Minimal equations for antiphase dynamics in multimode lasers

Thomas Erneux* and Paul Mandel*
Optique Nonlinéaire Théorique, Université Libre de Bruxelles, Campus Plaine, Code Postale 231, 1050 Bruxelles, Belgium

(Received 1 March 1995)

We consider two models of multimode lasers: the Tang-Statz-deMars equations modeling a solid-state Fabry-Pérot laser [C.L. Tang, H. Statz, and G. deMars, J. Appl. Phys. **34**, 2289 (1963)] and the rate equations modeling intracavity second-harmonic generation in a Nd:YAG ring laser (where YAG denotes yttrium aluminum garnet). In both models, the dynamics is dominated by a global coupling of the modes of the electromagnetic field in the cavity. Although the equations for these two problems are fairly different, we prove that a dominant asymptotic approximation can be determined in each case that leads to the same conservative problem. It depends on one parameter, which measures the strength of the global coupling, and admits a class of antiphase periodic solutions.

PACS number(s): 42.50.Ne; 42.60.Mi

I. INTRODUCTION

The antiphase state denotes a simple periodic, multiperiodic, or chaotic state that was observed recently in experiments, mostly in multimode lasers [1–5]. The simplest type of antiphase state is characterized by the fact that the intensity of each mode oscillates with a similar wave form but not with the same phase. A typical example of a periodic antiphase state is shown in Fig. 1: the three modes differ only by the relative phase, which is equal to one-third of the period. More complex antiphase states may occur. Recently, four broad classes of antiphase states have been described [6] and the simple case displayed in Fig. 1 belongs to the first type, referred to as AD1 in [6]. In addition to the fact that antiphase regimes are easily recognizable, they are characterized by surprising spectral properties. Of particular interest is the total intensity, which oscillates with only one frequency while the intensity of each mode oscillates with two or more frequencies [2,3,7–10]. This property remains verified in the presence of noise and in the chaotic regime as shown experimentally by Otsuka *et al.* [4,5]. For periodically driven lasers generalized forms of antiphased states have also been reported [11,12].

Antiphase states appear with high multiplicity because there is no preferential mode if all modes are equally coupled. In the case of intracavity second-harmonic generation, numerical studies have suggested that there are at least two distinct domains of parameters corresponding to either harmonic or pulsating antiphase periodic solutions [6,9,13]. Specific forms of antiphase solutions have been studied mathematically in the context of Josephson-junction arrays (splay states [14] or ponies on a merry-go-round [15]) and in the context of coupled laser arrays (splay phase states [16,17]). The effects of collective coupling were also explored for coupled nonlinear oscillators from a more general point of view [18–20].

In this paper, our purpose is to use asymptotic methods to formulate a minimal problem for a multimode laser exhibiting antiphase dynamics. The equations are derived from two multimode laser problems, which have been studied independently. This minimal problem is given by

$$x_p' = -y_p - \lambda \sum_{j=1}^{N} y_j , \tag{1.1}$$

$$y_p' = x_p(1+y_p), \tag{1.2}$$

where $p=1,2,\ldots,N$. In these equations x_p and y_p are proportional to the deviation of the population inversion and the intensity from their steady-state values, respectively. Equa-

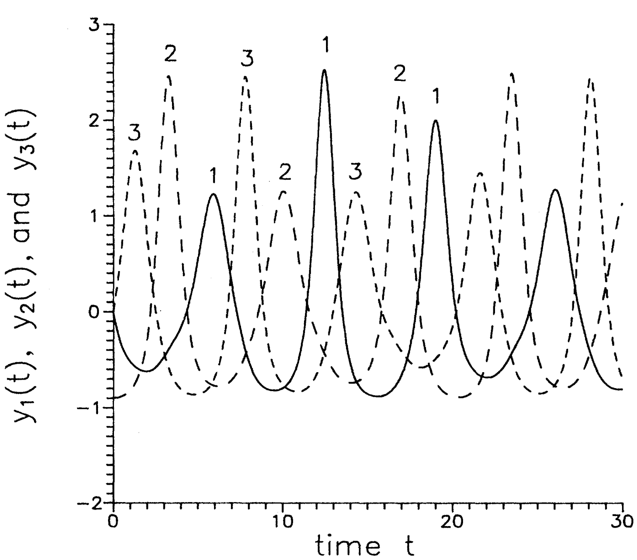

FIG. 1. Antiphase state for a three-mode laser. The figure represents the solution of Eqs. (1.1) and (1.2) with $N=3$ and $\lambda=0.4$. The initial conditions are given by $x_1(0)=-1$, $y_1(0)=0$, $x_2(0)=0$, $y_2(0)=-0.9$, $x_3(0)=1$, and $y_3(0)=0$. $y_1(t)$ is shown by a full line while $y_2(t)$ and $y_3(t)$ are shown by dotted lines. The labels 1, 2, and 3 denote the maxima for each mode. We note that these maxima always appear in sequence, which is a typical feature of an antiphase state.

