Number-phase Wigner function on Fock space

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We define a quasiprobability distribution $S_{\rm NP}(n,\theta)$ which describes the quantum statistics of the photon number and phase observables of a single-mode field (or, equivalently, a harmonic oscillator). The properties of $S_{\rm NP}(n,\theta)$ are the photon number and phase analogies of the properties of Wigner's original function, which describes the position and momentum observables. For example, the marginals of $S_{\rm NP}(n,\theta)$ are the continuous phase and the discrete photon-number probability distributions. We give examples of the $S_{\rm NP}(n,\theta)$ representation of various states and show, in particular, that $S_{\rm NP}(n,\theta)$ displays the quantum interference associated with Schrödinger cat states. We also describe how $S_{\rm NP}(n,\theta)$ can be determined from quantities that are, in principle, measurable.

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I. INTRODUCTION

It is now over 60 years since Wigner introduced his celebrated quasiprobability distribution for illustrating the difference between classical and quantum statistics [1]. Wigner's function expresses the quantum statistics of a pair of canonically conjugate observables that have continuous spectra. Over the intervening years a number of related quasiprobability distributions have been introduced mainly for quantum observables with discrete spectra and associated with finite-dimensional systems. For example, Agarwal [2] (see also [3]) introduced quasiprobability distributions for finite-dimensional systems based on the atomic coherent state formalism of Arecchi *et al.* [4]. Agarwal's quasiprobability distributions are functions of continuous variables in contrast to the discrete Wigner function defined by Wootters [5], which is a function of discrete variables for the same finite-dimensional systems. The discrete nature of Wootters's function reflects the discrete nature of the spectra of the underlying pair of general canonically conjugate observables. Mukunda [6] introduced a Wigner function for the canonically conjugate rotation angle and angular momentum observables; recently this function was further justified and studied comprehensively by Bizarro [7]. An interesting point about the Mukunda-Bizarro function is that one variable is continuous (angle) and the other is discrete (angular momentum) and so it strides both discrete and continuous domains.

Another pair of canonically conjugate observables having both a discrete and a continuous spectrum are the phase and photon-number observables of a single-mode field (or, equivalently, a harmonic oscillator). The definition of the phase observable has been studied extensively in recent years [8] and there are now a number of different formalisms for describing it that yield identical physical results; these are the Newton, Pegg-Barnett, Ban, Helstrom-Shiparo-Shepard probability-operator measure (POM), and the $H_{\rm sym}$ formalisms [9]. Each of the these formalisms is based on a different state space. Newton [10] uses a Hilbert space that contains negative as well as positive photon-number states. Pegg and Barnett [11–14] use a Hilbert space of finite dimension and a special procedure for taking the infinite-dimensional limit. It has been shown recently [15] that the special limiting procedure essentially builds a vector space E that is larger than the conventionally used Fock space. Ban's approach [16,17] employs a tensor product of two Fock spaces. In the POM approach of Helstrom [18] and Shapiro and Shepard [19] (see also [20,21]) only the conventionally used Fock space is required and phase operators are not considered explicitly. Instead only the phase probability distribution is defined and from this expectation values of a stochastic phase variable are found. The $H_{\rm sym}$ formalism [22] is based on an infinite-dimensional Hilbert space $H_{\rm sym}$, which is an extension of the conventionally used Fock space and contains vectors representing infinite photon number; $H_{\rm sym}$ has the distinction of supporting the strong limits of the Pegg-Barnett phase operators.

In this paper we wish to define a quasiprobability distribution that has properties analogous to those of Wigner's function, but is associated with the canonically conjugate [23] phase and photon-number observables rather than the position and momentum observables treated by Wigner [24]. For this we require a state space on which phase and photon-number probability distributions are defined. Fortunately, all the previously mentioned phase formalisms agree with respect to these distributions for physically relevant states. Moreover, it turns out that we do not require a phase operator explicitly for the definition of the quasiprobability distribution and so we have some latitude in our choice of the state space. Since the Fock space conventionally used to represent the state of a single-mode field does support the phase and photon-number probability distributions it provides a sufficient state space for our analysis. Thus we confine our attention to defining a quasiprobability distribution on the conventionally used Fock space in this paper.

