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Extended Gel'fand-Levitan method leading to exactly solvable Schrödinger equations with generalized Bargmann potentials

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We develop an extension of a class of degenerate Gel'fand-Levitan and Marchenko equations previously used in the theory of isospectral Hamiltonians and in the theory of continuum bound states. The formalism allows the generation of new exactly solvable Schrödinger equations, whose potentials extend the class of Bargmann potentials. The scattering theory is developed using the Jost solutions of the new Schrödinger equations, leading to exact expressions for the Jost function and the scattering phase shifts. The theory is illustrated by a simple example.

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I. INTRODUCTION

The purpose of this paper is to develop an extension to the particular class of degenerate Gel'fand-Levitan equations whose solutions we studied in Ref. [1]. We were motivated by a desire to extend the formalism so that it would apply to the Jost scattering solutions [2] of the Schrödinger equation as well as to solutions which satisfy physical boundary conditions. However, the extended formalism which we present here far exceeds this goal. Indeed, the formalism permits the creation of a class of exactly solvable Schrödinger equations similar to but much more extensive than those generated by the theory of isospectral Hamiltonians [3,4].

The Gel'fand-Levitan [5] and Marchenko [6] equations were first developed in the context of inverse scattering theory. These original applications are discussed in detail in the standard monographs on inverse scattering theory [7,8], and also in the monograph by Newton [9]. We follow the latter reference in our choice of notations. These integral equations relate the potentials, spectra, scattering data, and wave functions associated with two different Schrödinger equations. In each case, the kernel of the integral equation is expressed in terms of solutions of one of the Schrödinger equations (usually taken to be the free particle equation), together with the scattering data from both. The difference between the potentials associated with the two Schrödinger equations is directly related to the solution of the integral equation. Thus knowledge of the solutions of one Schrödinger equation together with knowledge of the scattering data (obtained from experiment) in principle permits the computation of the unknown potential associated with the second Schrödinger equation.

Our concern, however, is not with inverse scattering theory. It was realized quite early that the Gel'fand-Levitan equation could be applied in a much wider context. In these applications, the kernel is not constructed from experimentally obtained scattering data, but instead a degenerate kernel is chosen in order to construct simple theoretical models for studying selected physical phenomena. Moses and Tuan [10] used the Gel'fand-Levitan equation to create potentials which produced no scattering phase shift, and incidentally discovered a procedure for generating bound states with energies embedded in the continuum. The Gel'fand-Levitan equation has also been used by Meyer-Vernet [11] to study continuum bound states, while we [12] have used both the Gel'fand-Levitan and Marchenko equations for this purpose.

The Gel'fand-Levitan equation has also been used in the study of isospectral Hamiltonians. For quantum mechanics in one dimension, this application was investigated by Abraham and Moses [13]. For the same system, Luban and Pursey [3] compared the Gel'fand-Levitan-Abraham-Moses system of isospectral Hamiltonians with systems developed using the Darboux trick [14] (best known today as the basis of supersymmetric quantum mechanics [15]), while Pursey [4] extended the theory to include the Marchenko equation and explored the relationships between the several approaches. Of course the Abraham-Moses technique can be adapted to the semi-infinite line, as in the radial equation associated with rotationally symmetric problems in two or three dimensions. Luban *et al.* [16] used a two-dimensional approach of this kind to fit experimental results on a quantum dot system. In the context of the radial equation for spherically symmetric three-dimensional systems, we [1] developed a compact formalism for those degenerate Gel'fand-Levitan and Marchenko equations which are relevant to the theories of isospectral Hamiltonians and continuum bound states. It is this formalism which we extend in the present work.

The paper is organized as follows. For the sake of completeness, we use the next section to summarize and slightly generalize our earlier treatment [1] of degenerate Gel'fand-Levitan and Marchenko equations, omitting the proofs. Because we shall make use of the Jost solutions of the Schrödinger equation, and because definitions are not fully standardized, we summarize our definitions in Sec. III, again omitting all proofs. Our extension of the Gel'fand-Levitan formalism for degenerate kernels is developed in Secs. IV and V, and the new formalism is used in Sec. VI to discuss the Jost solutions and the Jost function associated with the new Schrödinger generated by the method. In Sec. VII, we sketch the manner in which Bargmann potentials [17] are generated as a special case of our formalism. To assist in understanding the method, we treat a simple example in Sec. VIII. We discuss the significance of this work in Sec. IX.

II. UNIFIED TREATMENT OF THE GEL'FAND-LEVITAN AND MARCHENKO EQUATIONS

In this section, we summarize the results of Ref. [1] on a particular class of degenerate Gel'fand-Levitan and Marchenko equations. As in that reference, we shall work with the radial equation associated with a spherically symmetric Schrödinger equation. For simplicity, we shall consider only s states. The generalization to higher angular momentum is straightforward. The treatment here differs only in minor respects from that of Ref. [1], and readers are referred to that paper for details.

We choose units such that $2m=1$ and $\hbar=1$, and use the radial equation in the form

$$\left[-\frac{d^2}{dr^2}+V(r)-E\right]\psi(r,E)=0. \tag{1}$$

We assume that $V(r)$ satisfies

$$\lim_{r\to 0} r|V(r)|<\infty. \tag{2}$$

Then physically acceptable solutions $\psi(r,E)$ must satisfy the boundary conditions

$$\psi(0,E)=0, \tag{3a}$$

$$\lim_{r\to\infty}|\psi(r,E)|<\infty. \tag{3b}$$

We shall use the Gel'fand-Levitan and Marchenko equations to relate solutions of Eq. (1) to those of

$$\left[-\frac{d^2}{dr^2}+V_c(r)-E\right]\varphi(r,E)=0, \tag{4}$$

where $V_c(r)$ is a comparison potential, and we assume that the solutions of Eq. (4) are completely known.