*Electronic address: TERNEUX(PMANDEL)@ULB.AC.BE

tions (1.1) and (1.2) combine two essential features of the multimode laser. First, each mode oscillates through its intrinsic dynamic described by

$$x' = -y, \tag{1.3}$$

$$y' = x(1+y). \tag{1.4}$$

These equations form a conservative system of equations that admits a one-parameter family of periodic solutions. This can be shown by determining a first integral or energy $E = x^2/2 + y - \ln(1+y)$ and by analyzing the periodic orbit in the phase plane. Second, the coupling term

$$S \equiv \sum_{p}^{N} y_p \tag{1.5}$$

is acting equally on all the laser modes and is the synchronization mechanism leading to antiphase dynamics. The main purpose of this paper is to show that Eqs. (1.1) and (1.2) are the common asymptotic limit of two different laser models and that they admit antiphase periodic solutions.

In Secs. II and III, we derive Eqs. (1.1) and (1.2) from the equations describing (i) a multimode Fabry-Pérot laser and (ii) intracavity second-harmonic generation in a Nd:YAG ring laser (where YAG denotes yttrium aluminum garnet). The asymptotic approximation is based on the experimental laser parameters as described in [1–5]. In Sec. IV, we show that our minimal model admits antiphase solutions. In Sec. V, we investigate the case $N=2$ in detail. In Sec. VI, we discuss our main results.

II. MULTIMODE FABRY-PÉROT LASER

In this section, we consider the Tang-Statz-deMars (TSD) equations for a N-mode Fabry-Pérot laser [21] and show that they can be reduced to Eqs. (1.1) and (1.2). In terms of normalized quantities, the TSD equations are given by

$$\frac{dn_0}{dt} = w - n_0 - \sum_{k=1}^{N} \gamma_k (n_0 - \tfrac{1}{2} n_k) I_k, \tag{2.1}$$

$$\frac{dn_m}{dt} = \gamma_m n_0 I_m - n_m \left(1 + \sum_{k=1}^{N} \gamma_k I_k \right), \tag{2.2}$$

$$\frac{dI_m}{dt} = \kappa[\gamma_m (n_0 - \tfrac{1}{2} n_m) - 1] I_m, \tag{2.3}$$

where $m = 1,2,\ldots,N$. In these equations, $n_0(t)$ represents the space average of the population inversion $n(x,t)$ while $n_m(t)$ denotes the Fourier integral of $n(x,t)$ with respect to mode $\cos(2k_m x)$:

$$n_0 \equiv \frac{1}{L} \int_0^L n(x,t)\, dx, \tag{2.4}$$

$$n_m \equiv \frac{2}{L} \int_0^L n(x,t) \cos(2k_m x)\, dx, \tag{2.5}$$

where k_m is the wave number of mode m and L is the length of the cavity, which is entirely filled with the active medium. The modes have average intensities I_m. The parameter w is the pump parameter and is defined so that $w=1$ corresponds to the threshold of the first lasing mode. The parameter γ_m is the ratio of the gain of mode m to the gain of the first mode. Detailed numerical simulations of the antiphase responses observed experimentally suggest that γ_m is close to 1. κ is the cavity decay rate in units of the population inversion decay rate. For simplicity, we assume the same κ for all modes. κ is typically large and may vary from 10^3 to 10^6 for solid-state lasers.

We now propose to reformulate the TSD equations in terms of deviations from a particular reference steady state. Setting $\gamma_m = 1$ $(m = 1,\ldots,N)$ in Eqs. (2.1)–(2.3), we determine an N-mode steady-state solution given by

$$I_m = J \geq 0, \tag{2.6}$$

$$n_m = \bar{n} \equiv 2(w - 1 - NJ), \tag{2.7}$$

$$n_0 = \bar{n}_0 \equiv w - NJ, \tag{2.8}$$

where J satisfies the quadratic equation

$$(2N^2 - N)J^2 + (w - 2Nw + 4N)J - 2(w-1) = 0. \tag{2.9}$$

From (2.9), we find that there is a unique real and positive solution if $w \geq 1$. We next introduce the variables y_m, N_0, and N_m defined by

$$I_m = J(1 + y_m), \tag{2.10}$$

$$n_0 = \bar{n}_0 + (J/\kappa)^{1/2} N_0, \tag{2.11}$$

$$n_m = \bar{n} + (J/\kappa)^{1/2} 2N_m \tag{2.12}$$

and a time variable s defined by

$$s = (\kappa J)^{1/2} t. \tag{2.13}$$

Inserting (2.10)–(2.13) into Eqs. (2.1)–(2.3) and grouping the leading $O(1)$ terms give

$$\frac{dN_0}{ds}=-\sum_k^N y_k-(\kappa J)^{-1/2}N_0-(J\kappa^{-1})^{1/2}\sum_k^N (N_0-N_k)(1+y_k)-\sum_k^N (\gamma_k-1)[1+(J\kappa^{-1})^{1/2}(N_0-N_k)](1+y_k), \tag{2.14}$$

