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Number-phase uncertainty product for generalized squeezed states arising from the Pegg-Barnett Hermitian phase operator formalism

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The number-phase uncertainty relation based on the Pegg-Barnett Hermitian phase operator formalism is discussed for *generalized squeezed states* of the harmonic oscillator. The corresponding number-phase uncertainty product is calculated for the magnitudes of the squeeze and displacement parameters ranging from 0 to $\frac{3}{2}$ in the former case and from 0 to 4 in the latter case for the first few classes of generalized squeezed states ($m=0$, 1, and 2) and for different values of their combined phases. It is found that for a given magnitude of the squeeze parameter, the number-phase uncertainty product tends to the fixed limiting value $m+\frac{1}{2}$ when the magnitude of the displacement parameter tends to infinity. On the other hand, for a fixed magnitude of the displacement parameter, the uncertainty product grows indefinitely as the magnitude of the squeeze parameter increases. It is also observed that the number-phase uncertainty product tends to zero for *few-photon* generalized squeezed states (when the magnitudes of both squeeze and displacement parameters tend to zero) so that, according to the Pegg-Barnett Hermitian phase formalism, it is possible to have generalized squeezed states with a number-phase uncertainty product smaller than $\frac{1}{2}$.

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I. INTRODUCTION

The question of the proper dynamical variable corresponding to the phase of a quantum field has been the subject of discussion for a long time. The problem appeared to be solved for the first time by Dirac in 1927 [1] only to be refuted much later by Susskind and Glogower [2,3]. They introduced two Hermitian dynamical variables analogous to the sine and cosine of the phase, but as the two variables do not commute this has also been regarded as an unsatisfactory solution. There have been numerous attempts to construct other, more satisfactory phase operators [4–10]. Recent theoretical work of Pegg and Barnett [7–10] concerning the Hermitian phase operator of the harmonic oscillator, based on a model of the single-mode electromagnetic field that involves a finite but arbitrary large state space, is of particular importance. Phase calculations, based on various phase formalisms, have been carried out for coherent [11], squeezed [12], and displaced number states [13]. Several experiments [14,15] were also reported in which phase differences and their fluctuations were measured as a function of average photon number, and attempts were made to test some of the definitions against experiment [16–18] but no clear conclusion emerged. The predictions based on the Pegg-Barnett Hermitian phase operator formalism appear to have been corroborated by experiment in several cases [18,19].

In this paper we present phase calculations, based on the Pegg-Barnett Hermitian phase operator formalism, for *generalized squeezed* states of the quantum oscillator. The importance and certain properties of these states have been discussed in Refs. [20,21]. Briefly, a single quantum oscillator, prepared initially in the mth number state (with $m=0,1,2,\dots$), and driven by a transient, spatially uniform external force represented in the Hamiltonian by the terms linear and *quadratic* in annihilation and creation operators, evolves to the corresponding generalized squeezed state. Different classes of generalized squeezed states correspond to different m values. The $m=0$ class coincides with the ordinary squeezed states [22]. These states have been first introduced and studied in the field of quantum optics with the ultimate aim to obtain a reduced fluctuation in one field quadrature, at the expense of an increased fluctuation in the other, leading to an increase in the signal to noise ratio in suitable experiments ranging from optical communication to detection of gravitational radiation. Generalized squeezed states within each class are not mutually orthogonal and form separately an overcomplete set. In Sec. II, and in the Appendix, we present the properties of the generalized squeezed states that are needed for the calculation of the corresponding number-phase uncertainty product. Subsequently, in Sec. III we describe phase calculations for generalized squeezed states based on the recent extensive work of Pegg and Barnett that is derived from the form of phase states in a finite but arbitrarily large state space. Their approach seems to have been corroborated by experiment in several cases, and it is advantageous in that it permits the existence of a Hermitian phase operator. The corresponding number-phase uncertainty product for generalized squeezed states is calculated, for first few classes of generalized squeezed states ($m=0$, 1, and 2), and for the magnitudes of the squeeze and displacement parameters ranging from 0 to $\frac{3}{2}$ in the former case, and from 0 to 4 in the latter case. We find numerically that, despite the fact that a generalized squeezed state is described, in the most general case, by *six* independent real parameters [23], the corresponding Pegg-Barnett number-phase uncertainty product depends only on *three* of these, namely on the magnitudes of the squeeze and displacement parameters and on their combined phases. We also discuss the asymptotic behavior, and certain special cases, of the uncertainty product. Finally, in Sec. IV, we present our conclusions.

II. GENERALIZED SQUEEZED STATES

We consider the most general *quadratic* Hamiltonian [23] ($\hbar=1$),

$$\hat{H}(t)=\omega(t)(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})+g(t)\hat{a}^2+g(t)^*\hat{a}^{\dagger 2}+h(t)\hat{a}+h(t)^*\hat{a}^\dagger, \tag{2.1}$$

which has important applications in quantum optics and molecular dynamics [24,25]. For definiteness, we shall think of a single quantum oscillator, with a variable frequency $\omega(t)$, driven by a transient, spatially uniform, external force represented in the Hamiltonian by the two given functions $g(t)$ and $h(t)$. One assumes $\omega(t)=\omega_0=\text{const}$ and $g(t)=h(t)=0$ before some arbitrarily chosen initial time $t=0$ and after some later time $t\geqslant\tau$. The corresponding time evolution operator is [23,26]

$$\hat{U}(t,0)=e^{i(\varepsilon-\phi/2)}\hat{B}(\phi,\xi)\hat{D}(\delta), \tag{2.2}$$

with

$$\hat{B}(\phi,\xi)=\exp(-i\phi\hat{a}^\dagger\hat{a})\exp[\tfrac{1}{2}(\xi^*\hat{a}^2-\xi\hat{a}^{\dagger 2})], \tag{2.3}$$