In a previous work [25] we defined a quasiprobability distribution $W_{N\phi}(n,\theta)$ associated specifically with the Pegg-Barnett phase formalism. This function, which we shall call the *discrete number-phase Wigner function*, is based on

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Wootters's definition of a discrete Wigner function for a finite-dimensional space (e.g., of dimension $s+1$). In accordance with the Pegg-Barnett formalism, the limit of infinite s is taken only after expectation values have been calculated. An unusual feature of $W_{N\phi}(n,\theta)$ is that for some states of the field (including coherent states) it undergoes a revival as the number variable n is increased beyond the value representing half the dimension $s/2$. Thus there appears to be no simple way of taking the infinite-dimensional limit of the Wigner function itself for these states. Lukš and Peřinová [26] attempted to overcome this problem by defining an analogous function $W_{n\varphi}(n,\theta)$ at half integer values of the photon-number variable. Although the function $W_{n\varphi}(n,\theta)$ at the fractional values of n has no direct physical relevance, nevertheless, the values of $W_{n\varphi}(n,\theta)$ at these values of n are required for normalization.

We take a fresh look at the problem in this paper by adopting a *first-principles* approach. We require the number-phase Wigner function to have the analogous properties of Wigner's original function as described by Hillery *et al.* [27]. The revivals found for the discrete number-phase Wigner function do not appear and there is no need for fractional values of n as used by Lukš and Peřinová. However, even after ensuring that the number-phase Wigner function has properties analogous to those of Wigner's original function as listed by Hillery *et al.*, we find that the number-phase Wigner function is not uniquely defined. We enlist an extra property, which is responsible for the quantum interference fringes displayed by Wigner's function for Schrödinger cat states, to lead us to the definition of the *special number-phase Wigner function* $S_{\rm NP}(n,\theta)$. A brief preliminary description of $S_{\rm NP}(n,\theta)$ was given recently in [28]. Here we give a more-detailed analysis. We illustrate $S_{\rm NP}(n,\theta)$ for various states and show, in particular, that it displays interference fringes for Schrödinger cat states in a manner similar to that of Wigner's original function. We also show that $S_{\rm NP}(n,\theta)$ can be determined from quantities that are, in principle, measurable.

The format of this paper is as follows. In Sec. II we define the number-phase Wigner function $S_{\rm NP}(n,\theta)$. In Sec. III we illustrate $S_{\rm NP}(n,\theta)$ for various states and in Sec. IV we discuss the $S_{\rm NP}$ representation of arbitrary operators. Then, in Sec. V we show how $S_{\rm NP}(n,\theta)$ for an unknown state of the field can be determined from measurable quantities. We end with a discussion in Sec. VI.

II. SPECIAL NUMBER-PHASE WIGNER FUNCTION $S_{\rm NP}$

We wish to give the special number-phase Wigner function $S_{\rm NP}(n,\theta)$ properties analogous to those of the position-momentum Wigner function $W(x,p)$ (for convenience we call these functions the NP-Wigner function and the PM-Wigner function, respectively). To this end we shall take the list of properties given by Hillery *et al.* [27] for the PM-Wigner function and transform them into analogous properties for the NP-Wigner function. We note that Wigner [29] has shown that the PM-Wigner function is uniquely determined by just five of these properties and also that O'Connell and Wigner [30] have shown that the PM-Wigner function is also uniquely determined by a different set of five properties. Our use of a superset of seven analogous defining properties will therefore yield a function that is acceptable as the Wigner function for the photon-number and phase observables. We use the symbol $S_{\rm NP}$ and the adjective "special" to distinguish the NP-Wigner function defined here from the discrete NP-Wigner function and the NP-Wigner function defined by Lukš and Peřinová [26].