$$\frac{dN_m}{ds}=\tfrac{1}{2}\bar{n}_0 y_m-\tfrac{1}{2}\bar{n}\sum_k^N y_k+(J\kappa^{-1})^{1/2}N_0(1+y_m)+\tfrac{1}{2}(\gamma_m-1)[\bar{n}_0+(J\kappa^{-1})^{1/2}N_0](1+y_m)$$
$$-\tfrac{1}{2}(\kappa J)^{-1/2}N_m(1+NJ)-[\bar{n}+2(J\kappa^{-1})^{1/2}N_m]\tfrac{1}{2}\sum_k^N y_k-[\bar{n}+2(J\kappa^{-1})^{1/2}N_m]\tfrac{1}{2}\sum_k^N (\gamma_k-1)(1+y_k), \tag{2.15}$$

$$\frac{dy_m}{ds}=[(N_0-N_m)+(\gamma_m-1)(J^{-1/2}+N_0-N_m)](1+y_m). \tag{2.16}$$

These equations were first derived in [4], but without the $O(\kappa^{-1/2})$ corrections terms. Finally, we consider the limit $\kappa\to\infty$ and assume $1-\gamma_m=O(\kappa^{-1/2})$. In terms of the difference (N_0-N_m) and y_m, Eqs. (2.14)–(2.16) reduce to

$$\frac{d}{ds}(N_0-N_m)=-\tfrac{1}{2}\bar{n}_0 y_m+(\tfrac{1}{2}\bar{n}-1)\sum_k^N y_k, \tag{2.17}$$

$$\frac{dy_m}{ds}=(N_0-N_m)(1+y_m). \tag{2.18}$$

It is easy to rewrite Eqs. (2.17) and (2.18) in the form of Eqs. (1.1) and (1.2). Introducing

$$S\equiv(\tfrac{1}{2}\bar{n}_0)^{1/2}s, \tag{2.19}$$

$$x_m\equiv(\tfrac{1}{2}\bar{n}_0)^{-1/2}(N_0-N_m) \tag{2.20}$$

into Eqs. (2.17) and (2.18) gives (1.1) and (1.2), where λ is defined by

$$\lambda=2(1-\tfrac{1}{2}\bar{n})/\bar{n}_0=2(2-\bar{n}_0)\bar{n}_0. \tag{2.21}$$

III. MULTIMODE INTRACAVITY SECOND-HARMONIC GENERATION

Roy and co-workers have observed antiphase dynamics in a Nd:YAG laser that contains a potassium titanyl phosphate (KTP) crystal that doubles the frequency of the light emitted by the Nd:YAG crystal [1]. They studied the response of the laser by using the rate equations [22]

$$\eta I_p'=\left[G_p-\alpha+\epsilon g I_p-2\epsilon g\sum_{j=1}^M I_j-2\epsilon(1-g)\sum_{j=1}^P J_j\right]I_p, \tag{3.1}$$

$$\eta J_q'=\left[H_q-\alpha+\epsilon g J_q-2\epsilon g\sum_{j=1}^P J_j-2\epsilon(1-g)\sum_{j=1}^M I_j\right]J_q, \tag{3.2}$$

$$G_p'=\gamma-G_p\left[1+(1-\beta)I_p+\beta\sum_{j=1}^M I_j+\beta\sum_{j=1}^P J_j\right], \tag{3.3}$$

$$H_q'=\gamma-H_q\left[1+(1-\beta)J_q+\beta\sum_{j=1}^P J_j+\beta\sum_{j=1}^M I_j\right], \tag{3.4}$$

where (I_p,G_p) $(1\leq p\leq M)$ and (J_q,H_q) $(1\leq q\leq P)$ denote the dependent variables for the two polarizations, respectively. I_p and G_p (J_q and H_q) represent the intensity and the gain associated with the p (q) longitudinal mode, respectively. A prime means differentiation with respect to the dimensionless time $\tau\equiv t/\tau_f$, where τ_f is the fluorescence time. The parameter $\eta\equiv\tau_c/\tau_f$ is the ratio of cavity round-trip time τ_c and the fluorescence lifetime τ_f. α and γ are the cavity loss parameter and the small signal gain, which we assumed to be equal for all modes. $0<\beta<1$ is the cross saturation parameter, $0<g<1$ is a geometrical parameter, and ϵ is a parameter associated with the KTP crystal.