representing the Bogoliubov operator, and

$$\xi=|\xi|e^{i(\phi+\theta)}, \quad |\xi|=\text{arctanh}(|\nu|/|\mu|) \tag{2.4}$$

denoting the complex squeeze parameter. The three complex and one real time dependent parameters, $\mu\equiv|\mu|e^{i\phi}$, $\nu\equiv|\nu|e^{i\theta}$, $\delta\equiv|\delta|e^{i\Delta}$, and ϵ, respectively, satisfy the following set of coupled equations of motion [26]

$$\dot{\mu}=i(\omega\mu-2g\nu), \quad \dot{\nu}=i(2g^*\mu-\omega\nu), \tag{2.5}$$

$$\dot{\delta}=-i(h^*\mu-h\nu), \quad \dot{\varepsilon}=-\text{Re}\{(h^*\mu-h\nu)\delta^*\}, \tag{2.6}$$

with the appropriate initial conditions $\mu(0)=1$, $\nu(0)=\delta(0)=\varepsilon(0)=0$. As is well known, in the general case of arbitrary ω, h, and g, these equations of motion cannot be integrated in closed form since they ultimately lead to a Ricatti equation [23]. On the other hand, the numerical integration of the equations of motion is easily performed and we shall, in what follows, assume that the functions μ, ν, δ, and ε are known. In (2.2), $\hat{D}(\delta)$ denotes the usual displacement operator [27].

The Bogoliubov operator (2.3) effects the homogeneous Bogoliubov transformation of the boson annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators, $\hat{b}\equiv\mu\hat{a}+\nu\hat{a}^\dagger$, for a pair of complex parameters μ and ν, obeying additionally $|\mu|^2-|\nu|^2=1$. This transformation is *canonical* since it leaves the commutator invariant, $[\hat{a},\hat{a}^\dagger]=[\hat{b},\hat{b}^\dagger]=\hat{1}$. A theorem of von Neumann [28] asserts that every canonical transformation can be represented as a *unitary* transformation; thus in particular $\hat{b}=\hat{B}\hat{a}\hat{B}^\dagger$. Equation (2.3) shows that, in the general case of a *complex* μ, the Bogoliubov operator differs from the squeeze operator, a fact of some importance for the theory of the (ordinary and generalized) squeezed states.

A quantum oscillator, described by the Hamiltonian (2.1), and initially prepared in a number state $|m\rangle$, with $m=0,1,2,\ldots$, evolves to the final, *generalized squeezed state*

$$|\text{generalized squeezed state}, \; m,t\rangle\equiv\hat{U}(t,0)|m\rangle. \tag{2.7}$$

In particular, by driving the oscillator initially prepared in the ground state ($m=0$) one obtains the usual squeezed states [22]. The generalized squeezed state has an expansion in terms of the number states with complex, time-dependent coefficients [cf. (2.2)]

$$U_{nm}(t)\equiv\langle n|\hat{U}(t,0)|m\rangle=e^{i(\varepsilon-\phi/2)}\sum_k^\infty B_{nk}(\phi,\xi)D_{km}(\delta). \tag{2.8}$$

The matrix elements, $B_{nk}(\phi,\xi)$, of the Bogoliubov operator in the number base are given in the Appendix. The matrix elements, $D_{km}(\delta)$, of the displacement operator are well known [27,29]. In the $m=0$ special case, the matrix elements $D_{k0}(\delta)$ represent the expansion coefficients for the coherent state, and (A17) and (2.8) lead immediately to the well-known expression for the expansion coefficients for the ordinary squeezed states

$$U_{n0}=e^A\frac{(\nu/2\mu)^{n/2}}{(n!)^{1/2}}H_n\left(\frac{\delta}{(2\mu\nu)^{1/2}}\right), \tag{2.9}$$

with

$$A=-\tfrac{1}{2}\ln|\mu|+\frac{\nu^*\delta^2}{2\mu}-\tfrac{1}{2}|\delta|^2+i\left(\varepsilon-\frac{\phi}{2}\right). \tag{2.10}$$

Occasionally, the ordinary squeezed states are defined via the expansion in terms of number states with complex coefficients that differ in the choice of the (time-dependent) phase of the normalizing factor e^A in (2.9). The expression (2.9) for the expansion coefficients is preferable, since only then do the squeezed states provide the solution of the time-dependent Schrödinger equation corresponding to the Hamiltonian (2.1) and thus only then do the states have a clear dynamic origin.

In the case of the generalized squeezed states ($m\neq 0$), with the help of (A14) and (A17), one reduces the infinite sum in (2.8) to a finite sum with at most m terms

$$U_{nm}=\alpha_{nm}e^A\times\sum_{k=0}^{\min(m,n)}\frac{H_{m-k}[i(|\mu|b^*-|\nu|b)]H_{n-k}(b)}{(m-k)!(n-k)!}\times\frac{(2/i|\nu|)^k}{k!}, \tag{2.11}$$

with

$$a_{nm} \equiv (n!m!)^{1/2}\left(\frac{|\nu|}{2|\mu|}\right)^{(n+m)/2} e^{i(n/2)(\theta-\phi)} e^{-i(m/2)(\theta+\phi-\pi)},$$

$$b \equiv \frac{\delta}{(2\mu\nu)^{1/2}}. \quad (2.12)$$

The form (2.11) of the matrix elements U_{nm} is of particular interest for efficient numerical calculation of the number-phase uncertainty product based on the number-state expansion of the generalized squeezed state.