The class of Gel'fand-Levitan equations which we shall consider have the form

$$K(r,r')=g(r,r')-\int_0^r d\xi\ K(r,\xi)g(\xi,r'), \tag{5}$$

where the (degenerate) kernel $g(r,r')$ is

$$g(r,r')=\sum_{j=1}^{n}\lambda_j^{-1}\varphi(r,E_j)\varphi(r',E_j), \tag{6}$$

and the $\varphi_j(r)\equiv\varphi(r,E_j)$ are solutions of Eq. (4) which satisfy the boundary condition (3a). As we showed in Ref. [1], the Marchenko equation with an analogous degenerate kernel can be written

$$\bar{K}(r,r')=\bar{g}(r,r')+\int_r^\infty d\xi\ \bar{K}(r,\xi)\bar{g}(\xi,r') \tag{7}$$

where $\bar{g}(r,r')$ is given by an expression similar to Eq. (6) except that now the solutions $\varphi_j(r)\equiv\varphi(r,E_j)$ must vanish as $r\to\infty$ sufficiently rapidly for $\int_r^\infty d\xi|\varphi(\xi,E_j)|^2<\infty$, $r>0$. Clearly Eqs. (5) and (7) can be combined into the single form

$$K(r,r')=g(r,r')-\int_a^r d\xi\ K(r,\xi)g(\xi,r'),\quad a=0,\infty \tag{8}$$

where $g(r,r')$ is given by Eq. (6) and the functions $\varphi(r,E_j)$ satisfy

$$\varphi_j(a)\equiv\varphi(a,E_j)=0. \tag{9}$$

The Gel'fand-Levitan equation is given by $a=0$, while the Marchenko equation is given by $a=\infty$; however, the following development remains valid for any non-negative a.

The formalism becomes more compact in a matrix notation. We define a row matrix $\tilde{\phi}(r)$ and a column matrix $\phi(r)$ by

$$\tilde{\phi}_j(r)=\phi_j(r)=\varphi_j(r)\equiv\varphi(r,E_j), \tag{10}$$

so that (in this and the next sections only) $\tilde{\phi}(r)$ is the transpose of $\phi(r)$:

$$\tilde{\phi}(r)=\phi(r)^T. \tag{11}$$

It follows that $\tilde{\phi}\phi$ is a scalar while $\phi\tilde{\phi}$ is an $n\times n$ square matrix. We also take the set $\{\lambda_j\}$ of parameters to be the diagonal elements of a diagonal $n\times n$ matrix λ; thus we define

$$\lambda_{jk}=\lambda_j\delta_{jk}. \tag{12}$$

Equation (6) may then be written

$$g(r,r')=\tilde{\phi}(r)\lambda^{-1}\phi(r'). \tag{13}$$

Equation (8) with the kernel of Eq. (13) has the unique solution

$$K(r,r')=\tilde{\phi}(r)\Lambda^{-1}(r)\phi(r'), \tag{14}$$

where $\Lambda(r)$ is the $n\times n$ matrix [18] defined by

$$\Lambda(r)=\lambda+\int_a^r d\xi\ \phi(\xi)\tilde{\phi}(\xi). \tag{15}$$

The potential in Eq. (1) is given by

$$V(r)-V_c(r)=-2\frac{d}{dr}K(r,r)=-2\frac{d^2}{dr^2}\ln[\det\Lambda(r)], \tag{16}$$

or

$$V(r)-V_c(r)=2[K(r,r)]^2-2\left[\tilde{\phi}(r)\Lambda^{-1}(r)\frac{d\phi(r)}{dr}+\frac{d\tilde{\phi}(r)}{dr}\Lambda^{-1}(r)\phi(r)\right]. \tag{17}$$

We now consider a solution $\varphi(r,E)$ of Eq. (4) which satisfies the boundary conditions

$$\varphi(a,E)=0,\quad a<\infty \tag{18a}$$

$$|\varphi(a,E)|<\infty,\quad a=\infty. \tag{18b}$$

With the Wronskian of two functions defined by

$$W[f(r),g(r)]=f(r)\frac{d}{dr}g(r)-g(r)\frac{d}{dr}f(r), \qquad (19)$$

it follows from Eqs. (9) and (18a) that

$$W[\varphi_j(a),\varphi(a,E)]=0, \quad j=1,\ldots,n. \qquad (20)$$

Then

$$\psi(r,E)=\varphi(r,E)-\int_a^r d\xi\, K(r,\xi)\varphi(\xi,E) \qquad (21)$$

is a solution of Eq. (1) which at $r=a$ satisfies the same boundary conditions, Eqs. (18a) as $\varphi(r,E)$.

If $E=E_j$ then $\varphi(r,E)$ must be proportional to $\varphi_j(r)$ because of Eqs. (9) and (18a). If we choose $\varphi(r,E)=\varphi_j(r)$, then

$$\psi_j(r)\equiv\psi(r,E_j)=\varphi_j(r)-\int_a^r d\xi\, K(r,\xi)\varphi_j(\xi). \qquad (22)$$

If we consider the functions $\psi_j(r)$ to be the elements of a row matrix $\tilde{\psi}(r)$, then simple algebra shows that

$$\tilde{\psi}(r)=\tilde{\phi}(r)\Lambda^{-1}(r)\lambda. \qquad (23)$$

It follows that the solution $K(r,r')$ of Eq. (8) may be written

$$K(r,r')=\tilde{\psi}(r)\lambda^{-1}\phi(r'). \qquad (24)$$

From Eqs. (1), (4), (12) and (24), it is clear that

$$\left(\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial r'^2}\right)K(r,r')=[V(r)-V_c(r')]K(r,r'). \qquad (25)$$

III. JOST SOLUTIONS OF THE SCHRÖDINGER EQUATION

We shall follow Newton's definitions of the Jost solutions and Jost function, found in Chapter 12 of Ref. [9], which differ slightly from the original definitions in Ref. [2]. A physical scattering wave function $\psi(r,E)$ is a solution of Eq. (1) which satisfies the boundary conditions

$$\psi(0,E)=0, \qquad (26a)$$

$$\lim_{r\to\infty}|\psi(r,E)|<\infty. \qquad (26b)$$

Newton completes the definition by imposing the normalization condition

$$\lim_{r\to 0}\left[\frac{d}{dr}\psi(r,E)\right]=1. \qquad (27)$$

The Jost solutions $f_\pm(k,r)$ with $k^2=E$ are solutions of Eq. (1) which satisfy the boundary conditions

$$\lim_{r\to\infty}[e^{\mp ikr}f_\pm(k,r)]=1. \qquad (28)$$

The physical solution $\psi(r,E)$ is a linear combination of the Jost solutions, given by

$$\psi(r,E)=\frac{1}{2ik}[\mathscr{F}_-(k)f_+(k,r)-\mathscr{F}_+(k)f_-(k,r)], \qquad (29)$$

where

$$\mathscr{F}_\pm(k)=W[f_\pm(k,r),\psi(r,E)] \qquad (30)$$

and $\mathscr{F}(k)\equiv\mathscr{F}_+(k)$ is the Jost function. Provided that the potential $V(r)$ satisfies some fairly mild constraints discussed in detail by Newton, $\mathscr{F}(k)$ regarded as a function of complex k contains full information regarding the energy spectrum and scattering data. In particular, a zero of $\mathscr{F}(k)$ with k on the positive imaginary axis corresponds to a bound state, while other complex zeros, if well isolated, may correspond to resonances. The scattering phase shift $\delta(E)$ is the negative of the phase of $\mathscr{F}(k)$:

$$\mathscr{F}(k)=|\mathscr{F}(k)|e^{-i\delta(E)}. \qquad (31)$$

From Eq. (30) together with the boundary condition (26), it is clear that

$$\mathscr{F}_\pm(k)\equiv f_\pm(k,0). \qquad (32)$$

While one may extract all necessary scattering information directly from the physical wave function $\psi(r,E)$, it is clearly desirable to work with the Jost solution $f_+(k,r)$. However, this function satisfies the condition of Eqs. (18a) only if $a=\infty$. This makes it impossible to generate the Jost solutions of Eq. (1) from those of Eq. (4) except when $a=\infty$, that is, when the relevant integral equation is the Marchenko equation. This difficulty provided the initial motivation for the extension of the Gel'fand-Levitan equation described in the next section.