Both η and ϵ are small parameters that suggest investigating the double limit $\epsilon\to 0$ and $\eta\to 0$. Equations (3.1)–(3.4) with $\epsilon=0$ admit a unique, nonzero steady-state intensity given by

$$G_p=H_q=\alpha, \tag{3.5}$$

$$I_p=J_q=I, \tag{3.6}$$

where I is defined by

$$I\equiv\frac{\gamma-\alpha}{\alpha(1-\beta+\beta N)} \tag{3.7}$$

and $N=M+P$. We next introduce the deviations x_p, y_p, u_q, and v_q defined by

$$I_p=I(1+y_p),\quad J_q=I(1+v_q), \tag{3.8}$$

$$G_p=\alpha+\mu x_p,\quad G_q=\alpha+\mu u_q \tag{3.9}$$

and the time s given by

$$s=\delta\tau. \tag{3.10}$$

The coefficients μ and δ are determined so that the equations for x_p, y_p, u_q, and v_q do not have the parameter η multiplying a time derivative. We find

$$\mu \equiv \eta^{1/2}[(1-\beta)I\alpha]^{1/2}, \tag{3.11}$$

$$\delta \equiv \eta^{-1/2}[(1-\beta)I\alpha]^{1/2}. \tag{3.12}$$

Inserting (3.8)–(3.12) into Eqs. (3.1)–(3.4) leads to the following equations for the variables x_p, y_p, u_q, and v_q:

$$x_p' = -y_p - \lambda\sum_{j=1}^{M} y_j - \lambda\sum_{j=1}^{P} v_j - Ax_p\left[I^{-1} + (1-\beta)(1+y_p) + \beta\sum_{j=1}^{M}(1+y_j) + \beta\sum_{j=1}^{P}(1+v_j)\right] \tag{3.13}$$

$$y_p' = (1+y_p)\left\{x_p + B\left[g(1+y_p) - 2g\sum_{j=1}^{M}(1+y_j) - 2(1-g)\sum_{j=1}^{P}(1+v_j)\right]\right\} \tag{3.14}$$

$$u_q' = -v_q - \lambda\sum_{j=1}^{M} y_j - \lambda\sum_{j=1}^{P} v_j - Au_q\left[I^{-1} + (1-\beta)(1+v_q) + \beta\sum_{j=1}^{M}(1+y_j) + \beta\sum_{j=1}^{P}(1+v_j)\right], \tag{3.15}$$

$$v_q' = (1+v_q)\left\{u_q + B\left[g(1+v_q) - 2g\sum_{j=1}^{P}(1+v_j) - 2(1-g)\sum_{j=1}^{M}(1+y_j)\right]\right\}, \tag{3.16}$$

where λ, A, and B are defined by

$$\lambda \equiv \frac{\beta}{1-\beta}, \tag{3.17}$$

$$A \equiv \delta^{-1}I = \eta^{1/2}I[(1-\beta)I\alpha]^{-1/2}, \tag{3.18}$$

$$B \equiv \epsilon\mu^{-1}I = \epsilon\eta^{-1/2}I[(1-\beta)I\alpha]^{-1/2}. \tag{3.19}$$

We now investigate the limit

$$\epsilon = 0(\eta) \to 0$$

keeping α, β, and γ fixed. This implies $A = 0(B) \to 0$ and setting $A = B = 0$ in Eqs. (3.13)–(3.16) leads to the following equations for (x_p, y_p) and (u_q, v_q):

$$x_p' = -y_p - \lambda\sum_{j=1}^{M} y_j - \lambda\sum_{j=1}^{P} v_j, \tag{3.20}$$

$$y_p' = (1+y_p)x_p \tag{3.21}$$

$$u_p' = -v_q - \lambda\sum_{j=1}^{M} y_j - \lambda\sum_{j=1}^{P} v_j, \tag{3.22}$$

$$v_q' = (1+v_q)u_q. \tag{3.23}$$

We note that (x_p, y_p) and (u_p, v_q) satisfy the same equations. If we redefine (x_p, y_p) so that $(x_p, y_p) = (u_q, v_q)$ with p running from $M+1$ to $N = M+P$, we obtain the minimal equations (1.1) and (1.2). Note that the asymmetry between the two polarizations, which is modeled by the factor g, does not appear in the leading-order problem.

IV. ANTIPHASE DYNAMICS

In this section, we show that Eqs. (1.1) and (1.2) admit antiphase periodic solutions of the type AD1. Specifically, we seek a solution of Eqs. (1.1) and (1.2) of the form

$$x_p = X(t+\phi_p), \tag{4.1}$$

$$y_p = Y(t+\phi_p), \tag{4.2}$$

where the functions X and Y are T-periodic functions of t and ϕ_p is a constant phase. The AD1 solutions are characterized by the property that each mode oscillates with the same amplitude. Introducing (4.1) and (4.2) into Eqs. (1.1) and (1.2) and changing the time variable for each mode from t to $\xi_p \equiv t+\phi_p$ lead to the following problem for the functions $X(\xi_p)$ and $Y(\xi_p)$:

$$\frac{dX(\xi_p)}{d\xi_p} = -Y(\xi_p) - \lambda S_p, \tag{4.3}$$

$$\frac{dY(\xi_p)}{d\xi_p} = [1+Y(\xi_p)], \tag{4.4}$$

where ψ_{jp} and S_p are defined by

$$\psi_{jp} \equiv \phi_j - \phi_p, \tag{4.5}$$

$$S_p \equiv \sum_{j=1}^{N} Y(\xi_p + \psi_{jp}). \tag{4.6}$$

Equations (4.3) and (4.4) are the same equations for each p provided S_p does not depend on p. Then, $S = S_p$ and the functions X and Y satisfy