With the help of (2.11), one also obtains the time-dependent coordinate-space wave function that represents the generalized squeezed state in closed form

$$\Psi^{(m)}(x,t) \equiv \langle x|\text{generalized squeezed state, } m,t\rangle$$

$$= \sum_{n=0}^{\infty} U_{nm}(t) u_n(x)$$

$$= N_m\left(\frac{\mu}{\mu-\nu}\right)^{1/2}\left(\frac{|\mu-\nu|}{\mu-\nu}\right)^m \exp\left(A + \frac{\delta^2}{2\mu(\mu+\nu)}\right) H_m\left(\frac{x-2\sigma \mathrm{Re}\{(\mu-\nu)\delta^*\}}{2^{1/2}\sigma|\mu-\nu|}\right) \exp\left[-\frac{1}{4\sigma^2}\left(\frac{\mu+\nu}{\mu-\nu}\right)\left(x-\frac{2\sigma\delta}{\mu+\nu}\right)^2\right], \quad (2.13)$$

with $u_n(x) = \langle x|n\rangle$ denoting the usual energy eigenfunction in position space [30]. In (2.13), the normalizing constant is

$$N_m^{-1} = (2\pi)^{1/4}\sigma^{1/2}(2^m m!)^{1/2}, \quad \sigma = (2m_0\omega_0)^{-1/2}, \quad (2.14)$$

with m_0 denoting the mass and ω_0 the constant angular frequency of the free oscillator. Various limiting cases of (2.13) can be observed: (i) initially, for $t=0, \Psi^{(m)}(x,0) \to u_m(x)$; (ii) in the $m=0$ special case one obtains the wave function $\Psi^{(0)}(x,t)$ representing the ordinary squeezed states; (iii) in the case $g(t) \equiv 0$ [cf. Eq. (2.1)], $\Psi^{(m)}(x,t)$ reduces to the wave function representing the *displaced number states* [31,32] corresponding to the complex displacement parameter δ.

Equation (2.13) leads, in the case of the generalized squeezed states, to the correct choice of the reference phase on which the Pegg-Barnett Hermitian phase operator depends, as discussed in the next section. The proof of (2.13) is obtained with the help of a generalization of the Mehler formula ($k=0,1,2,\ldots$)

$$(1-w^2)^{-(k+1)/2} H_k\left(\frac{v-uw}{\sqrt{1-w^2}}\right) e^{[2uvw-(u^2+v^2)w^2]/(1-w^2)}$$

$$= \sum_{n=0}^{\infty} \frac{H_n(u)H_{n+k}(v)}{2^n n!} w^n, \quad (2.15)$$

and another useful relation [33]

$$\sum_{k=0}^{m} \frac{H_k(u)H_{m-k}(v)}{k!(m-k)!} w^k = \frac{(1+w^2)^{m/2}}{m!} H_m\left(\frac{uw+v}{\sqrt{1+w^2}}\right). \quad (2.16)$$

In the last two equations, u, v, and w denote complex quantities, with $|w|<1$ in the case of (2.15), and $w^2 \neq -1$ in the case of (2.16).

The dispersion of the number operator in the generalized squeezed state (2.7), or equivalently (2.13), is a quadratic function of the quantum number m (denoting the initial number state):

$$(\delta n)^2 = \langle \hat{n}^2\rangle - \langle \hat{n}\rangle^2 = A'm^2 + B'm + C', \quad (2.17)$$

with

$$A' \equiv 2|\mu|^2|\nu|^2, \quad (2.18)$$

$$B' \equiv 2[(|\mu|^4+|\nu|^4)|\delta|^2 + |\mu|^2|\nu|^2(1+6|\delta|^2) - 4|\mu|\,|\nu|\,|\delta|^2(|\mu|^2+|\nu|^2)\cos(\phi+\theta-2\Delta)], \quad (2.19)$$

and

$$C' \equiv (|\mu|^4+|\nu|^4)|\delta|^2 + 2|\mu|^2|\nu|^2(1+3|\delta|^2) - 4|\mu|\,|\nu|\,|\delta|^2(|\mu|^2+|\nu|^2)\cos(\phi+\theta-2\Delta). \quad (2.20)$$

It is seen that δn depends on the phases ϕ, θ, and Δ only through the combination $\alpha \equiv \phi+\theta-2\Delta$.

III. PEGG-BARNETT NUMBER-PHASE UNCERTAINTY PRODUCT

Pegg and Barnett [7,8] introduced a mathematical model of the single-mode electromagnetic field that involves a finite but arbitrary large state space. The dimensionality, $N+1$, of this space is allowed to tend to infinity only after calculation of expectation values is made. The finiteness of the state space means that the operators involved may have somewhat different properties than those of their infinite space counterparts. Such differences do not lead to detectable physical differences when the limit $N \to +\infty$ is eventually taken. The advantage of their approach is that it permits the existence of a Hermitian phase operator. First, the phase states $|\varphi_k\rangle$ with well-defined phase φ_k are introduced via an expansion in number states $|n\rangle$, which span the $(N+1)$-dimensional state space [34,35],

$$|\varphi_k\rangle \equiv \frac{1}{(N+1)^{1/2}} \sum_{n=0}^{N} e^{in\varphi_k}|n\rangle, \quad (3.1)$$

with

$$\varphi_k \equiv \varphi_0 + \frac{2\pi k}{N+1} \quad (k=0,1,2,\ldots,N), \quad (3.2)$$

and φ_0 representing a reference phase. For each φ_0, the $N+1$ phase states $|\varphi_k\rangle$ form an alternative, orthonormal and complete, basis in this space. The Hermitian phase operator $\hat{\varphi}$ is then introduced as the operator that has eigenstates that are phase states with corresponding eigenvalues that are equal to the phase of the state. In fact, a function of the phase operator, $f(\hat{\varphi})$, can be constructed from the phase states in the usual way,

$$f(\hat{\varphi}) \equiv \sum_{k=0}^{N} |\varphi_k\rangle f(\varphi_k)\langle\varphi_k|. \quad (3.3)$$

