Nonlinear dynamics of the Kingdon equation

Stefan J. Linz
Theoretische Physik I, Institut für Physik, Universität Augsburg, D-86135 Augsburg, Germany
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The dynamic Kingdon trap recently proposed by Blümel [Phys. Rev. A **51**, R30 (1995)] is a storage device for charged particles. The equation of motion for one trapped ion, the so-called Kingdon equation $\ddot{r}+\gamma\dot{r}+(1-2\eta\cos 2t)r^{-1}=0$, allows stable limit cycle solutions for a wide range of parameters. In this Brief Report we calculate the periodic orbit of a trapped ion approximately and by using a singular perturbation analysis, and discuss its main properties.

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INTRODUCTION

In a recent paper, Blümel [1,2] had proposed a design for the permanent storage of charged particles in traps that is different from the standard Paul trap. The dynamical Kingdon trap [1,2] consists of a metal filament surrounded by a metal cylinder with a dc *and* an ac voltage applied between the filament and the cylinder. In contrast to the static case originally studied by Kingdon [3], Blümel's dynamic variant is able to confine an ion permanently in such a way that the ion oscillates in a radial direction about a point between the filament and the cylinder (for the physical details we refer to Ref. [2]). For the simplest realization of this trap, Blümel has derived the "dynamic Kingdon equation" governing the motion of a trapped ion. In appropriately scaled units, this equation reads [1,2]

$$\ddot{r}+\gamma\dot{r}+(1-2\eta\cos 2t)\frac{1}{r}=0 , \tag{1}$$

with $r(t)>0$ representing the distance of the ion from the center of the trap. The entering control parameters are the damping coefficient γ, which mimics the additional effect of laser cooling, and the ac voltage modulation amplitude 2η. Blümel [1,2] found numerically that stable limit cycles exist in the damped Kingdon equation for large enough modulation amplitudes $\eta>3.5$. Lowering η from 3.5 to 2.9, he also observed a period-doubling scenario of the limit cycle. Using a pseudopotential approach [4], Blümel has shown why the forcing can generate dynamics in a system that—without forcing—has divergent dynamics.

The focus of this Brief Report is the basic limit cycle of (1), which determines—if stable—the long-time permanent motion of the trapped ion in the simplest version of a dynamic Kingdon trap. After stating some general remarks, we present a simple approximate solution for the limit cycle that is surprisingly accurate. Then we show that this solution can be derived more rigorously through singular perturbation analysis. Besides its physical realization, the Kingdon equation (1) also has its own rights as a simple nonlinear system with surprising dynamic properties.

Our analysis starts from rewriting Eq. (1) in form

$$r(\ddot{r}+\gamma\dot{r})=-1+2\eta\cos 2t , \tag{2}$$

which seems to be more adequate for the following analytical analysis. Equations (1) and (2) possess several symmetries: in particular, they are invariant under a shift in time by π and under the transformations $(t\to -t,\gamma\to-\gamma)$ and $(t\to t+\pi/2,\ \eta\to-\eta)$. Note, however, that these invariances hold for the differential equation (1) and (2) and not necessarily for their solutions; they might be broken.

BASIC LIMIT CYCLE

Since Eqs. (1) and (2) do not possess a natural frequency if the forcing is absent $\eta=0$, it is quite natural to suppose that the period of forcing determines the basic frequency of (1) and (2). This can also be seen by a numerical integration of the system (1). Therefore, the solution $r(t)$ can be expanded in a Fourier series in terms of $\sin 2nt$, $\cos 2nt$, $n=0,1,2,\dots$. Due to the nonlinearity in (2), insertion of the Fourier expansion leads to an infinite hierarchy of coupled algebraic equations for the Fourier coefficients. To find an approximate solution for the limit cycle, one has to truncate the system by closure relations. To give a systematic treatment of the successive approximations is far beyond the scope of this paper. Here we use the simplest closure and represent $r(t)$ by the approximation

$$r(t)\simeq r_0+a_1\sin 2t+a_2\cos 2t , \tag{3}$$

omitting all higher harmonics proportional to $\cos 2nt$ and $\sin 2nt$ with positive integer $n>1$. Inserting (3) in (2), multiplying the resulting equation successively with 1, $\sin 2t$, and $\cos 2t$, and integrating them over one period $T=\pi$, we obtain

$$a_1^2+a_2^2=\tfrac{1}{2} , \tag{4a}$$

$$2a_1+\gamma a_2=0 , \tag{4b}$$

$$r_0\gamma a_1-2r_0a_2-\eta=0 . \tag{4c}$$

The nonlinear system of algebraic equations (4) can be solved easily. Taking into account that r has to be positive, the relevant solutions read