IV. THE EXTENDED GEL'FAND-LEVITAN EQUATION

In this section, we generalize the degenerate kernel of the Gel'fand-Levitan equation of Eq. (8) in two ways. We define the kernel to be a bilinear combination of products of functions drawn from two possibly different sets of solutions of Eq. (1), thus dropping the condition Eq. (11), and we drop the constraint on the functions at $r=a$ given in Eq. (9). To avoid breaks in the continuity of the discussion, we relegate the more tedious proofs to a series of appendices. We choose a set of solutions $\{\tilde{\varphi}_j(r), j=1,\ldots,n\}$ of Eq. (1) with corresponding energy eigenvalues $\{\tilde{E}_j, j=1,\ldots,n\}$, and a second set $\{\varphi_j(r), j=1,\ldots,n\}$ with energy eigenvalues $\{E_j, j=1,\ldots,n\}$. Some members of the set $\{\tilde{E}_j\}$ of energy eigenvalues may coincide with members of the other set $\{E_j\}$; if so, we require that the corresponding functions $\tilde{\varphi}$ and φ also coincide (at least up to a constant factor). If m such energies and functions are common to both collections, it is convenient to order the sets so that

$$\tilde{E}_j=E_j \quad \text{and} \quad \tilde{\varphi}_j(r)=\varphi_j(r) \quad \text{for} \quad 1\leq j\leq m, \qquad (33)$$

where $0 \leqslant m \leqslant n$. As before, we shall consider the $\tilde{\varphi}_j(r)$ to be elements of a $1 \times n$ row matrix $\tilde{\phi}(r)$, and the $\varphi_j(r)$ to be elements of an $n \times 1$ column matrix $\phi(r)$.

We consider the Gel'fand-Levitan equation (8) with the extended (but still degenerate) kernel

$$g(r,r') = \tilde{\phi}(r)\lambda^{-1}\phi(r). \tag{34}$$

Here λ is an $n \times n$ matrix, no longer diagonal, subject to certain constraints which we shall state presently. The unique solution of Eq. (8) with kernel given by Eq. (34) is given by

$$K(r,r') = \tilde{\phi}(r)\Lambda^{-1}(r)\phi(r'), \tag{35}$$

where $\Lambda(r)$ is the $n \times n$ matrix

$$\Lambda(r) = \lambda + \int_a^r d\xi\ \phi(\xi)\tilde{\phi}(\xi), \tag{36}$$

so that

$$\lambda = \Lambda(a). \tag{37}$$

We define two diagonal $n \times n$ matrices E and $\tilde{\mathsf{E}}$ by

$$\mathsf{E}_{jk} = E_j \delta_{jk}, \tag{38a}$$

$$\tilde{\mathsf{E}}_{jk} = \tilde{E}_j \delta_{jk}, \tag{38b}$$

and an $n \times n$ Wronskian matrix $\mathsf{W}(r)$ by

$$\mathsf{W}(r) = \phi(r)\tilde{\phi}'(r) - \phi'(r)\tilde{\phi}(r). \tag{39}$$

We now impose the constraint

$$\mathsf{E}\lambda - \lambda\tilde{\mathsf{E}} = \mathsf{W}(a), \tag{40}$$

which determines the elements of the matrix λ except for λ_{jj} in the special case when $\tilde{E}_j = E_j$, that is, for $j \leqslant m$. The undefined diagonal elements λ_{jj} for $j \leqslant m$, together with the choices of solutions $\tilde{\varphi}(r)$ and of $\varphi(r)$ of Eq. (4) for all values of j, are constrained only by the requirement that

$$\det\Lambda(r) \neq 0, \quad 0 \leqslant r < \infty \tag{41}$$

As a consequence of Eqs. (36) and (40) we have

$$\mathsf{E}\Lambda(r) - \Lambda(r)\tilde{\mathsf{E}} = \mathsf{W}(r). \tag{42}$$

Equation (42) shows that $\Lambda_{jk}(r)$ is independent of the lower limit a in Eq. (36) except when $E_j = \tilde{E}_k$. If $E_j = \tilde{E}_j$ for $j \leqslant m \leqslant n$, then a must be chosen so that the integrals in Eq. (36) are convergent. Since changes due to different choices of a can be absorbed into the free parameter λ_{jj}, we may even choose a to be different for different $j \leqslant m \leqslant n$. For most purposes, it is convenient to choose $a = 0$. Because of Eq. (42), $K(r,r')$ still satisfies Eq. (25) with $V(r)$ defined by Eq. (16).

We define column matrices $w(r,E)$, $\gamma(E)$, and $\Gamma(r,E)$ by

$$w(r,E) = W[\phi(r), \varphi(r,E)], \tag{43}$$

$$(\mathsf{E} - E\mathbb{1})\gamma(E) = w(a,E), \tag{44}$$

where $\mathbb{1}$ is the $n \times n$ unit matrix, and

$$\Gamma(r,E) = \gamma(E) + \int_a^r d\xi\ \phi(\xi)\varphi(\xi,E). \tag{45}$$

Then provided that E is neither one of the $\tilde{E}_j$ nor one of the E_j, the solution $\psi(r,E)$ of Eq. (1) which corresponds to the solution $\varphi(r,E)$ of Eq. (4) is

$$\psi(r,E) = \varphi(r,E) - \int_a^r d\xi\ K(r,\xi)\varphi(\xi,E) - \tilde{\phi}(r)\Lambda^{-1}(r)\gamma(E) \tag{46a}$$

$$= \varphi(r,E) - \tilde{\phi}(r)\Lambda^{-1}(r)\Gamma(r,E). \tag{46b}$$

It follows from Eqs. (44) and (45) that

$$(\mathsf{E} - E\mathbb{1})\Gamma(r,E) = w(r,E). \tag{47}$$

Provided $E \neq E_k$, Eq. (47) shows that $\Gamma_k(r,E)$ is independent of the lower limit a in Eq. (45). If $E = E_k$, the limit a must be chosen so that the integral in Eq. (45) is convergent. As for the integral in Eq. (36) with $E_j = \tilde{E}_k$, it is usually convenient to choose $a = 0$. We shall treat the special cases when either $E = \tilde{E}_j$ or $E = E_j$ for some j in the next section.

