such a theory, this being a major motive for I and this work. Thus, we are proceeding just as in the nonrelativistic case. One attempts to employ the conventional rules of quantizing classical descriptions, and external interactions are added in the same way. This last step automatically suppresses the possibility of these models yielding cosmological theories and ignores the questions asked by Hoyle and Narliker.

There is another fundamental point that has not yet been adequately addressed in any of the general discussions [2,24]. This is the question of the physical content or interpretation of any particular form of the Lagrangian, that is, the integrand, appearing in the variational integral. We have shown [5,25] that these Lagrangians can be written as

$$L=L_f+\frac{1}{2}(V_{12}^R+V_{12}^A) \quad (4.1a)$$

$$=L_f+\frac{1}{2}(V_{21}^R+V_{21}^A)+dF_1/dt \quad (4.1b)$$

$$=L_f+\frac{1}{4}(V_{12}^R+V_{12}^A+V_{21}^R+V_{21}^A)+dF_2/dt \quad (4.1c)$$

$$=L_f+\frac{1}{2}(V_{12}^R+V_{21}^R)+dF_3/dt \quad (4.1d)$$

$$=L_f+\frac{1}{2}(V_{12}^A+V_{21}^A)+dF_4/dt. \quad (4.1e)$$

Here L_f represents the two free-particle parts; $V_{ij}^{R(A)}$ represents the interaction of the ith particle with the retarded (advanced) fields of the jth particle. The expressions have been written for two particles merely for simplicity. All arguments can be extended straightforwardly to any number of particles. Note that, similarly, all advanced times can be removed from the model problem considered herein (see I). Each of the divergences satisfies the generalized Euler-Lagrange equations of motion exactly; that is, they produce identities that can be removed. Now, the straightforward application of the variational principle to (4.1a) for particle 1 leads to the standard form of the equation of motion; that is,

$$\frac{d\vec{p}_1}{dt}=\frac{1}{2}(\vec{F}_{12}^R+\vec{F}_{12}^A). \quad (4.2)$$

Next, when the variational principle is applied to (4.1a) for particle 2, its equation of motion does not take a form similar to (4.2) until the identity contained in dF_1/dt is removed, in one way or another. Now, asserting that (4.2) has a valid physical interpretation is equivalent to claiming that (4.1a) is physically meaningful for particle 1 but contains an unphysical part for particle 2 through dF_1/dt. We consider this position to be unacceptable and that, in fact, a particle symmetric form will most likely produce the correct physical interpretation. Quite probably (4.1d) without dF_3/dt will be the physical form. It contains no advanced terms and yet retains all of the features of the other forms. In this case, the radiation reaction comes from terms like $(V_{21}^R - V_{12}^R)/2$ coupled with the characteristics of the local universe.

Although many of these comments are speculative, they are allowed within the context of these models; we consider them more physically acceptable than invoking the concept of the future affecting the present, a concept that is unnecessary in all of these models. We also consider this feature of the theory to be fundamental in making it deterministic and causal. Although it is often convenient to display and work with the time-symmetric equations of motion, they can clearly be cast in time-asymmetric forms with no advanced terms appearing by adding identities. Both forms of the equations of motion necessarily have identical solutions. This raises the possibility that absorber theory may contain a nonphysical component and that subsequent conclusions may be unphysical.

We close by noting that Schulman [26] has recently examined, from a completely different perspective, another one-dimensional one-particle double–time-delay model system to the one discussed in I. His model also belongs to the same class as the Fokker-Wheeler-Feynman–like models in the sense that the advanced times can be removed from the Lagrangian by adding divergences.

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APPENDIX A

To consider the constants of the motion, it is necessary to assume that the exact Lagrangian, that is, the integrand of (2.1), can be converted to an infinite order Lagrangian. This requires the solution to belong to C^∞, being consistent with (2.3), (2.4), and (2.6). It is thus convenient to set

$$x_i(t+\tau)=x_i(t)+\Delta x_i^+ = x_i(t)+\sum_{n=1}^{\infty}\frac{\tau^n}{n!}x_i^{(n)}(t) \quad \text{(A1)}$$

and

$$x_i(t-\tau)=x_i(t)+\Delta x_i^- = x_i(t)+\sum_{n=1}^{\infty}(-1)^n\frac{\tau^n}{n!}x_i^{(n)}(t), \quad \text{(A2)}$$

where $x_i^{(n)}(t)$ represents the nth time derivative of $x_i(t)$. The Lagrangian then becomes

$$L=\tfrac{1}{2}m_1[x_i^{(1)}]^2+\tfrac{1}{2}m_2[x_2^{(1)}]^2+\tfrac{1}{4}k[x_1(t)-x_2(t)-l_0]^2+\tfrac{1}{8}k[x_1(t)-x_2(t)-l_0]\{[\Delta x_1^+ +\Delta x_1^-]-[\Delta x_2^+ +\Delta x_2^-]\} + \tfrac{1}{16}k\{[\Delta x_1^+]^2+[\Delta x_1^-]^2+[\Delta x_2^+]^2+[\Delta x_2^-]^2\}. \quad \text{(A3)}$$