Consider then the Pegg-Barnett Hermitian phase operator $\hat{\varphi}$ and the corresponding number-phase uncertainty relation. Since, in the $(N+1)$-dimensional state space, the operators $\hat{n}$ and $\hat{\varphi}$ do not commute [8], the number of quanta and the phase are incompatible observables and the usual uncertainty relation holds $\delta n\ \delta\varphi \geq \frac{1}{2}|\langle[\hat{n},\hat{\varphi}]\rangle|$ with $(\delta\varphi)^2 = \langle\hat{\varphi}^2\rangle - \langle\hat{\varphi}\rangle^2$ [36]. In order to find $\delta\varphi$ for a general time-dependent state, given by the expansion over number states in the $(N+1)$-dimensional state space, one calculates the matrix elements $\langle n|\hat{\varphi}|n'\rangle$ and $\langle n|\hat{\varphi}^2|n'\rangle$ with the help of (2.11) and the expressions given in [10,13].

The Pegg-Barnett Hermitian phase operator depends on the choice of the reference phase φ_0 in that its eigenvalues, Eq. (3.2), range from φ_0 to $\varphi_0 + 2\pi N/(N+1)$ [8,10]. This reference phase can be, in principle, assigned any value. The average value and the dispersion of the phase have meaning only if the particular window of phase eigenvalues is specified. These quantities will be reasonably insensitive to the variations in the precise choice of φ_0 *only* if such variations do not take φ_0 too close to the average value of the phase in a given state. Once a generalized squeezed state has been created [for $t \geq \tau$, when $\omega(t) = \omega_0 =$ const and $g(t) = h(t) = 0$] the center of the wave packet $|\langle x|\text{generalized squeezed state},m,t\rangle|^2 = |\Psi^{(m)}(x,t)|^2$, representing this state, evolves in time with simple harmonic motion with phase $\varphi - \omega_0(t-\tau)$. Indeed, denoting the values of the three complex parameters μ, ν, and δ, attained at $t=\tau$, by μ', ν', and δ', respectively, one obtains from (2.5), (2.6), and the general result $\langle\hat{x}\rangle = 2\sigma\ \text{Re}\{(\mu-\nu)\delta^*\}$, that $\langle\hat{x}\rangle = X\cos[\varphi - \omega_0(t-\tau)]$ with

$$X^2 = (2\sigma|\delta'|)^2[|\mu'|^2 + |\nu'|^2 - 2|\mu'|\ |\nu'|\cos(\phi' + \theta' - 2\Delta')], \quad (3.4)$$

and

$$\tan\varphi = -\frac{|\mu'|\sin(\phi'-\Delta') + |\nu'|\sin(\theta'-\Delta')}{|\mu'|\cos(\phi'-\Delta') - |\nu'|\cos(\theta'-\Delta')}. \quad (3.5)$$

Therefore, the average value of the phase and the maximum of the *phase probability distribution* for a generalized squeezed state, $|\langle\varphi_k|\text{generalized squeezed state},m,t\rangle|^2$, are both in the vicinity of the corresponding classical phase. For large magnitudes of the displacement and squeeze parameters, the phase probability distribution is sharply peaked at the classical phase. As the values of these parameters decrease, the distribution becomes broader until, for $|\delta|^2 = |\xi|^2 = 0$ [when a generalized squeezed state degenerates into the corresponding number state $|\text{generalized squeezed state}, m,t\rangle \rightarrow |m\rangle$], it becomes uniform $[|\langle\varphi_k|m\rangle|^2 = 1/(N+1)]$, cf. Eq. (3.1). (In fact, the phase probability distribution, as a function of the quasicontinuous variable φ_k ranging from $-\infty$ to $+\infty$ is periodic with period 2π.) The choice $\varphi_0 = \varphi - \omega_0(t-\tau) - \pi$ that we make provides, then, the phase window $\varphi - \omega_0(t-\tau) \mp \pi$ in which the phase distribution for generalized squeezed states is always centered.

Finally, the first two moments of the number operator $(l=1,2)$

$$\langle\hat{n}^l\rangle = \sum_{n=0}^{N} n^l |U_{nm}|^2, \quad (3.6)$$

provide the corresponding dispersion $(\delta n)^2$ that agrees with (2.17) in the limit $N \rightarrow +\infty$.

Typical numerical results for the number-phase uncertainty product for generalized squeezed states that arise when the oscillator is initially prepared in the ground state, and in the first and second number state ($m=0$, 1, and 2), are presented in Figs. 1–3. Certain general features are readily apparent. First, despite the fact that in the general case a generalized squeezed state is characterized by six independent real parameters [23], one finds that, according to the Pegg-Barnett Hermitian theory, the corresponding number-phase uncertainty product depends only on three of these; the magnitudes of the squeeze parameter $|\xi|$ (or equivalently $|\nu| = \sinh|\xi|$), and displacement parameter $|\delta|$, and on the following phase combination of these two complex parameters, $\alpha \equiv \phi + \theta - 2\Delta$. This last was already hinted at by the expressions (2.17)–(2.20) for the dispersion of the number operator. Second, for a given $|\xi|$, the number-phase uncertainty product tends to the fixed limiting value $m + \frac{1}{2}$ when $|\delta| \rightarrow +\infty$. This implies that asymptotically, for $|\delta|^2 \gg 1$, the phase dispersion falls off as

$$(\delta\varphi)^2 \rightarrow \frac{m+\frac{1}{2}}{2B''|\delta|^2}, \quad (3.7)$$

with

$$B'' \equiv |\mu|^4 + |\nu|^4 + 6|\mu|^2|\nu|^2 - 4|\mu|\ |\nu|(|\mu|^2 + |\nu|^2)\cos\alpha. \quad (3.8)$$