A. Seven basic properties

We begin by ensuring that $S_{\rm NP}(n,\theta)$ is a bilinear functional of the state vector by specifying that it is the expectation value of the NP-Wigner operator $\hat{S}_{\rm NP}(n,\theta)$, which we represent as

$$\hat{S}_{\rm NP}(n,\theta)=\frac{1}{2\pi}\sum_{p,q=0}^{\infty}\Omega_{p,q}(n,\theta)|p\rangle\langle q| \qquad (1)$$

in the Fock basis for $n=0,1,2,\dots$ and for a real value of θ. Our task is to specify the elements $\Omega_{p,q}(n,\theta)$ of the matrix $\mathbf{\Omega}(n,\theta)$ that give $S_{\rm NP}$ its required properties. Most of these properties will be expressed in terms of expectation values; such expressions can be put into the form

$$\sum_{p,q=0}^{\infty} M_{p,q}f_p^*f_q=0 \ ,$$

where $M_{p,q}$ are the matrix elements of an operator $\hat{M}$ in the Fock basis and $f_n=\langle n|f\rangle$ are the Fock state coefficients of an arbitrary vector $|f\rangle$ in the Fock space. Since $|f\rangle$ is arbitrary it follows that the operator $\hat{M}$ itself vanishes. In the following, this allows us to translate the required properties of $S_{\rm NP}$ into requirements for the NP-Wigner operator $\hat{S}_{\rm NP}(n,\theta)$ and the associated matrix elements $\Omega_{p,q}(n,\theta)$. We now specify seven properties of $S_{\rm NP}$ using the same numbering scheme as Hillery *et al.* to aid a comparison of $S_{\rm NP}$ with the PM-Wigner function $W(x,p)$.

(i) $S_{\rm NP}(n,\theta)$ should be real and so $\hat{S}_{\rm NP}(n,\theta)$ should be Hermitian. Thus

$$\Omega_{p,q}(n,\theta)^*=\Omega_{q,p}(n,\theta) \ . \qquad (2)$$

(ii) The marginal distributions of $S_{\rm NP}$ should be the (normalized) number and phase probability distributions

$$\int_{2\pi}S_{\rm NP}(n,\theta)d\theta=\langle|n\rangle\langle n|\rangle \ , \qquad (3)$$

$$\sum_{n=0}^{\infty} S_{\rm NP}(n,\theta)=\langle|\theta\rangle\langle\theta|\rangle \ . \qquad (4)$$

The phase state $|\theta\rangle\equiv(2\pi)^{-1/2}\sum_{n=0}^{\infty}\exp(in\theta)|n\rangle$ in the expression (4) belongs to a rigged Hilbert space [31]. It yields the now-well-established [10–13,15–22,32–34] definition of the phase probability distribution $P(\theta)=\langle|\theta\rangle\langle\theta|\rangle$ for states belonging to the conventionally used infinite-dimensional Fock space. Thus we require the NP-Wigner operator to satisfy

$$\int_{2\pi} \hat{S}_{\rm NP}(n,\theta)d\theta = |n\rangle\langle n| \ ,$$

$$\sum_{n=0}^{\infty} \hat{S}_{\rm NP}(n,\theta) = |\theta\rangle\langle\theta| \ .$$

These expressions imply that

$$\frac{1}{2\pi}\int_{2\pi} \Omega_{p,q}(n,\theta)d\theta = \delta_{p,n}\delta_{q,n} \quad (5)$$

and

$$\sum_{n=0}^{\infty} \Omega_{p,q}(n,\theta) = e^{i(p-q)\theta} \ , \quad (6)$$

respectively, where $\delta_{n,m}$ is the Kronecker delta function.