Now, in this form, the equations of motion become [7,8,12–14]

$$\frac{dp_i^1}{dt}=\frac{\partial L}{\partial x_i}, \quad \text{(A4)}$$

where the first-order generalized linear momentum is defined by

$$p_i^1=\sum_{m=0}^{\infty}(-1)^m\frac{d^m}{dt^m}\frac{\partial L}{\partial x_i^{(m+1)}}. \quad \text{(A5)}$$

It follows immediately from (A3) and (A4) that $P=p_1^1+p_2^1$ is a constant of the motion. It is useful to verify this relationship and to examine the form of P.

Because $x_i^{(1)}$ appears uniquely in (A3), it is convenient to evaluate the partial derivative of L with respect to this quantity separately. The manipulations are straightforward, and one obtains

$$\frac{\partial L}{\partial x_i^{(1)}}=m_i x_i^{(1)}+\frac{k\tau}{8}\sum_{n=1}^{\infty}\frac{\tau^n}{n!}[1-(-1)^n]x_i^{(n)}. \quad \text{(A6)}$$

Further, for $m>0$,

$$\frac{\partial L}{\partial x_i^{(m+1)}}=\frac{k\tau^{m+1}}{8(m+1)!}\left\{[1+(-1)^{m+1}](x_1-x_2-l_0)S_i + \sum_{n=1}[1+(-1)^{n+m+1}]\frac{\tau^n}{n!}x_i^{(n)}\right\}, \quad \text{(A7)}$$

where $S_1=1$ and $S_2=-1$. Thus,

$$\frac{d^m}{dt^m}\frac{\partial L}{\partial x_i^{(m+1)}}=\frac{k\tau^{m+1}}{8(m+1)!}\left\{[1+(-1)^{m+1}][x_1^{(m)}-x_2^{(m)}]S_i + \sum_{n=1}[1+(-1)^{n+m+1}]\frac{\tau^n}{n!}x_i^{(n+m)}\right\}. \quad \text{(A8)}$$

Hence,

$$p_i^1=m_i x_i^{(1)}+\tfrac{1}{8}k\tau\sum_{n=1}[1-(-1)^n]\frac{\tau^n}{(n+1)!}x_j^{(n)}+\tfrac{1}{8}k\tau^2 x_i^{(1)}$$

$$+\tfrac{1}{8}k\tau\sum_{n=2}[1-(-1)^n]\frac{\tau^n}{(n+1)!}\left\{n-\sum_{q=0}^{n-2}\frac{(-1)^q(n+1)!}{(q+2)!(n-q-1)!}\right\}x_i^{(n)}$$

$$=m_i x_i^{(1)}+\tfrac{1}{8}k\tau\sum_{n=1}[1-(-1)^n]\frac{\tau^n}{(n+1)!}[x_i^{(n)}+x_j^{(n)}]. \tag{A9}$$

The last step follows by recognizing that the sum can be evaluated as $n+(-1)^n$ by comparing to $(1-1)^{n+1}$, that only odd n contributes, and by combining terms. Finally,

$$P=\sum_{i=1}^{2}\left\{m_i x_i^{(1)}+\tfrac{1}{4}k\sum_{n=1}[1-(-1)^n]\frac{\tau^{n+1}}{(n+1)!}x_i^{(n)}\right\}=\sum_{i=1}^{2}\left\{m_i x_i^{(1)}+\tfrac{1}{2}k\sum_{n=0}\frac{\tau^{2n+2}}{(2n+2)!}x_i^{(2n+1)}\right\}. \tag{A10}$$

Now, using the solutions (2.3), (2.4), and (2.6), we see that (2.14) follows.