$$\frac{dX}{d\xi} = -Y - \lambda S, \tag{4.7}$$

$$\frac{dY}{d\xi} = X(1+Y), \tag{4.8}$$

where

$$S \equiv \sum_{j=1}^{N} Y(\xi_p + \psi_{jp}). \tag{4.9}$$

In Appendix A, we use the periodicity of the function Y and show that S is indeed independent of p provided

$$\psi_{jp} = (j-p)\frac{mT}{N} \quad (m=0,\ldots,N-1), \tag{4.10}$$

where m denotes a fixed wave number. If $m=0$, all modes are oscillating in phase. If $m \neq 0$, each mode oscillates with the same period but experiences a phase shift equal to a rational fraction of the period. As a result, each y_p becomes maximum at different times and the sequence of maxima depends on m. For example, if $N=3$, $m=1$ implies that y_3 is maximum first, then y_2 and finally y_1. On the other hand, $m=2$ implies that y_2 is maximum first and then y_3 and y_1.

The expression of $S=S_p$ in terms of X and Y is easy to find if $m=0$ (all modes are in phase). Then $S=NY$ and Eqs. (4.7)–(4.9) become

$$\frac{dX}{d\xi} = -(1+\lambda N)Y, \tag{4.11}$$

$$\frac{dY}{d\xi} = X(1+Y). \tag{4.12}$$

If $m \neq 0$ (all modes are out of phase), we cannot evaluate S exactly. In the next section, we determine an approximation for S for the case $N=2$.

V. $N=2$

In this section we investigate the case $N=2$ in detail and illustrate the properties of the antiphase periodic solutions discussed in Sec. IV. We construct a small-amplitude solution and show that the general solution of Eqs. (1.1) and (1.2) is in first approximation a linear combination of an in-phase periodic solution and an out-of-phase periodic solution. It is this out-of-phase solution that is responsible for the antiphase dynamics. We also show that the solution remains bounded for all time. Its amplitude is determined by the initial conditions and never grows or decays. This suggests that our minimal problem verifies a conservative property, which we demonstrate by determining a first integral.

Linearizing Eqs. (4.7)–(4.9) with $N=2$, we find that the solution is quasiperiodic and depends on two frequencies

$$\omega_1 = 1, \quad \omega_2 = \sigma = (1+2\lambda)^{1/2}. \tag{5.1}$$

We construct a small-amplitude solution by using a standard multitime perturbation method [23]. From Eqs. (1.1) and (1.2), we obtain the following approximations for y_1 and y_2:

$$\begin{aligned} y_1 \approx{}& (Ae^{i\sigma t} + \text{c.c.}) + (Be^{it} + \text{c.c.}) + (A^2 p_1 e^{2i\sigma t} + \text{c.c.}) \\ &+ (B^2 p_2 e^{2it} + \text{c.c.}) + (ABp_3 e^{i(\sigma+1)t} + \text{c.c.}) \\ &+ (A\bar{B} p_4 e^{(\sigma-1)t} + \text{c.c.}), \end{aligned} \tag{5.2}$$

$$\begin{aligned} y_2 \approx{}& (Ae^{i\sigma t} + \text{c.c.}) - (Be^{it} + \text{c.c.}) + (A^2 p_1 e^{2i\sigma t} + \text{c.c.}) \\ &+ (B^2 p_2 e^{2it} + \text{c.c.}) - (ABp_3 e^{i(\sigma+1)t} + \text{c.c.}) \\ &- (A\bar{B} p_4 e^{(\sigma-1)t} + \text{c.c.}), \end{aligned} \tag{5.3}$$

where the next correction is proportional to cubic powers of A and B. The constants $p_1, p_2, \ldots$ are defined by

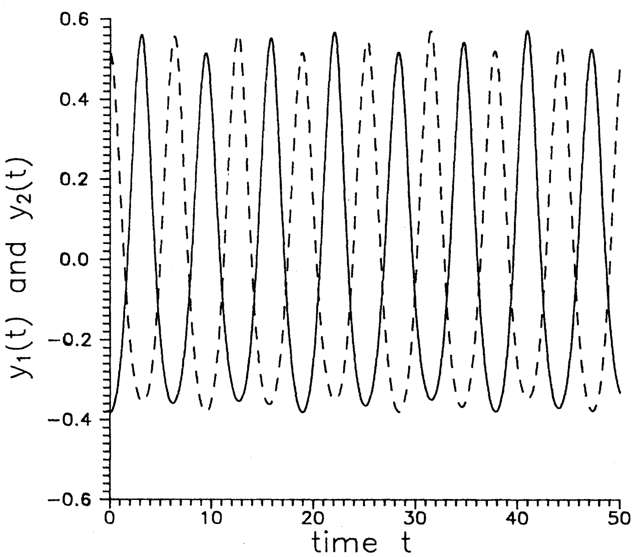

FIG. 2. Antiphase periodic state for a two-mode laser. The figure represents the solution of Eqs. (1.1) and (1.2) for $N=2$ and $\lambda=0.4$. The initial conditions are $x_1(0)=x_2(0)=0$, $y_1(0)=-0.38$, and $y_2(0)=0.52$.