Third, for a finite, fixed magnitude of the displacement parameter, $|\delta| =$ const, the uncertainty product grows indefinitely as the magnitude of the squeeze parameter increases. This is due to the growth of the photon number dispersion with increasing $|\xi|$ as is apparent from (2.17)–(2.20). We also note that in the special case $g(t) \equiv 0$, when the corresponding displaced number states are obtained, the number-

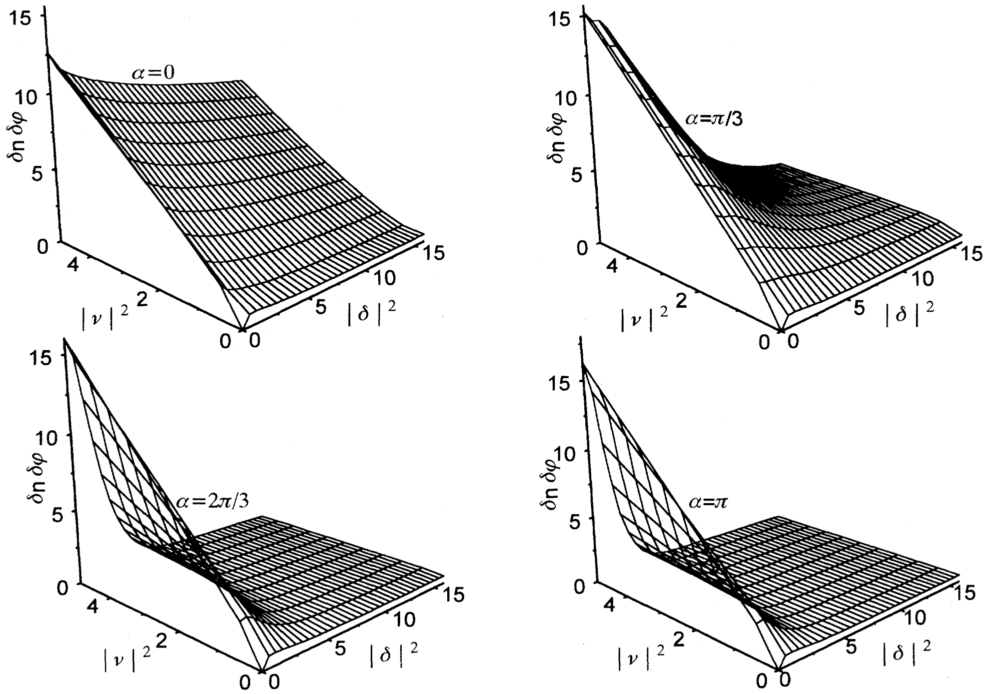

FIG. 1. The dependence of the Pegg-Barnett number-phase uncertainty product $\delta n\ \delta\varphi$, for the ordinary squeezed states ($m=0$), on the magnitudes of the displacement, $|\delta|$, and squeeze, $|\xi|=\text{arcsinh}|\nu|$, parameters for different values of their combined phases $\alpha\equiv\phi+\theta-2\Delta$. Shown are the $\alpha=0$, $\pi/3$, $2\pi/3$, and π surfaces (the surfaces for α and $2\pi-\alpha$ are the same). The $|\nu|=0$ section corresponds to the case of the coherent states [19,23].

phase uncertainty product agrees with previous calculations [13] and thus is, in the $m=0$ special case, also in good agreement with available experimental results [19]. For the general reasons analogous to the case of the displaced number states [13], and quite independently of the Pegg-Barnett theory, it is reasonable to expect that the number-phase uncertainty product for the generalized squeezed states tends to zero as both $|\delta|^2\rightarrow 0$ and $|\xi|^2\rightarrow 0$. This is indeed observed by the Pegg-Barnett Hermitian phase formalism, as Figs. 1–3 illustrate. According to the Pegg-Barnett Hermitian phase theory, then, it is possible to have generalized squeezed states with the number-phase uncertainty product smaller than $\frac{1}{2}$. In the Carruthers-Nieto formalism [3] this value defines the *minimum* number-phase uncertainty states; it is seen that there is no such lower bound according to the Pegg-Barnett Hermitian phase theory. Carruthers and Nieto [11], Jackiw [49], and Lukš and Peřinová [50], among others, have sought these minimum number-phase uncertainty states. Carruthers and Nieto have shown that the *coherent* state approximately minimizes the corresponding uncertainty product with the approximation improving in the limit $|\delta|\rightarrow+\infty$. Jackiw applied more sophisticated techniques to examine this question in detail, obtaining, in the framework of the Susskind-Glogower theory [2], exact solutions. However, the minimum number-phase uncertainty states found thereby do not seem to possess much physical significance. On the other hand, present numerical results show that, according to the Pegg-Barnett Hermitian phase formalism, it is possible to have generalized squeezed states with a number-phase uncertainty product smaller than $\frac{1}{2}$ thus depriving the search for the minimum number-phase uncertainty states of its goal.

Finally, we remark that the ordinary squeezed states (corresponding to the $m=0$ special case) have been already produced [37–40]. The generalized squeezed states can, in principle, be obtained in much the same way by driving two-photon processes with a classical source [cf. Eq. (2.1)], the only difference being in the preparation in the initial state [$m\neq 0$, cf. Eq. (2.7)]. With the experimental realization of generalized squeezed states in the realm of quantum optics, it will hopefully become possible to test these general features of the number-phase uncertainty product for these states.