(iii) $S_{\rm NP}$ should be Galilei invariant in the sense that shifts in phase $\langle\theta|f\rangle \mapsto \langle\theta+\Delta|f\rangle$ and photon number $\langle n+1|f\rangle \mapsto \langle n|f\rangle$ should produce the corresponding shifts $S_{\rm NP}(n,\theta) \mapsto S_{\rm NP}(n,\theta+\Delta)$ and $S_{\rm NP}(n+1,\theta) \mapsto S_{\rm NP}(n,\theta)$, respectively, in $S_{\rm NP}$. A phase shift of Δ is generated by the operator $\exp(i\hat{N}\Delta)$, where $\hat{N}$ is the photon-number operator. Thus we require that

$$e^{i\hat{N}\Delta}\hat{S}_{\rm NP}(n,\theta)e^{-i\hat{N}\Delta} = \hat{S}_{\rm NP}(n,\theta+\Delta) \ .$$

Expressing the NP-Wigner operator in terms of the coefficients given by Eq. (1) and equating matrix elements in the Fock basis yields

$$\Omega_{p,q}(n,\theta)e^{i(p-q)\Delta} = \Omega_{p,q}(n,\theta+\Delta)$$

and thus we have

$$\Omega_{p,q}(n,\theta) = \Omega_{p,q}(n)e^{i(p-q)\theta} \ , \quad (7)$$

where we have defined

$$\Omega_{p,q}(n) \equiv \Omega_{p,q}(n,0) \ . \quad (8)$$

Substituting $\Omega_{p,q}(n,\theta)$ from Eq. (7) into Eqs. (2), (5), and (6) gives

$$\Omega_{p,q}(n)^* = \Omega_{q,p}(n) \ , \quad (9)$$

$$\Omega_{p,p}(n) = \delta_{p,n} \ , \quad (10)$$

$$\sum_{n=0}^{\infty} \Omega_{p,q}(n) = 1 \ , \quad (11)$$

respectively.

The shift in photon number $\langle n+1|f\rangle \mapsto \langle n|f\rangle$ is produced by the operator, $\widehat{e^{-i\phi}} = \sum_{n=0}^{\infty}|n+1\rangle\langle n|$, i.e., $\langle n+1|\widehat{e^{-i\phi}}|f\rangle = \langle n|f\rangle$. (We note that $\widehat{e^{-i\phi}}$ is the Susskind-Glogower [35] exponential phase operator, which is well known to be nonunitary. However, we need only the "upward" number-shifting property of $\widehat{e^{-i\phi}}$; the nonunitary nature of $\widehat{e^{-i\phi}}$ is of no consequence here because the operation $|f\rangle \mapsto \widehat{e^{-i\phi}}|f\rangle$ preserves the norm.) Thus, for the number-shift invariance property we require that

$$\widehat{e^{i\phi}}\hat{S}_{\rm NP}(n+1,\theta)\widehat{e^{-i\phi}} = \hat{S}_{\rm NP}(n,\theta) \ ,$$

where $\widehat{e^{i\phi}} = \widehat{e^{-i\phi}}^{\dagger}$, from which we find that

$$\Omega_{p,q}(n) = \Omega_{p+1,q+1}(n+1) \quad (12)$$

for non-negative integers p,q,n. This translational property allows a more-convenient representation of the matrix elements $\Omega_{p,q}(n)$ as follows: we extend the definition of $\Omega_{p,q}(0)$ to negative integer values of p and q according to

$$\Omega_{p-n,q-n}(0) = \Omega_{p,q}(n) \quad (13)$$

for $n > p,q$. That this gives a unique definition of $\Omega_{p,q}(0)$ for negative integers p and q is easily proved from Eq. (12). Thus, specifying $\Omega_{p,q}(0)$ for all integers p and q is equivalent to specifying $\Omega_{p,q}(n)$ for all non-negative integers p,q,n. Equation (11) can now be written as

$$\sum_{n=0}^{\infty} \Omega_{p-n,q-n}(0) = 1 \ , \quad (14)$$

which holds for all non-negative integers p,q.