$$p_1 = \frac{2}{3}, \quad p_2 = \frac{2}{(3-2\lambda)}, \quad p_3 = \frac{(\sigma+1)^2}{\sigma(\sigma+2)}, \quad p_4 = \frac{(\sigma-1)^2}{\sigma(\sigma-2)}. \tag{5.4}$$

A and B now denote amplitudes that are determined uniquely from the initial conditions. For a particular set of initial conditions, we may have

$$A = 0, \tag{5.5}$$

which implies, using (5.2) and (5.3), a *pure antiphase solution*. An approximation for the initial conditions leading to this antiphase periodic solution can be found from (5.2) and (5.3) with $A=0$. We find the conditions

$$\begin{gathered} y_2(0) \approx -y_1(0), \quad x_2(0) \approx -x_1(0), \\ B \approx \tfrac{1}{2}[y_1(0) - ix_1(0)], \quad \bar{B} \approx \tfrac{1}{2}[y_1(0) + ix_1(0)]. \end{gathered} \tag{5.6}$$

Figure 2 shows a periodic antiphase periodic solution obtained numerically from Eqs. (1.1) and (1.2) with $N=2$. The initial conditions satisfy (5.6).

With the expression (5.2) and (5.3) we may also compute the time-dependent contribution of the total intensity $S=y_1+y_2$:

$$S \approx 2(Ae^{i\sigma t} + \text{c.c.}) + 2(A^2 p_1 e^{2i\sigma t} + \text{c.c.}) + 2(B^2 p_2 e^{2it} + \text{c.c.}). \tag{5.7}$$

Equation (5.7) shows the contribution of frequency σ, but no contribution from frequency 1. This feature has been observed numerically from the original laser equations [7–9]. In the particular case of $A=0$ leading to a pure antiphase periodic solution, S oscillates with respect to a multiple of the basic frequency. Its leading approximation is given by

$$S=2(B^2p_2e^{2it}+\text{c.c.})+\cdots. \tag{5.8}$$

Note that S is proportional to B^2 and not B. This explains why the numerically determined value of S is relatively small.

The solution (5.2) and (5.3) is determined from the initial conditions and suggests that the minimal problem does not include a dissipation mechanism. This can be substantiated analytically by determining a first integral of Eqs. (1.1) and (1.2). We find

$$E=\sum_p^2 \{\tfrac{1}{2}x_p^2+C[y_p-\ln(1+y_p)]\}+Dx_1x_2, \tag{5.9}$$

where

$$C\equiv(1+2\lambda)/(1+\lambda), \quad D\equiv-\lambda/(1+\lambda). \tag{5.10}$$

E is the constant of integration (or energy) and its value is determined from the initial conditions. We determine (5.9) by using the expression of the energy when $\lambda=0$ [namely, $E=x^2/2+y-\ln(1+y)$] and by introducing a mixed term x_1x_2. The two coefficients C and D are obtained from the condition $dE/dt=0$, using first (5.9) and then Eqs. (1.1) and (1.2).

VI. GENERALIZATION AND DISCUSSION

We have formulated a minimal problem for two distinct multimode lasers and showed that it admits antiphase periodic solutions. The asymptotic approximation is based on the experimental laser parameters as described in [1–5]. A common small parameter for both laser systems is the ratio of the population inversion decay rate and the cavity decay rate. The other key parameters are specific to each of the two problems. Although the two problems differ considerably by their dissipative corrections to the minimal problem, the laser oscillations are described by the same leading-order nonlinear equations, namely, Eqs. (1.1) and (1.2). This analysis of two problems known to exhibit antiphase dynamics allows us to identify key mechanisms leading to antiphase responses. For example, we note that the global coupling term only appears in the x_p equations (associated with the population inversion), but not in the y_p equations (associated with the intensities). This particular feature is significant if we consider coupled solid-state lasers [16,17]. We have found that the original equations can be reduced to the following problem for x_p and y_p:

$$x_p'=-y_p-\tfrac{1}{2}y_p^2, \tag{6.1}$$

$$y_p'=x_p(1+y_p)+\lambda\sum_{j\neq p}^N (y_j-y_p). \tag{6.2}$$

The $\lambda=0$ problem is conservative and admits periodic solutions as for Eqs. (1.1) and (1.2), but the coupling term now appears in the equations for the intensities y_p. No antiphase periodic solutions of the form $(x_p,y_p)=[x_p(t+\theta_p),y_p(t+\theta_p)]$ has been found yet.