To conclude this section, we note that the energies E_j and $\tilde{E}_j$ and the parameters λ_{jk} need not be real, although the condition that $V(r)$ should be real imposes constraints on the acceptable choices of parameters and of the sets of solutions $\varphi(r)$ and $\tilde{\varphi}(r)$ of Eq. (4).

V. SPECIAL SOLUTIONS OF THE NEW SCHRÖDINGER EQUATION

In the present section we investigate the exceptional cases when E is either one of the E_j or one of the $\tilde{E}_j$.

We first consider $E = \tilde{E}_k \notin \{E_j\}$. By Eqs. (40) and (44), if $\varphi(r,\tilde{E}_k) = \tilde{\varphi}_k(r)$ then

$$\gamma_j(\tilde{E}_k) = \lambda_{jk}, \tag{48}$$

and by Eqs. (42) and (47),

$$\Gamma_j(r,\tilde{E}_k) = \Lambda_{jk}(r). \tag{49}$$

Substitution of $\varphi(r,\tilde{E}_k) = \tilde{\varphi}_k(r)$ into Eq. (46b) then yields

$$\tilde{\psi}_k(r) \equiv \tilde{\varphi}_k(r) - [\tilde{\phi}(r)\Lambda^{-1}(r)\Lambda(r)]_k = 0. \tag{50}$$

Therefore in order to construct a solution of Eq. (1) with energy $\tilde{E}_k \notin \{E_j\}$ we must start from a solution $\varphi(r,\tilde{E}_k) \equiv \tilde{\varphi}_k^{(2)}(r)$ of Eq. (4) which is distinct from $\tilde{\varphi}_k(r)$. Then

$$\tilde{\psi}_k^{(2)}(r) = \tilde{\varphi}_k^{(2)}(r) - \tilde{\phi}(r)\Lambda^{-1}(r)\Gamma^{(2)}(r,\tilde{E}_k), \tag{51}$$

with

$$\Gamma_j^{(2)}(r,\tilde{E}_k)=\frac{W[\varphi_j(r),\tilde{\varphi}_k^{(2)}(r)]}{E_j-\tilde{E}_k}. \quad (52)$$

Since any other solution of Eq. (4) with energy $\tilde{E}_k$ is a linear combination of $\tilde{\varphi}_k(r)$ and $\tilde{\varphi}_k^{(2)}(r)$, and since $\psi(r,E)$ depends linearly on $\varphi(r,E)$, different choices of $\tilde{\varphi}_k^{(2)}(r)$ can lead only to different multiples of the same solution of Eq. (1), namely, that of Eq. (51).

We next consider the case of Eqs. (46) with $E=E_k\notin\{\tilde{E}_j\}$. With $\varphi(r,E_k)=\varphi_k(r)$, Eq. (44) fails to define $\gamma_k\equiv\gamma_k(E_k)$, and for $\Gamma_k(r,E_k)$ we must use

$$\Gamma_k(r,E_k)=\gamma_k+\int_a^r d\xi\,[\varphi_k(\xi)]^2, \quad (53)$$

with γ_k a free parameter. Then

$$\psi(r,E_k)=\varphi_k(r)-\tilde{\phi}(r)\Lambda^{-1}(r)\Gamma(r,E_k) \quad (54)$$

is a solution of Eq. (1) with $E=E_k$. Different values of the parameter γ_k will lead to different linearly independent solutions, differing only by multiples of

$$\psi_k^{(2)}(r)=[\tilde{\phi}(r)\Lambda^{-1}(r)]_k, \quad (55)$$

which must also be a solution of Eq. (1) with energy E_k. If we regard the functions $\psi_k^{(2)}(r)$ defined by this equation as elements of a row matrix $\tilde{\psi}^{(2)}(r)$, then we may write

$$K(r,r')=\tilde{\psi}^{(2)}(r)\phi(r'), \quad (56)$$

which leads to a proof of Eq. (25).

Still with $E=E_k\notin\{\tilde{E}_j\}$, we may ask what happens if we choose $\varphi(r,E_k)$ to be a solution of Eq. (4) distinct from $\varphi_k(r)$. The Wronskian of $\varphi_k(r)$ and $\varphi(r,E_k)$ will then be a nonzero constant, say c_k:

$$c_k\equiv W[\varphi_k(r),\varphi(r,E_k)]\neq 0. \quad (57)$$

In this case, Eqs. (46) cannot be directly applied, since Eq. (47) cannot be satisfied with a finite value of $\Gamma_k(E_k)$. However if $\varphi(r,E)$ and its second derivative are analytic functions of E in a small region of the complex plane containing $E=E_k$, then the same is true for $\psi(r,E)$ and for the left side of Eq. (1). Hence if $\mathscr{C}$ is a small contour in the complex E plane containing E_k, then

$$\frac{1}{2\pi i}\oint_{\mathscr{C}} dE\left[-\frac{d^2}{dr^2}+V(r)-E\right]\psi(r,E)=0. \quad (58)$$

From Eqs. (46), (47), and the residue theorem, $\psi(r,E_k)$ is a solution of Eq. (1), where

$$\psi(r,E_k)=c_k[\tilde{\phi}(r)\Lambda^{-1}(r)]_k.$$

This is just a multiple of the solution $\psi_k^{(2)}(r)$ given by Eq. (55).

It remains to consider the case when $E=\tilde{E}_k=E_k$, which also requires that $\tilde{\varphi}_k(r)=c\varphi_k(r)$ where c is a constant which may be taken to be 1 without loss of generality. In this case, with $\varphi(r,E_k)=\tilde{\varphi}_k(r)=\varphi_k(r)$ both λ_{kk} and $\gamma_k\equiv\gamma_k(E)$ are free parameters, and the relation between $\Lambda_{jk}(r)$ and $\Gamma_j(r,E_k)$ becomes

$$\Gamma_j(r,E_k)=\Lambda_{jk}(r)+\delta_{jk}(\gamma_k-\lambda_{kk}). \quad (59)$$

Equation (46b) then yields

$$\psi_k(r)\equiv\psi(r,E_k)=(\lambda_{kk}-\gamma_k)[\tilde{\phi}(r)\Lambda^{-1}(r)]_k, \quad (60)$$

a multiple of $\psi_k^{(2)}(r)$ defined in Eq. (55). Any attempt to generate a second solution using Eq. (51) fails because of the singularity in $\Gamma_k^{(2)}(r,\tilde{E}_k)$ when $E_k=\tilde{E}_k$.