(iv) $S_{\rm NP}$ should be invariant with respect to a reflection in time and a phase shift of π rad. That is, if $\langle n|f\rangle \mapsto \langle n|f\rangle^*$, then $S_{\rm NP}(n,\theta) \mapsto S_{\rm NP}(n,-\theta)$, and if $\langle\theta|f\rangle \mapsto \langle\theta+\pi|f\rangle$, then $S_{\rm NP}(n,\theta) \mapsto S_{\rm NP}(n,\theta+\pi)$.

In the former invariance, the transformation is a time reflection in the sense that if $f_n(t) \equiv \langle n|f(t)\rangle = \langle n|\hat{U}(t)|f(0)\rangle = e^{-in\omega t}f_n(0)$, where $\hat{U}(t) = e^{-i\hat{N}\omega t}$ is the time evolution operator at time t, i.e., $|f(t)\rangle = \hat{U}(t)|f(0)\rangle$, then $|f(t)^*\rangle \equiv \Sigma f_n(t)^*|n\rangle = \hat{U}(-t)|f(0)^*\rangle$. This invariance implies that

$$\sum_{p,q=0}^{\infty} \Omega_{p,q}(n,\theta)f_p f_q^* = \sum_{p,q=0}^{\infty} \Omega_{p,q}(n,-\theta)f_p^* f_q$$

for arbitrary vector $|f\rangle$ and so

$$\Omega_{p,q}(n,\theta) = \Omega_{q,p}(n,-\theta) \ .$$

Combining this result with Eq. (7) yields

$$\Omega_{p,q}(n) = \Omega_{q,p}(n) \quad (15)$$

and so from Eq. (9) we obtain

$$\Omega_{p,q}(n) = \Omega_{p,q}(n)^* \ , \quad (16)$$

that is, $\mathbf{\Omega}(n)$ is a real symmetric matrix. We can now express the matrices $\mathbf{\Omega}(n)$ in a more convenient form by identifying the diagonals of the matrix $\mathbf{\Omega}(0)$ as follows: let $\mathbf{\Lambda}(r)$ be the rth diagonal of $\mathbf{\Omega}(0)$, where

$$\Lambda_p(r) \equiv \Omega_{p,p-r}(0) = \Omega_{p-r,p}(0) \quad (17)$$

for integers p,q and $r = 0,1,2,\ldots$, and thus, from Eq. (13),

$$\Omega_{p-n,q-n}(0) = \Omega_{p,q}(n) = \begin{cases} \Lambda_{p-n}(p-q) & \text{for } p \geq q \\ \Lambda_{q-n}(q-p) & \text{for } p < q \end{cases} \quad (18)$$

for non-negative integers p, q, and n.

The latter invariance under a phase shift of π rad is the analogy of the invariance of the PM-Wigner function to a spatial reflection, i.e., where $W(x,p)\mapsto W(-x,-p)$, which is a rotation by π rad in the x-p plane about the origin. However, this invariance supplies no extra restriction on $S_{\rm NP}$ as the invariance under phase shifts follows automatically from the Galilei invariance property (iii).

(v) The equation of motion of $S_{\rm NP}$ for a free oscillator should be the classical one. Under free evolution the wave function experiences the phase shift $\langle\theta|f\rangle\mapsto\langle\theta+\omega t|f\rangle$ and so from the Galilei invariance property (iii) we find that $S_{\rm NP}(n,\theta,t)\mapsto S_{\rm NP}(n,\theta+\omega t,0)$, which is the expected evolution of the corresponding classical phase-space distribution. [Here $S_{\rm NP}(n,\theta,t)$ is the Wigner function at time t.] Thus the classical equation of motion follows automatically from the Galilei invariance under phase shifts.