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$$r_0 = \frac{|\eta|\sqrt{2}}{\sqrt{4+\gamma^2}} , \tag{5a}$$

$$a_1 = \frac{1}{\sqrt{2}} \frac{\text{sgn}(\eta)\gamma}{\sqrt{4+\gamma^2}} , \tag{5b}$$

$$a_2 = -\frac{\sqrt{2}\,\text{sgn}(\eta)}{\sqrt{4+\gamma^2}} , \tag{5c}$$

where sgn denotes the signum function. Note that there are two different solutions, depending on the sign of η. Both solutions (3) and (5) represent the same orbit in the $(r,\dot{r})$ phase space; they are phase shifted in time by half of the modulation period $\pi/2$, reflecting the two possible ways of forcing with positive or negative η.

COMPARISON WITH NUMERICAL RESULTS

Although our approximate solution is quite simple, its accuracy in comparison to the numerical findings of Blümel [1,2] is surprisingly good. For fixed $\gamma=0.001$, Blümel obtained numerically $r_0(\eta=4)=2.889$, $r_0(\eta=6)=4.283$, $r_0(\eta=8)=5.687$, and $r_0(\eta=10)=7.095$, whereas our approximation gives $r_0(\eta=4)=2.828$, $r_0(\eta=6)=4.243$, $r_0(\eta=8)=5.657$, and $r_0(\eta=10)=7.071$. These values differ only by less than 2%. The accuracy is even better for larger modulation amplitudes 2η; for $\eta=10$ one obtains a difference of about 0.3%. For the oscillation amplitude Blümel obtained 0.707 plus corrections in the fourth decimal caused by a weak dependence on η. Our approximation gives a constant value of 0.707 for fixed $\gamma=0.001$.

SOME PROPERTIES OF THE LIMIT CYCLE

From (3) and (5) the following properties hold. (i) The moduli of oscillation amplitudes a_1 and a_2 are independent of the modulation amplitude 2η and depend only on the damping constant γ. The modulation amplitude 2η affects the location of the center of the periodic orbit and the sign of η selects one of the solutions in (3) and (5). In Fig. 1 we show the dependence of $|a_1|$ and $|a_2|$ on the damping constant γ; the dependence of r_0 and γ follows directly from $|a_2|=r_0/|\eta|$. (ii) Assuming, for example, that η is positive, the solutions (3) and (5) reads $r(t)=r_0+(1/\sqrt{2})\cos(2t-\pi/2+\phi)$ with a phase shift $\phi=-\arctan(2/\gamma)$. For small γ, the phase shift ϕ is given by $-\pi/2$. Increasing the friction coefficient increases ϕ until it reaches zero for $\gamma\to\infty$. This implies that the response $r(t)$ follows the forcing $-2\eta\cos(2t)$ simultaneously for $\gamma\to 0$. (iii) Since $r(t)$ has to be non-negative, the limit cycle ceases to exist when the modulus of the amplitude of the oscillatory part of (3) equals r_0. A lower limit for the existence of the limit cycle is $|\eta|=1+\gamma^2/4$. (iv) The limit of vanishing friction leads to $r_0=\eta/\sqrt{2}$, $a_1=0$, and $a_2=-1/\sqrt{2}$. This compares well with Blümel's estimates based on the method of pseudopotentials [1,4]. Note that the Kingdon equation also possesses a uniquely determined limit cycle in the undamped limit $\gamma=0$. The limit cycle, however, is marginally stable; it takes an infinite time to reach it. (v) A limit cycle can exist for either sign of the damping constant γ. Some numerical tests we have performed indicate that the limit cycle is most likely unstable for $\gamma<0$.