While in general the extended Gel'fand-Levitan method can be used to generate two independent solutions of Eq. (1) with energy eigenvalue E from a corresponding pair of independent solutions of Eq. (4), we have shown that this is not so when $E=\tilde{E}_k$ for some k, whether or not $\tilde{E}_k$ coincides with E_k. In order to obtain the second solution of Eq. (1) for energy $\tilde{E}_k$, we may use the fact that the Wronskian of two independent solutions is a constant. If

$$W[\psi^{(1)}(r,E),\psi^{(2)}(r,E)]=\text{const}=c\neq 0, \quad (61)$$

then

$$\psi^{(2)}(r,E)=c\,\psi^{(1)}(r,E)\int^r d\xi\,[\psi^{(1)}(\xi,E)]^{-2}. \quad (62)$$

VI. JOST SOLUTIONS OF THE NEW SCHRÖDINGER EQUATION

In Sec. II, a solution $\psi(r,E)$ of Eq. (1) generated from a solution $\varphi(r,E)$ of Eq. (4) was guaranteed to satisfy a boundary condition at $r=a$. As a result, with either $a=0$ or $a=\infty$ a solution $\psi(r,E)$ generated in this way was guaranteed to satisfy physical boundary conditions at at least one end of the physical range of r. In Sec. IV, however, we abandoned any constraints at $r=a$ on the functions $\varphi(r,E)$, including those in the sets $\{\tilde{\varphi}_j(r)\}$ and $\{\varphi_j(r)\}$. While the functions $\psi(r,E)$ given by Eqs. (46) are certainly solutions of Eq. (1), there is no guarantee that they satisfy any physically relevant boundary conditions even if a is chosen to be either 0 or ∞. Except for the special cases when E is one of the E_j or one of the $\tilde{E}_j$, we may generate two independent solutions of Eq. (1) with energy eigenvalue E from two independent solutions of Eq. (4) with the same energy eigenvalue, and use a suitable linear combination of these to satisfy boundary conditions at $r=0$ or as $r\to\infty$. It will remain an open question whether such a linear combination will also satisfy physical boundary conditions at the other end of the range. The Jost solutions of Eq. (4) provide an especially convenient starting point, leading as they do to the Jost solutions of Eq. (1), and this will be the topic of this section. We shall use $f_{c\pm}(k,r)$ to denote the Jost solutions of Eq. (4) and $f_\pm(k,r)$ to denote those of Eq. (1). We use the technique of Sec. IV with $\varphi(r,E)$ chosen to be one or other of $f_{c\pm}(k,r)$. This yields the solutions $\psi_\pm(k,r)$ of Eq. (1) given by

$$\psi_\pm(k,r)=f_{c\pm}(k,r)-\tilde{\phi}(r)\Lambda^{-1}(r)\Gamma^{(\pm)}(r,k), \quad (63)$$

where from Eq. (47),

$$\Gamma_j^{(\pm)}(r,k)=\frac{1}{E_j-k^2}W[\varphi_j(r),f_{c\pm}(k,r)]. \quad (64)$$

Hence

$$\lim_{r\to\infty}[e^{\mp ikr}\psi_\pm(k,r)]=C_\pm(k), \quad (65)$$

where

$$C_\pm(k)=1+\lim_{r\to\infty}\{\tilde{\phi}(r)\Lambda^{-1}(r)(\mathsf{E}-k^2\mathbb{1})^{-1}\times[\phi'(r)\mp ik\phi(r)]\}, \quad (66)$$

and the prime denotes differentiation with respect to r. It follows that

$$f_\pm(k,r)=C_\pm^{-1}(k)\psi_\pm(k,r), \quad (67)$$

where $\psi_\pm(k,r)$ is given by Eq. (63). The new Jost function is then

$$\mathscr{F}(k)=f_+(k,0)=C_+^{-1}(k)F(k), \quad (68)$$

where

$$F(k)=\mathscr{F}_c(k)-\tilde{\phi}(0)\Lambda^{-1}(0)\Gamma^{(+)}(0,k). \quad (69)$$

If $\mathscr{F}(k)\neq 0$ then the physical scattering solution of Eq. (1) with energy $E=k^2$ may be constructed using Eq. (29). If $\mathscr{F}(k)=0$ with k on the positive imaginary axis, then $f_+(k,r)$ is the wave function of a bound state of Eq. (1) with energy $E=k^2=-|k|^2$. We illustrate these results with a simple example in Sec. VIII.

VII. RELATION TO BARGMANN POTENTIALS

In Sec. 14.7 of Ref. [9], Newton discussed the Bargmann potentials using a generalization of a treatment due to Theis [19]. In this section, we show that Newton's treatment is a special case of the formalism we have presented in Sec. IV. We shall continue to restrict our discussion to the case of zero angular momentum, although the generalization to arbitrary angular momentum is easy. Newton seeks to create a potential such that the new S matrix is related to the old by

$$S(k)=S_c(k)\frac{R(-k)}{R(k)}, \quad (70)$$

where $R(k)$ is a rational function having simple poles at $k=\beta_j$ and simple zeros at $k=\alpha_j$, where j runs from 1 to n. We order the set of α's so that

$$\mathrm{Im}(\alpha_j)\begin{cases}<0, & 1\leq j\leq m\leq n\\ >0, & m<j\leq n.\end{cases} \quad (71)$$

In this section, $\varphi(k,r)$ denotes that solution of Eq. (4) with $E=k^2$ which satisfies the boundary condition of Eqs. (26), while $f_c(k,r)$ is the solution of Eq. (4) with $E=k^2$ which satisfies $\lim_{r\to\infty}e^{-ikr}f_c(k,r)=1$. We choose the n functions $\varphi_j(r)$ to be

$$\varphi_j(r)=\varphi(\beta_j,r), \qquad 1\leq j\leq n \quad (72)$$

and the n functions $\tilde{\varphi}_j(r)$ to be

$$\tilde{\varphi}_j(r)=f_c(-\alpha_j,r), \qquad 1\leq j\leq m \quad (73a)$$

$$\tilde{\varphi}_j(r)=f_c(-\alpha_j,r)-C_j\varphi(\alpha_j,r), \quad m<j\leq n. \quad (73b)$$

The C_j are to be chosen in such a way that the matrix $\Lambda(r)$ is nonsingular.