IV. CONCLUSIONS

In this paper we have discussed the number-phase uncertainty product for generalized squeezed states of the harmonic oscillator. In particular, we presented the results based on the promising Pegg-Barnett Hermitian phase formalism. We calculated the corresponding number-phase uncertainty product for the magnitudes of the squeeze and displacement parameters ranging in the physically most interesting intervals (where nonclassical behavior is expected), from 0 to $\frac{3}{2}$ in the former, and from 0 to 4 in the latter case for first few classes of generalized squeezed states ($m=0$, 1, and 2, Figs. 1–3). It is found numerically that, despite the fact that a generalized squeezed state is described by six independent real parameters, the number-phase uncertainty product depends only on *three* of these, specifically on the magnitudes of the squeeze and displacement parameters and on their combined phases $\alpha\equiv\phi+\theta-2\Delta$. Additionally, it is obtained that, for a given magnitude of the squeeze parameter, $|\xi|$, the number-phase uncertainty product tends to the fixed limiting value $m+\frac{1}{2}$ when $|\delta|\to\infty$. Also, for a fixed magnitude of the displacement parameter, $|\delta|=\text{const}$, it is found that the uncertainty product grows indefinitely as the magnitude of the squeeze parameter increases. In the special case $g(t)\equiv 0$, when the corresponding displaced number states are obtained, the number-phase uncertainty product agrees with previous calculations [13] and, in the $m=0$ special case, it is in agreement with available experimental results [19]. It is observed that the number-phase uncertainty product tends to zero for *few-photon* generalized squeezed states (when both $|\delta|^2$ and $|\xi|^2$ tend to zero), so that, according to the Pegg-Barnett Hermitian phase formalism the generalized squeezed states provide another instance (in addition to the displaced number states [13]) of states with the number-phase uncertainty product smaller than $\frac{1}{2}$. With the experimental realization of generalized squeezed states it will, hopefully become possible to test these general features against experiment.

APPENDIX

In this Appendix we outline the derivation of the matrix elements of the Bogoliubov unitary operator, and of the expansion coefficients (2.11) of the generalized squeezed states. The matrix element $B_{mn}\equiv\langle m|\hat{B}|n\rangle$ of the Bogoliubov unitary operator (2.3) in the number base $|n\rangle$, with $\hat{a}^\dagger\hat{a}|n\rangle=n|n\rangle$, can be expressed in terms of the following polynomials:

$$C^{(m,n)}(z)\equiv\sum_k \frac{z^k}{k![(m-k)/2]![(n-k)/2]!}=C^{(n,m)}(z). \tag{A1}$$

Here z represents a complex variable and m,n denote nonnegative integers of *equal parity* $[C^{(m,n)}(z)\equiv 0$ otherwise]. In (2.1), $k=0,2,4,\ldots,\min(m,n)$ for m and n both even, or $k=1,3,5,\ldots,\min(m,n)$ for m and n both odd. Thus, the polynomial $C^{(m,n)}(z)$ has the parity of m (and n), and degree $\min(m,n)$. In particular, for $m=n$, the polynomials $C^{(n,n)}(z)$ are related to the ordinary Legendre polynomials [41]

$$C^{(n,n)}(z)=\frac{(z^2-4)^{n/2}}{n!}P_n\left(\frac{z}{\sqrt{z^2-4}}\right). \tag{A2}$$

For the sake of completeness, we first enumerate relevant properties of these polynomials, and then clarify their relation to the hypergeometric function and to the (generalized) Gegenbauer polynomials.

The generating function for the polynomials $C^{(m,n)}(z)$ is

$$f_n(z,w)\equiv\frac{H_n(izw/2)}{i^n n!}e^{w^2}=\sum_k^{\infty} C^{(n,k)}(z)w^k, \tag{A3}$$

with w a complex parameter, $H_n(\zeta)$ denoting the usual Hermite polynomial of a complex argument ζ, and k running over all even or odd values according to the parity of n. A simple proof of (A3) is obtained by dividing (A3) by w^{m+1} and integrating in the complex w plane along a closed contour, in the positive sense, about the origin. Using the explicit expression for the Hermite polynomials [42] and the residue theorem one obtains, after simple transformations, the expression (A1). With the help of the well-known recurrence relation for the Hermite polynomials one gets from (A3) the recurrence

$$(m+1)C^{(m+1,n)}=zC^{(m,n-1)}+2C^{(m-1,n)}. \tag{A4}$$

Here, obviously, a nontrivial result is obtained only if m and n have different parities. The generating function (A3) and the relation $H_n'(\zeta)=2nH_{n-1}(\zeta)$ lead together to

$$\frac{dC^{(m,n)}(z)}{dz}=C^{(m-1,n-1)}(z), \tag{A5}$$

and thus immediately to

$$\frac{d^kC^{(m,n)}(z)}{dz^k}=C^{(m-k,n-k)}(z), \tag{A6}$$

for the kth derivative $[k\le\min(m,n)]$. From (A4) one then finds the differential equation

$$(4-z^2)\frac{d^2C^{(m,n)}}{dz^2}+(m+n-1)z\frac{dC^{(m,n)}}{dz}-mnC^{(m,n)}=0. \tag{A7}$$