APPENDIX B

This type of system also has a generalized Hamiltonian, which is a constant of the motion, by definition [7,8,12–17,19], and which is

$$H=\sum_i\sum_{m=1}x_i^{(m)}p_i^m-L, \tag{B1}$$

where

$$p_i^m=\sum_{l=0}(-1)^l\frac{d^l}{dt^l}\frac{\partial L}{\partial x_i^{(l+m)}}. \tag{B2}$$

The expressions for $m=1$ are given in (A9); those for $m>1$ can be readily found by using (A7) and are

$$p_i^m=\frac{k}{8}[1+(-1)^m]\frac{\tau^m}{m!}(x_1-x_2-l_0)S_i+\frac{k}{8}\sum_{n=1}[(-1)^m+(-1)^n]\frac{\tau^{m+n}}{(m+n)!}[x_i^{(n)}-x_j^{(n)}]S_i$$

$$+\frac{k}{8}\sum_{n=1}[(-1)^m+(-1)^n]\frac{\tau^{m+n}}{(m+n)!}x_i^{(n)}\sum_{q=0}^{n-1}(-1)^{(m+q)}\frac{(m+n)!}{(m+q)!(n-q)!}$$

$$=\frac{k}{8}[1+(-1)^m]\frac{\tau^m}{m!}(x_1-x_2-l_0)S_i-\frac{k}{8}\sum_{n-1}[(-1)^m+(-1)^n]\frac{\tau^{m+n}}{(m+n)!}\left[x_j^{(n)}+x_i^{(n)}(-1)^{m-1}\frac{(m+n-1)!}{(m-1)!n!}\right]. \tag{B3}$$

We derive the first expression by identifying the coefficients of $x_i^{(n)}$ and the second expression by evaluating the sum, which can be done as follows. Recognize that

$$\sum_{q=0}^{n-1}(-1)^{q+m}\frac{(m+n)!}{(m+q)!(n-q)!}=-(-1)^{m+n}-\sum_{l=0}^{m-1}(-1)^l\frac{(m+n)!}{l!(m+n-l)!} \tag{B4}$$

by comparing to $(1-1)^{m+n}$. Next evaluate the remaining sum by noting that

$$\sum (m+n-1,m)=\sum_{l=0}^{m-1}(-1)^l\frac{(m+n-1)!}{l!(m+n-1-l)!}$$
$$=\sum_{l=0}^{m-1}(-1)^l\frac{(m+n-1)!}{l!(m+n-1-l)!}\frac{(m+n-l)}{(m+n-l)}$$
$$=\sum (m+n,m)+\sum (m+n-1,m)-(-1)^{m-1}\frac{(m+n-1)!}{(m-1)!n!}. \tag{B5}$$

This gives the desired sum, $\Sigma(m+n,m)$.

Now, the generalized Hamiltonian becomes

$$H=\tfrac{1}{2}m_1[x_1^{(1)}]^2+\tfrac{1}{2}m_2[x_2^{(1)}]^2-\tfrac{1}{4}k(x_1-x_2-l_0)^2-\tfrac{1}{4}k\sum_{m=1}\sum_{n=1}[(-1)^m+(-1)^n]\frac{\tau^{m+n}}{(m+n)!}x_1^{(n)}x_2^{(m)}. \tag{B6}$$

For the solutions of the form (2.3), (2.4), and (2.6),

$$x_i=X_i(t)+S_iA_i\cos(\omega t+\delta)=X_i(t)+S_iy_i; \tag{B7}$$

then

$$H=\tfrac{1}{2}[m_1+m_2+\tfrac{1}{2}k\tau^2]V^2+\left\{m_1y_i^{(1)}-m_2y_2^{(1)}+\tfrac{1}{4}k\sum_{n=1}[1-(-1)^n]\frac{\tau^{n+1}}{(n+1)!}(y_1^{(n)}-y_2^{(n)})\right\}V+\tfrac{1}{2}m_1[y_1^{(1)}]^2$$
$$+\tfrac{1}{2}m_2[y_2^{(1)}]^2-\tfrac{1}{4}k(y_1+y_2)^2+\tfrac{1}{4}k\sum_{m=1}\sum_{n=1}[(-1)^m+(-1)^n]\frac{\tau^{m+n}}{(m+n)!}y_1^{(n)}y_2^{(m)}. \tag{B8}$$

The linear term sums to zero by virtue of (2.7), and thus, H reduces to the sum of a center-of-momentum contribution and of a relative contribution, where the former is

$$H_{\rm CM}=\tfrac{1}{2}(m_1+m_2+\tfrac{1}{2}k\tau^2)V^2 \tag{B9}$$

and the latter is

$$H_{\rm rel}=\tfrac{1}{2}m_1[y_1^{(1)}]^2+\tfrac{1}{2}m_2[y_2^{(1)}]^2-\tfrac{1}{4}k(y_1+y_2)^2+\tfrac{1}{4}k\sum_{m=1}\sum_{n=1}[(-1)^m+(-1)^n]\frac{\tau^{m+n}}{(m+n)!}y_1^{(n)}y_2^{(m)}$$
$$=\tfrac{1}{2}m_1\omega^2A_1^2+\tfrac{1}{2}m_2\omega^2A_2^2-\tfrac{1}{4}k\omega\tau\sin(\omega\tau)A_1A_2. \tag{B10}$$

The last form comes by substituting the solutions for $y_i(t)$ into the first form.

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