With these choices, the theory of Sec. IV reproduces Newton's results in Sec. 14.7 of Ref. [9]. In particular, Newton's matrix x in his Eqs. (14.59) is identical with our matrix $\Lambda(r)$, and Newton's $\Sigma_\beta K_\beta(r)\varphi_l^{(0)}(\beta,r)$ is identical with our $-K(r,r)$. Hence our formalism includes and extends the class of Bargmann potentials.

VIII. A SIMPLE EXAMPLE

As an example, we choose $V_c(r)=0$ so that Eq. (4) corresponds to a free particle. We choose $n=1$, and for convenience we drop the indices on λ, Λ, γ, and Γ, since these matrices have only one entry. We choose $a=0$ and

$$\tilde{\varphi}_1(r)=\varphi_1(r)=\sqrt{2\kappa}(e^{\kappa r}\cos\alpha+e^{-\kappa r}\sin\alpha). \quad (74)$$

This choice can be used in the conventional Gel'fand-Levitan equation only if $\tan\alpha=-1$, and can be used in the Marchenko equation only if $\cos\alpha=0$. If $\tan\alpha<0$ then the slight generalization of the Gel'fand-Levitan equation treated in Sec. II may be used provided a is chosen so that $2\kappa a=\ln(-\tan\alpha)$. If $\tan\alpha>0$, then the only acceptable procedure is that presented in Sec. IV.

With $\varphi_1(r)$ given by Eq. (74),

$$g(r,r')=\frac{2\kappa}{\lambda}(e^{\kappa r}\cos\alpha+e^{-\kappa r}\sin\alpha)\times(e^{\kappa r'}\cos\alpha+e^{-\kappa r'}\sin\alpha), \quad (75)$$

and

$$\Lambda(r)=\lambda-\cos2\alpha+e^{2\kappa r}\cos^2\alpha+2\kappa r\sin2\alpha-e^{-2\kappa r}\sin^2\alpha, \quad (76)$$

so that

$$K(r,r')=\frac{2\kappa(e^{\kappa r}\cos\alpha+e^{-\kappa r}\sin\alpha)(e^{\kappa r'}\cos\alpha+e^{-\kappa r'}\sin\alpha)}{\lambda-\cos2\alpha+e^{2\kappa r}\cos^2\alpha+2\kappa r\sin2\alpha-e^{-2\kappa r}\sin^2\alpha}. \quad (77)$$

In general, the condition that $\Lambda(r)$ be free of zeros requires that $\lambda>0$, but if $\cos\alpha=0$ the condition may also be satisfied by $\lambda<-1$.

The potential in Eq. (1) is

$$V(r,\alpha)=-8\kappa^2\left\{\frac{e^{2\kappa r}\cos^2\alpha-e^{-2\kappa r}\sin^2\alpha}{\lambda-\cos 2\alpha+e^{2\kappa r}\cos^2\alpha+2\kappa r\sin 2\alpha-e^{-2\kappa r}\sin^2\alpha}-\frac{(e^{\kappa r}\cos\alpha+e^{-\kappa r}\sin\alpha)^4}{(\lambda-\cos 2\alpha+e^{2\kappa r}\cos^2\alpha+2\kappa r\sin 2\alpha-e^{-2\kappa r}\sin^2\alpha)^2}\right\}. \tag{78}$$

If $\alpha=\pi/2$, so that $\cos\alpha=0$, this reduces to

$$V\left(r,\frac{\pi}{2}\right)=\frac{8\kappa^2(\lambda+1)e^{-2\kappa r}}{(\lambda+1-e^{-2\kappa r})^2}. \tag{79}$$

With the substitutions $\kappa\to b$ and $(\lambda+1)\to-\beta^{-1}$, this becomes the special case of a Bargmann potential given in Eq. (14.81) of Ref. [9]. We shall comment on this result at the end of this section.

Since we have chosen $V_c(r)=0$, the Jost solutions of Eq. (4) are just $e^{\pm ikr}$. Hence

$$\psi_\pm(k,r)=e^{\pm ikr}\left\{1+\frac{\varphi_1(r)[\pm ik\varphi_1(r)-\varphi_1'(r)]}{(\kappa^2+k^2)\Lambda(r)}\right\} \tag{80}$$

and we readily find that

$$C_\pm(k)=\begin{cases}1, & \cos\alpha=0\\ \dfrac{k\pm i\kappa}{k\mp i\kappa}, & \cos\alpha\neq 0.\end{cases} \tag{81}$$

From Eqs. (68) and (69), if $\cos\alpha=0$ then the Jost function is

$$\mathscr{F}(k)=1+\frac{2\kappa}{\lambda(\kappa-ik)}=\frac{\lambda k+i(\lambda+2)\kappa}{\lambda(k+i\kappa)}, \tag{82}$$

while if $\cos\alpha\neq 0$ then

$$\mathscr{F}(k)=\frac{1}{\lambda(k+i\kappa)^2}\{\lambda k^2+2i(1+\sin 2\alpha)k\kappa+(\lambda-2\cos 2\alpha)\kappa^2\}. \tag{83}$$

We temporarily ignore the special case $\cos\alpha=0$. Hence λ must be positive. The zeros of $\mathscr{F}(k,\alpha)$ are then

$$k_\pm=-\frac{i\kappa}{\lambda}[1+\sin 2\alpha\pm\sqrt{\lambda^2-2\lambda\cos 2\alpha+(1+\sin 2\alpha)^2}]. \tag{84}$$

The roots of $\omega(\lambda)\equiv\lambda^2-2\lambda\cos 2\alpha+(1+\sin 2\alpha)^2$ are $\cos 2\alpha\pm(\sin\alpha+\cos\alpha)\sqrt{-2\sin 2\alpha}$, which are real only if 2α is in the third or the fourth quadrant. Both roots are negative if 2α is in the third quadrant, while both are positive if 2α is in the fourth quadrant. If 2α is in the fourth quadrant and if λ is between the roots of $\omega(\lambda)$, then the zeros of $\mathscr{F}(k)$ are complex with negative imaginary parts and symmetrically located about the imaginary axis. Otherwise, both zeros of the Jost function are on the imaginary axis. Except when $0<\lambda<2\cos 2\alpha$ (which requires that 2α be in the first or fourth quadrants), one of these roots will be on the positive imaginary axis, leading to a bound state of Eq. (1). If $\lambda>2\cos 2\alpha$ so that a bound state exists, then the bound state wave function is given (up to a normalization constant) by $f_+(k,r)=[(\kappa-ik)/(\kappa+ik)]\psi_+(k,r)$ with $\psi_+(k,r)$ defined in Eq. (80), evaluated at the zero of the Jost function. The factor $(\kappa-ik)/(\kappa+ik)$ is needed only if $\alpha=-\pi/4$.