This, with the help of (A6), leads to another useful recurrence

$$(m+1)(n+1)C^{(m+1,n+1)}=(m+n+1)zC^{(m,n)}+(4-z^2)C^{(m-1,n-1)}. \tag{A8}$$

Additionally, (A7) shows that $C^{(m,n)}(z)$ can, like many other functions, be expressed in terms of the hypergeometric function. Specifically

$$C^{(m,n)}(z)=\gamma_{mn}\,{}_2F_1\left[-m,-n,\frac{1-m-n}{2},\frac{1}{2}\left(1+\frac{z}{2}\right)\right], \tag{A9}$$

with

$$(\gamma_{mn})^{-1}=\left(\frac{m}{2}\right)!\left(\frac{n}{2}\right)!\,{}_2F_1\left(-m,-n,\frac{1-m-n}{2},\frac{1}{2}\right), \tag{A10}$$

for m and n even, and

$$(\gamma_{mn})^{-1}=-\left(\frac{m+n+1}{2}\right)\left(\frac{m-1}{2}\right)!\times\left(\frac{n-1}{2}\right)!\,{}_2F_1\left(-m-1,-n-1,\frac{-m-n-1}{2},\frac{1}{2}\right) \tag{A11}$$

for m and n odd. Furthermore, a change of variable $\zeta=-z/2$ reveals a close resemblance of (A9) to the Gegenbauer (ultraspherical) polynomials, which are given in terms of the hypergeometric function as [43]

$$C_m^\lambda(\zeta)=\frac{(2\lambda)_m}{m!}{}_2F_1\left(-m,m+2\lambda,\lambda+\frac{1}{2},\frac{1-\zeta}{2}\right), \tag{A12}$$

with $(2\lambda)_m=2\lambda(2\lambda+1)\cdots(2\lambda+m-1)$ being the Pochhammer symbol. The agreement between (A9) and (A12) is

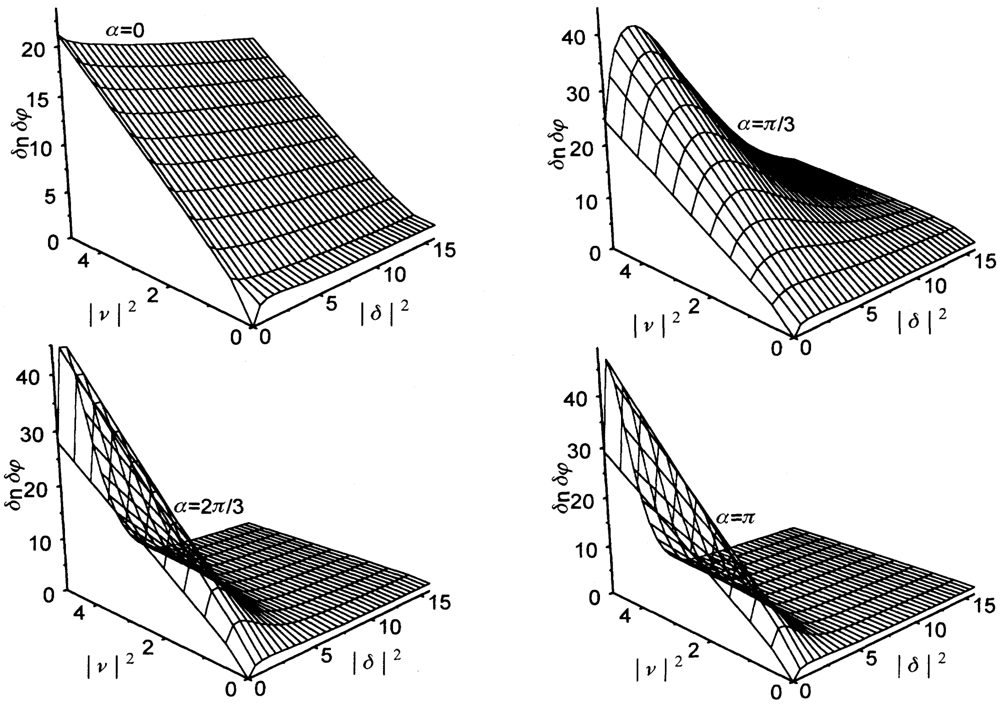

FIG. 2. The dependence of the Pegg-Barnett number-phase uncertainty product $\delta n\ \delta\varphi$, for the $m=1$ class of the generalized squeezed states, on the magnitudes of the displacement and squeeze parameters, for different values of their combined phases $\alpha=0$, $\pi/3$, $2\pi/3$, and π. The $\delta n\ \delta\varphi$ surfaces for α and $2\pi-\alpha$ are the same. The uncertainty product depends only on $|\delta|$, $|\xi|$, α, and on the quantum number m of the initial state. The $|\nu|=0$ section corresponds to the case of displaced number states with the same m and $|\delta|$ values [13].

obtained when $\lambda=-(m+n)/2$. Usually, the λ values in $C_m^\lambda(\zeta)$ are restricted to $\mathrm{Re}\{\lambda\}>-\frac{1}{2}$ and $\lambda\neq 0$ [43,44] thus apparently preventing a possible relation between the $C^{(m,n)}(z)$ and $C_m^\lambda(\zeta)$ polynomials. The first of the two restrictions on the allowable λ values is *not* essential; *if one defines* the Gegenbauer polynomials via (A12), a definite relation between the two sets of polynomials does exist, namely,

$$C^{(m,n)}(z)=\frac{m!\,\gamma_{mn}}{(2\lambda)_m}C_m^\lambda\left(-\frac{z}{2}\right), \tag{A13}$$

with $\lambda=-(m+n)/2\neq 0$. The polynomials $C^{(m,n)}(z)$, albeit being a special case of the Gegenbauer polynomials possess sufficiently specific properties and applications to warrant a separate consideration.

Finally we mention another useful relation involving the $C^{(m,n)}(z)$ polynomials

$$\sum_k^\infty C^{(n,k)}(z)L_m^{k-m}(|\delta|^2)w^k =\frac{(iw)^m e^{w^2}}{i^n}\times\sum_{k=0}^{\min(m,n)}\frac{H_{m-k}[i(|\delta|^2/2w-w)]H_{n-k}(izw/2)}{(m-k)!(n-k)!}\frac{z^k}{k!} \tag{A14}$$

which is valid for a complex z and w, and a real $|\delta|^2\geqslant 0$. In (A14), L_m^k denotes the associated Laguerre polynomial as *defined* e.g., in [45,46]. Thus, the right-hand side of (A14) represents the generating function for the product of the $C^{(m,n)}(z)$ and associated Laguerre polynomials. In particular, for $m=0$, (A14) reduces to (A3). The proof of (A14) is based on (A1) and the explicit expression for the associated Laguerre polynomials.