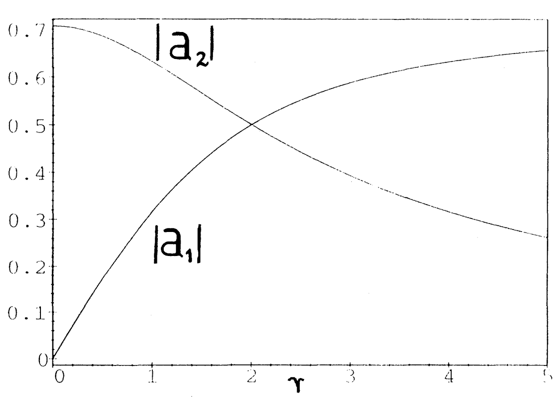

FIG. 1. Dependence of the moduli of a_1 and a_2 [cf. Eqs. (5b) and (5c)] on the damping coefficient γ. Note that the modulus of a_2 also equals $r_0/|\eta|$.

SINGULAR PERTURBATION THEORY

The accuracy of the approximations (3) and (5) raises the question whether it can be explained by using a sound mathematical argument. Let us now demonstrate that the approximate limit cycle solutions (3) and (5), which we found by simple Fourier truncation, represents the leading-order terms in a singular perturbation analysis in terms of the inverse modulation amplitude

$$\delta = \frac{1}{2\eta} \tag{6}$$

for small moduli of δ. To see this, we first rewrite Eq. (2) as

$$\delta r(\ddot{r}+\gamma\dot{r}) = -\delta + \cos 2t . \tag{7}$$

Trying to match the terms on the left- and the right-hand side of (7) by perturbation expansions in δ, one finds that the ansatz

$$r(t) = \frac{1}{\delta}[h_0 + \delta h_1(t) + \delta^2 h_2(t) + O(\delta^3)] \tag{8}$$

can fulfill Eq. (7) if the first-order term h_0 is constant in time. Inserting (8) in (7), one finds in order δ^0

$$\ddot{h}_1 + \gamma\dot{h}_1 = h_0^{-1}\cos 2t \tag{9a}$$

and in order δ

$$\ddot{h}_2 + \gamma\dot{h}_2 = -h_0^{-1}(1 + h_1\ddot{h}_1) . \tag{9b}$$

Both equations have to be solved with the constraints that $h_1(t)$ and $h_2(t)$ are periodic in π to obtain the basic limit cycle. The solution of (9a) yields

$$h_1(t) = \frac{1}{2h_0} \frac{\gamma\sin 2t - 2\cos 2t}{4+\gamma^2} , \tag{10a}$$

with h_0 being undetermined in this order. To find h_0, we insert (10a) in (9b), solve it, and observe that h_2 contains a secular term in t reading

$$-\frac{1}{4}\frac{4h_0^2(4+\gamma^2)-2}{h_0^3\gamma(4+\gamma^2)}t \; . \tag{10b}$$

The periodicity constraint requires that this term has to be zero, implying that $h_0^2=[2(4+\gamma^2)]^{-1}$. Going back to the original variable $r(t)$, one immediately infers the equivalence of the approximate solutions (3) and (5) and the singular perturbation expansion above. Therefore, one has to expect that the solutions (3) and (5) approximates the exact solution of (1) or (2) well if $|\delta|$ is small in comparison to unity, as it is for the relevant values of η in Blümel's work [1,2]. Higher harmonics of the ansatz (3) enter in higher orders in δ. In general, one has to expect that the Fourier coefficients belonging to $\sin 2nt$ and $\cos 2nt$ scale like $\delta^n=\frac{1}{2}\eta^{-n}$.

VARIATIONAL PRINCIPLE

One can easily infer that the damped Kingdon equation can be derived from a variational principle $\delta\int G(r,\dot r,t)dt=0$ with the function

$$G(r,\dot r,t)=e^{\gamma t}[\tfrac{1}{2}\dot r^2-(1-2\eta\cos 2t)\ln r] \; . \tag{11}$$

This formal result might be useful as a starting point for more sophisticated approaches based on variational methods for nonlinear forced oscillations [5]. Note, however, that the similarity of (11) with a Lagrangian function cannot be used for a mechanical reinterpretation of (1) if the damping constant γ is nonzero, quite similar to the case of the damped harmonic oscillator [6,7].

SUMMARY AND DISCUSSION

We have presented a surprisingly accurate approximate limit cycle solution for the damped (and also the undamped) Kingdon equation. We have also shown that this solution is in fact the leading term expansion of a singular perturbation analysis for small inverse forcing amplitudes. In this Brief Report, we did not address the question of how the limit cycle can undergo period doublings and lose its stability. This requires different approaches [5] that are planned for the future. Finally, we note that Eq. (2) has some formal relationship to equations describing the oscillations of a spherical bubble in a liquid in an external sinusoidal pressure field [5].

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