The minimal problem exhibits a first integral (energy), which we determine for arbitrary N in Appendix B. The general expression of the energy is given by

$$E\equiv\sum_p^N \{\tfrac{1}{2}x_p^2+C[y_p-\ln(1+y_p)]\}+\tfrac{1}{2}D\sum_p^N\sum_q^N x_px_q, \tag{6.3}$$

where C and D are defined by

$$C=\frac{1+\lambda N}{1+\lambda(N-1)}, \quad D=\frac{-\lambda}{1+\lambda(N-1)}. \tag{6.4}$$

The expression (6.3) allows us to discuss the case of weak coupling $(\lambda\to0)$ and the case of a large number of modes $(N\to\infty)$. For a weak coupling, we find $C\approx1$ and $D\approx-\lambda$ and the total energy is in first approximation a linear combination of the individual modal energies. For a large number of modes, we find $C\approx1$ and $D\approx-N^{-1}$ as $N\to\infty$ with λ fixed. Increasing the number of modes therefore has the same effect as a weak coupling. The total energy is again a sum of all the individual modal energies.

The expression of the energy is useful if we wish to determine the long-time effect of the damping terms. Indeed, we may obtain an equation for E by differentiating (6.3) and by using the original equations for x_p and y_p in order to eliminate x_p' and y_p'. The right-hand side of E' will then be proportional to the smaller parameter (the decay rate ratio) since E is defined as a first integral with this small parameter equal to zero. By using standard averaging techniques, we may then obtain a condition on the dissipative correction terms, which reduce to a solvability condition for the periodic solutions.

ACKNOWLEDGMENTS

This research was supported by the U.S. Air Force Office of Scientific Research Grant No. AFOSR-93-1-0084, the National Science Foundation Grant No. DMS-9308009, the Fonds National de la Recherche Scientifique (Belgium), and the InterUniversity Attraction Pole program of the Belgian government.

APPENDIX A: SUM OF INTENSITIES

We introduce the sum S_p defined as

$$S_p\equiv\sum_{j=1}^N Y(\xi_p+\psi_{jp}), \tag{A1}$$

where $Y(\xi_p)$ is T periodic and

$$\psi_{jp}=(j-p)\frac{mT}{N} \quad (m=0,\ldots,N-1). \tag{A2}$$

Our goal is to show that S_p is independent of p. To this end, we show that S_p with $p\neq1$ can be rewritten as S_1. S_1 and S_p are given by

$$S_1=Y(\xi_1)+Y(\xi_1+m\theta)+Y(\xi_1+2m\theta)+\cdots+Y(\xi_1+(N-1)m\theta), \quad \text{(A3)}$$

$$S_p=Y(\xi_p-(p-1)m\theta)+Y(\xi_p-(p-2)m\theta)+\cdots+Y(\xi_p+(N-p)m\theta), \quad \text{(A4)}$$

where $\theta\equiv T/N$ and $p\neq 1$. We now rewrite the first $p-1$ terms (i.e., all the terms that have a negative phase) using the fact that Y is T periodic. The T periodicity of $Y(\xi)$ implies that

$$Y(\xi+\psi)=Y(\xi+mT+\psi)=Y(\xi+mN\theta+\psi) \quad \text{(A5)}$$

and then that

$$\sum_k^{p-1} Y(\xi_p-(p-k)m\theta)=\sum_k^{p-1} Y(\xi_p-(p-k)m\theta+mN\theta)=\sum_k^{p-1} Y(\xi_p+m(N-p+k)). \quad \text{(A6)}$$

Using (A6), we obtain from (A4) that

$$S_p=\sum_k^{p-1} Y(\xi_p+m(N-p+k))+Y(\xi_p)+Y(\xi_p+\theta)+\cdots+Y(\xi_p+(N-p)m\theta), \quad \text{(A7)}$$

which is matching S_1 with $\xi_1=\xi_p$. Consequently,

$$S_p=S\equiv\sum_{j=1}^{N} Y(\xi+\psi_{j1}). \quad \text{(A8)}$$

APPENDIX B: FIRST INTEGRAL

In this appendix, we show that

$$E=\sum_p^N \{\tfrac{1}{2}x_p^2+C[y_p-\ln(1+y_p)]\}+\tfrac{1}{2}D\sum_p^N \sum_q^N x_p x_q \quad \text{(B1)}$$

is a first integral of Eqs. (1.1) and (1.2) and determine the coefficients C and D. The form of (B1) is motivated by the first integral of the laser equations $E=x^2/2+y-\ln(1+y)$ if $\lambda=0$ and an analysis of the case $N=2$ (Sec. V), which suggests the introduction of mixed products $x_p x_q$. The two coefficients C and D are obtained by using the condition $dE/dt=0$. Using (B1), we have