The derivation of the matrix elements of the Bogoliubov unitary operator is based on its normal ordered form

$$\hat{B}=|\mu|^{-1/2}\exp\left(-\frac{\nu}{2\mu}\hat{a}^{\dagger 2}\right)\exp(-\ln\mu\hat{a}^\dagger\hat{a})\exp\left(\frac{\nu^*}{2\mu}\hat{a}^2\right), \tag{A15}$$

and the matrix elements

$$\langle m|\exp(\alpha\hat{a}^2)|n\rangle=\left(\frac{n!}{m!}\right)^{1/2}\frac{\alpha^{(n-m)/2}}{[(n-m)/2]!}. \tag{A16}$$

In (A17), α denotes a complex parameter, while m and n (with $n\geqslant m$) represent non-negative integers of equal parity [$\langle m|\exp(\alpha\hat{a}^2)|n\rangle=0$, otherwise]. The completeness of the number states leads then, with the help of (A1), (A9)–(A11), and (A13), to

$$\begin{aligned}B_{mn}&=\beta_{mn}C^{(m,n)}\left(\frac{2}{i|\nu|}\right)\\&=\beta_{mn}\gamma_{mn}\,{}_2F_1\left[-m,-n,\frac{1-m-n}{2},\frac{1}{2}\left(1-\frac{i}{|\nu|}\right)\right]\\&=\frac{m!\,\beta_{mn}\gamma_{mn}}{(-m-n)_m}C_m^{-(m+n)/2}\left(\frac{i}{|\nu|}\right)\end{aligned} \tag{A17}$$

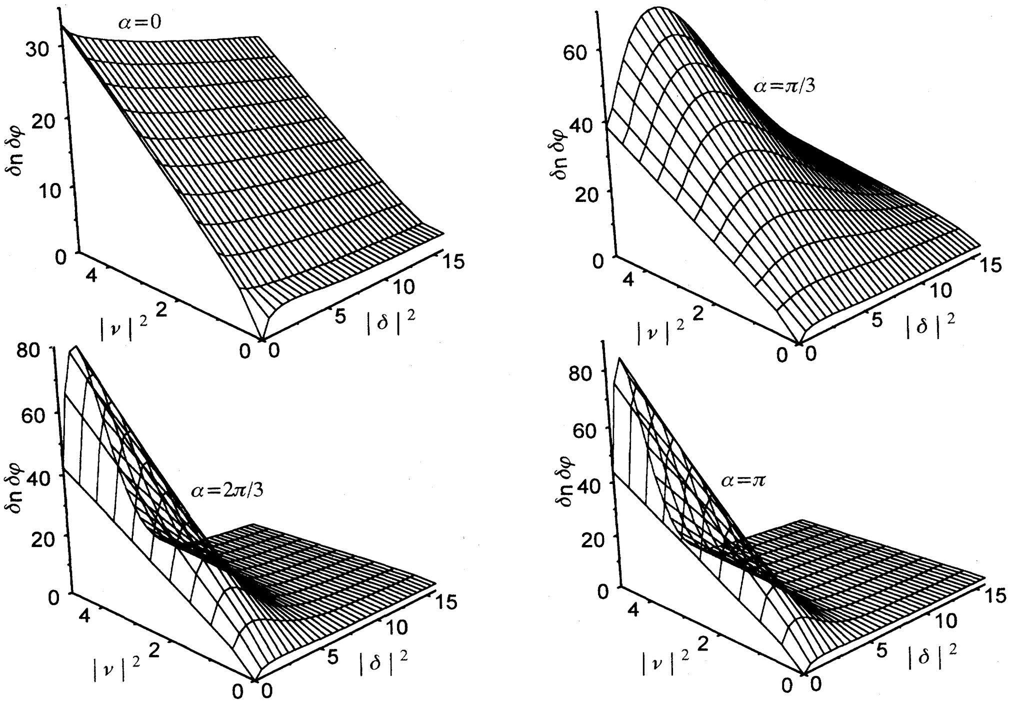

FIG. 3. The dependence of the Pegg-Barnett number-phase uncertainty product $\delta n\ \delta\varphi$, for the $m=2$ class of the generalized squeezed states, on the magnitudes of the displacement and squeeze parameters, for different values of their combined phases $\alpha=0$, $\pi/3$, $2\pi/3$, and π. The $\delta n\ \delta\varphi$ surfaces for α and $2\pi-\alpha$ are the same. The $|\nu|=0$ section corresponds to the case of displaced number states with the same m and $|\delta|$ values [13].

(the last expression being valid only for $m\neq n$), and with

$$\beta_{mn}=\left(\frac{m!n!}{2^{m+n}}\right)^{1/2}\frac{|\nu|^{(m+n)/2}}{|\mu|^{(m+n+1)/2}}e^{i(m/2)(\theta-\phi+\pi)}e^{-i(n/2)(\theta+\phi)}. \qquad \text{(A18)}$$

In particular, the diagonal matrix elements are simply

$$B_{nn}=\frac{e^{-in\phi}}{|\mu|^{1/2}}P_n\left(\frac{1}{|\mu|}\right). \qquad \text{(A19)}$$

A rather complicated expression for the matrix elements of the Bogoliubov operator, for the special case $\phi=\theta=0$ (when both μ and ν are real) was first obtained by Tanabe [47] and later a simpler expression was given by Rashid [48]. The special case $\phi=0$, when $\mu=|\mu|=$real, was provided by Satyanarayana [20]. Finally, with the help of (A14) and (A17), the validity of (2.11) is readily established.

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