The scattering phase shift is given by Eq. (31). From this and Eq. (83) (still with $\cos\alpha\neq 0$ and $\lambda>0$) we see that

$$\delta(k)=\pi-2\arctan x-\arctan\tau(x), \tag{85}$$

where $x=k/\kappa$ and

$$\tau(x)=\frac{2(1+\sin 2\alpha)x}{\lambda(x^2+1)-2\cos 2\alpha}. \tag{86}$$

Hence

$$\delta(0)-\delta(\infty)=\pi-\lim_{x\to 0}\arctan\tau(x). \tag{87}$$

If $0<\lambda<2\cos 2\alpha$, then $\tau(x)$ is negative for $x<\sqrt{2\lambda^{-1}\cos 2\alpha-1}$, so that $\lim_{x\to 0}\arctan\tau(x)=\pi$ and $\delta(0)-\delta(\infty)=0$. If $\lambda=2\cos 2\alpha$ then $\lim_{x\to 0}\arctan\tau(x)=\frac{1}{2}\pi$ and $\delta(0)-\delta(\infty)=\frac{1}{2}\pi$. If $\lambda>2\cos 2\alpha$, then $\tau(x)$ is positive for all x, $\lim_{x\to 0}\arctan\tau(x)=0$, and $\delta(0)-\delta(\infty)=\pi$. These results verify Levinson's theorem [20] for this example. Physical scattering wave functions are given by Eq. (29), together with Eqs. (32), (67), (80), and (81).

We next consider the special case $\cos\alpha=0$, in which case either $\lambda>0$ or $\lambda<-1$. From Eq. (82), the Jost function has a single zero, at $k=-i(\kappa/\lambda)(\lambda+2)$. If $-2<\lambda<-1$, this zero is on the positive imaginary axis, and there is a bound state with energy $-(\kappa/\lambda)^2(\lambda+2)^2$. The phase shift is given by

$$\delta(k)=\pi-\arctan\left[\frac{2x}{\lambda x^2+\lambda+2}\right]. \tag{88}$$

From this, it is easy to verify Levinson's theorem once again.

We now return to a discussion of the potential in the special case $\cos\alpha=0$. We have obtained Eq. (79) using the extended Gel'fand-Levitan method with $n=1$ and $\tilde{\varphi}_1(r)=\varphi_1(r)=\sqrt{2\kappa}e^{-\kappa r}$. However, in Newton's approach, as described in the preceding section, one should choose $\varphi_1(r)=\kappa^{-1}\sinh\kappa r$, while the choice of $\tilde{\varphi}_1(r)$ should be $\tilde{\varphi}_1(r)=e^{-\kappa' r}$ where $\kappa'=(\kappa/\lambda)(\lambda+2)$. This yields Newton's example provided $1+2/\lambda>0$. It is surprising that two quite different choices of $\varphi_1(r)$ and $\tilde{\varphi}_1(r)$ should yield the same potential.

The example discussed in this section illustrates the analytic structure of the Jost functions discussed in detail in Chapter 12 of Ref. [9].

IX. CONCLUSION

The classical exactly solvable Schrödinger equations have played an important role in the history of quantum mechanics, and are the starting point for many approximation procedures. It has long been realized that the class of exactly solvable Schrödinger equations can be extended using the techniques of supersymmetric quantum mechanics or techniques based on the Gel'fand-Levitan equation or on the Marchenko equation, and indeed these methods are fundamental to inverse scattering theory [7,8] and to the theory of isospectral Hamiltonians [3,4]. The development presented here provides yet a further extension of the class of exactly solvable Schrödinger equations. As demonstrated in Sec. VII, our method creates a class of potentials which extends the class of Bargmann potentials [17].

As with the original Gel'fand-Levitan and Marchenko equations, our generalization is motivated by the desire to attack a specific physical problem. We are interested in whether continuum bound states (that is, bound states with energies embedded in the continuous spectrum), first discovered by von Neumann and Wigner [21] and recently investigated in some detail in Ref. [12], can be regarded as the zero-width limit of a resonance in the scattering from a perturbed von Neumann–Wigner potential. To investigate this, we have studied several simple perturbations of a simple von Neumann–Wigner potential. Reference [22], which is based directly on the techniques of this paper, reports the results of one of our studies.

Some aspects of the extended Gel'fand-Levitan technique remain to be explored. While the restriction to a degenerate kernel in the extended Gel'fand-Levitan equations is sufficient for the immediate purposes of Ref. [22], it would be interesting to investigate the possibility of such a generalization with a non-degenerate kernel.

APPENDIX A: SOLUTION OF THE EXTENDED GEL'FAND-LEVITAN EQUATION

The solution of Eq. (8) with the degenerate kernel given by Eq. (34) follows the standard procedure for a degenerate integral equation. The equation can be written as

$$K(r,r')=[\tilde{\phi}(r)-\tilde{K}(r)]\lambda^{-1}\phi(r'), \tag{A1}$$

where $\tilde{K}(r)$ is the row matrix defined by

$$\tilde{K}(r)=\int_a^r d\xi\, K(r,\xi)\tilde{\phi}(\xi). \tag{A2}$$

Hence

$$\tilde{K}(r)=[\tilde{\phi}(r)-\tilde{K}(r)]\lambda^{-1}I(r), \tag{A3}$$

where $I(r)$ is the $n\times n$ square matrix defined by

$$I(r)=\int_a^r d\xi\, \phi(\xi)\tilde{\phi}(\xi)=\Lambda(r)-\lambda. \tag{A4}$$

Equation (A3) may be rewritten as

$$\tilde{K}(r)\lambda^{-1}\Lambda(r)=\tilde{\phi}(r)\lambda^{-1}[\Lambda(r)-\lambda], \tag{A5}$$

which has the solution

$$\tilde{K}(r)=\tilde{\phi}(r)-\tilde{\phi}(r)\Lambda^{-1}(r)\lambda. \tag{A6}$$

Equation (35) follows by substitution of this solution into Eq. (89).

APPENDIX B: PROOF OF EQ. (25)

Although Eq. (25) appears to be an immediate consequence of Eqs. (1) and (56), the proof that $\psi_k^{(2)}(r)=[\tilde{\phi}(r)\Lambda^{-1}(r)]_k$ is a solution of Eq. (1) with $E=E_k$ depends on Appendix C, which in turn depends on the key result of this appendix, namely, that given in Eq. (B6) below. Since Eq. (25) follows immediately from Eq. (B6), the use of Eq. (56) does not really shorten the proof of Eq. (25).