$$\frac{dE}{dt}=\sum_p^N \frac{dx_p}{dt}\left(x_p+D\sum_q^N x_q\right)+C\sum_p^N \frac{dy_p}{dt}y_p(1+y_p)^{-1}=0, \quad \text{(B2)}$$

where $q\neq p$ in the first summation. Using (1.1) and (1.2), Eq. (B2) becomes

$$\sum_p^N \left[\left(-y_p-\lambda\sum_k^N y_k\right)\left(x_p+D\sum_q^N x_q\right)+Cx_p y_p\right]=0 \quad \text{(B3)}$$

or, equivalently,

$$\sum_p^N \left[x_p\left(-y_p-\lambda\sum_k^N y_k\right)-Dy_p\sum_q^N x_q -\lambda D\sum_k^N y_k\sum_q^N x_q+Cx_p y_p\right]=0. \quad \text{(B4)}$$

Note that

$$\sum_p^N \sum_q^N x_q=(N-1)\sum_p^N x_p, \quad \text{(B5)}$$

which allows us to simplify the λD term in (B4):

$$\sum_p^N \left[x_p\left(-y_p-\lambda\sum_k^N y_k\right) -Dy_p\sum_q^N x_q-\lambda D(N-1)x_p\sum_k^N y_k+Cx_p y_p\right] =0, \quad \text{(B6)}$$

Separating the terms $x_p y_p$ and $x_p y_q$, we rewrite (B6) as

$$\sum_p^N x_p y_p[-1-\lambda+C-\lambda D(N-1)] +\sum_p^N \sum_q^N x_p y_q[-\lambda-D-\lambda D(N-1)]=0, \quad \text{(B7)}$$

which implies that the coefficients are zero. From these conditions, we obtain C and D

$$C=\frac{1+\lambda N}{1+\lambda(N-1)}, \quad D=\frac{-\lambda}{1+\lambda(N-1)}. \quad \text{(B8)}$$

[1] K. Wiesenfeld, C. Bracikowski, G. James, and R. Roy, Phys. Rev. Lett. **65**, 1749 (1990); G. E. James, E. M. Harrell, and R. Roy, Phys. Rev. A **41**, 2778 (1990).
[2] K. Otsuka, P. Mandel, S. Bielawski, D. Derozier, and P. Glorieux, Phys. Rev. A **46**, 1692 (1992).
[3] S. Bielawski, D. Derozier, and P. Glorieux, Phys. Rev. A **46**, 2811 (1992).
[4] K. Otsuka, M. Georgiou, and P. Mandel, Jpn. J. Appl. Phys. **31**, L1250 (1992).
[5] K. Otsuka, P. Mandel, M. Georgiou, and C. Etrich, Jpn. J. Appl. Phys. **32**, L318 (1993).
[6] J. Wang, P. Mandel, and T. Erneux, Quant. Semiclassical Opt. **7**, 169 (1995).
[7] P. Mandel, M. Georgiou, K. Otsuka, and D. Pieroux, Opt. Commun. **100**, 341 (1993).
[8] D. Pieroux, P. Mandel, and K. Otsuka, Opt. Commun. **108**, 273 (1994).
[9] J. Wang and P. Mandel, Phys. Rev. A **48**, 671 (1993).
[10] D. Pieroux and P. Mandel, Opt. Commun. **107**, 245 (1994).
[11] N. Ba An and P. Mandel, Opt. Commun. **112**, 235 (1994).
[12] N. B. Abraham, L. L. Everett, C. Iwata, and M. B. Janicki, SPIE Proc. **2095**, 16 (1994).
[13] P. Mandel and J. Wang, Opt. Lett. **19**, 533 (1994).
[14] K. Y. Tsang, R. E. Mirollo, S. H. Strogatz, and K. Wiesenfeld, Physica D **48**, 102 (1991).
[15] D. G. Aronson, M. Golubitsky, and J. Mallet-Paret, Nonlinearity **4**, 903 (1991).
[16] M. Silber, L. Fabiny, and K. Wiesenfeld, J. Opt. Soc. Am. B **10**, 1121 (1993).
[17] W.-J. Rappel, Phys. Rev. E **49**, 2750 (1994).
[18] S. Watanabe and S. H. Strogatz, Phys. Rev. Lett. **70**, 2391 (1993).
[19] G. B. Ermentrout, Physica D **41**, 219 (1990).
[20] P. C. Matthews and S. H. Strogatz, Phys. Rev. Lett. **65**, 1701 (1990).
[21] C. L. Tang, H. Statz, and G. deMars, J. Appl. Phys. **34**, 2289 (1963).
[22] R. Roy, C. Bracikowski, and G. James, in *Proceedings of the International Conference on Quantum Optics*, edited by R. Inguva and G. S. Agarwal (Plenum, New York, 1993), p. 309.
[23] J. Kervokian and J. D. Cole, *Perturbation Methods in Applied Mathematics*, edited by F. John, J. P. LaSalle, and L. Sirovich, Applied Mathematical Sciences Vol. 34 (Springer-Verlag, Berlin, 1981).