With

$$L(r,r')\equiv\left(\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial r'^2}\right)K(r,r') \tag{B1}$$

and with

$$\tilde{M}(r)\equiv\tilde{\phi}(r)\Lambda^{-1}(r) \tag{B2}$$

we have

$$L(r,r')=\left[\frac{d^2}{dr^2}\tilde{M}(r)\right]\phi(r')+\tilde{M}(r)[\mathsf{E}-V_c(r')\mathbb{1}]\phi(r'). \tag{B3}$$

We shall first compute $d^2\tilde{M}(r)/dr^2$. We simplify the notation by suppressing the argument r and denoting differentiation by primes. Then

$$\tilde{M}''=\tilde{\phi}(V_c\mathbb{1}-\tilde{\mathsf{E}})\Lambda^{-1}+2\tilde{\phi}\Lambda^{-1}\phi\tilde{\phi}\Lambda^{-1}\phi\tilde{\phi}\Lambda^{-1} \tag{B4a}$$

$$-2\tilde{\phi}'\Lambda^{-1}\phi\tilde{\phi}\Lambda^{-1}-\tilde{\phi}\Lambda^{-1}(\phi\tilde{\phi}'+\phi'\tilde{\phi})\Lambda^{-1}. \tag{B4b}$$

Line (B4b) of this equation can be rearranged to be

$$-2\tilde{M}(\tilde{\phi}'\Lambda^{-1}\phi+\tilde{\phi}\Lambda^{-1}\phi')-\tilde{\phi}\Lambda^{-1}\mathsf{W}\Lambda^{-1}, \tag{B5}$$

where W is the $n\times n$ wronskian matrix defined in Eq. (39). Line (B4a) and the first term of line (B5) combine to yield $\tilde{M}(V\mathbb{1}-\mathsf{E})+\tilde{\varphi}(\Lambda^{-1}\mathsf{E}-\tilde{\mathsf{E}}\Lambda^{-1})$, while from Eq. (42) the second term of line (B5) is $-\tilde{\varphi}(\Lambda^{-1}\mathsf{E}-\tilde{\mathsf{E}}\Lambda^{-1})$. Hence

$$\tilde{M}''(r)=\tilde{M}(r)[V(r)\mathbb{1}-\mathsf{E}]. \tag{B6}$$

Equation (25) follows immediately.

APPENDIX C: SOLUTIONS OF THE NEW SCHRÖDINGER EQUATION

In this section, we verify that Eqs. (46) do indeed yield a solution of the new Schrödinger equation. Equation (46b) may be written

$$\psi=\varphi-\tilde{M}\Gamma, \tag{C1}$$

where $\tilde{M}(r)$ and $\Gamma(r,E)$ were defined in Eqs. (B2) and (45), respectively, and for compactness we have again omitted the arguments r and E. Then

$$-\psi''+(V-E)\psi=(V-V_c)\varphi-N, \tag{C2}$$

where

$$N(E,r)=\left(-\frac{d^2}{dr^2}+V-E\right)\tilde{M}\Gamma. \tag{C3}$$

From Eq. (B6),

$$N=\{\tilde{M}(\mathsf{E}-E\mathbb{1})\Gamma \tag{C4a}$$

$$-2\tilde{M}'\phi\varphi-\tilde{M}(\phi\varphi'+\phi'\varphi)\}, \tag{C4b}$$

and the second line of this equation is

$$2[K(r,r)]^2\varphi-2(\tilde{\phi}'\Lambda^{-1}\phi+\tilde{\phi}\Lambda^{-1}\phi')\varphi-\tilde{M}w, \tag{C5}$$

where the column matrix w was defined in Eq. (43). The first two terms of Eq. (C5) combine to give $(V-V_c)\varphi$, while from Eq. (47) the third term is $\tilde{M}(E\mathbb{1}-\mathsf{E})\Gamma$. Hence $N=(V-V_c)\varphi$ and the right side of Eq. (C2) is zero, as we wished to prove.

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[1] D. L. Pursey and T. A. Weber, Phys. Rev. A **50**, 4472 (1994).
[2] R. Jost, Physica **12**, 509 (1946); Helv. Phys. Acta **20**, 256 (1947); R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952).
[3] Marshall Luban and D. L. Pursey, Phys. Rev. D **33**, 431 (1986).
[4] D. L. Pursey, Phys. Rev. D **33**, 1048, (1986); **33**, 2267 (1986); **36**, 1103 (1987).
[5] I. M. Gel'fand and B. M. Levitan, Dokl. Akad. Nauk SSSR **77**, 557 (1951) [Math. Rev. **13**, 240 (1952)]; Isv. Akad. Nauk SSSR Ser. Math. **15**, 309 (1951); Am. Math. Soc. Transl. **1**, 253 (1955).
[6] V. A. Marchenko, Dokl. Akad. Nauk SSSR **72**, 457 (1950) [Math. Rev. **12**, 183 (1951)]; **104**, 695 (1955) [**17**, 740 (1956)]; Trudy Moskov. Mat. Obsc. **1**, 327 (1952); **2**, 3 (1953); **52**, 739 (1960); **77**, 139 (1968).
[7] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem in Scattering Theory* (Gordon and Breach, New York, 1963).
[8] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer-Verlag, New York, 1977).
[9] Roger G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Springer-Verlag, New York, 1982).
[10] H. E. Moses and San Fu Tuan, Nuovo Cimento **13**, 197 (1959).
[11] N. Meyer-Vernet, Am. J. Phys. **50**, 354 (1982).
[12] T. A. Weber and D. L. Pursey, Phys. Rev. A **50**, 4478 (1994).
[13] P. B. Abraham and H. E. Moses, Phys. Rev. A **22**, 1333 (1980).
[14] G. Darboux, C. R. Acad. Sci. **94**, 1456 (1882).
[15] E. Witten, Nucl. Phys. **B188**, 513 (1981).
[16] Marshall Luban, James H. Luscombe, Mark A. Reed, and D. L. Pursey, Appl. Phys. Lett. **54**, 1997 (1989).
[17] V. Bargmann, Phys. Rev. **75**, 301 (1949); Rev. Mod. Phys. **21**, 488 (1949).
[18] We have changed the notation here, motivated by Eq. (37). In Ref. [1], the matrix $\Lambda(r)$ was denoted by $\Delta(r)$.
[19] W. R. Theis, Z. Naturforsch; Teil A **11**, 889 (1956).
[20] N. Levinson, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **25**, No. 9, (1949).
[21] J. von Neumann and E. Wigner, Z. Phys. **30**, 465 (1929).
[22] D. L. Pursey and T. A. Weber, following paper, Phys. Rev. A **52**, 3932 (1995).