(vi) $S_{\rm NP}$ should have the overlap property

$$2\pi\int_{2\pi}d\theta\sum_{n=0}^{\infty}S_{\rm NP}(n,\theta)S'_{\rm NP}(n,\theta)=|\langle f|g\rangle|^2 \ ,$$

where $S_{\rm NP}(n,\theta)$ and $S'_{\rm NP}(n,\theta)$ are the NP-Wigner functions for the pure states $|f\rangle$ and $|g\rangle$, respectively. This property should also extend to mixed states. We note that because this property is a special case of the next property (vii) we need only consider the latter more-general property.

(vii) $S_{\rm NP}$ should give the trace of a product of general operators as

$$\mathrm{tr}(\hat A\hat B)=2\pi\int_{2\pi}d\theta\sum_{n=0}^{\infty}S_{\rm NP}^{(A)}(n,\theta)S_{\rm NP}^{(B)}(n,\theta) \ , \tag{19}$$

where $S_{\rm NP}^{(A)}$ and $S_{\rm NP}^{(B)}$ are the NP-Wigner representations

$$S_{\rm NP}^{(A)}(n,\theta)=\mathrm{tr}[\hat S_{\rm NP}(n,\theta)\hat A] \ , \tag{20}$$

$$S_{\rm NP}^{(B)}(n,\theta)=\mathrm{tr}[\hat S_{\rm NP}(n,\theta)\hat B] \tag{21}$$

of any two linear operators $\hat A$ and $\hat B$ on the Fock space. We have introduced a different notation in these expressions to distinguish the Wigner function $S_{\rm NP}$, which is the expectation value of the NP-Wigner operator $\hat S_{\rm NP}$ and has all the properties discussed in this section, from the Wigner representation $S_{\rm NP}^{(A)}$ of an arbitrary linear operator $\hat A$, which is given by Eq. (20) and only need satisfy the trace property of Eq. (19). In the special case where $\hat A$ and $\hat B$ are density operators, Eqs. (20) and (21) are identical to the definition of the Wigner functions given by Eq. (1) and so we can drop the superscripts (A) and (B); thus Eq. (19) extends property (vi) to the general situation, which includes mixed states. Substituting for $\hat S_{\rm NP}$ in Eqs. (20), and (21), evaluating the trace in the Fock basis, making use of Eq. (7) and performing the integral over θ yields

$$\sum_{p,q=0}^{\infty}A_{p,q}B_{q,p}=\sum_{p,q=0}^{\infty}\sum_{p',q'=0}^{\infty}\sum_{n=0}^{\infty}\Omega_{p,q}(n)\Omega_{p',q'}(n)\times\delta_{p-q+p'-q',0}A_{q,p}B_{q',p'} \ ,$$

where $A_{p,q}\equiv\langle p|\hat A|q\rangle$ and $B_{p,q}\equiv\langle p|\hat B|q\rangle$. Since $\hat A$ and $\hat B$ are arbitrary operators we may choose the matrix elements $A_{p,q}$ and $B_{p,q}$ at will; thus choosing $A_{j,k}$ and $B_{r,s}$ as the only nonzero matrix elements gives

$$A_{j,k}B_{r,s}\delta_{k,r}\delta_{j,s}=\sum_{n=0}^{\infty}\Omega_{k,j}(n)\Omega_{s,r}(n)\,\delta_{k-j+s-r,0}A_{j,k}B_{r,s}$$

and thus

$$\delta_{k,r}\delta_{j,s}=\sum_{n=0}^{\infty}\Omega_{k,j}(n)\Omega_{s,r}(n)\,\delta_{k-j,r-s} \ . \tag{22}$$

Setting $k=j+r-s$ and $r\geqslant s$ and making use of Eq. (18) gives

$$\delta_{j,s}=\sum_{n=0}^{\infty}\Lambda_{j+r-s-n}(r-s)\Lambda_{r-n}(r-s) \ . \tag{23}$$

The expression for $s>r$ is obtained on interchanging r and s.

Hillery *et al.* also listed an eighth property concerning a symmetry between the position and momentum representations of the PM-Wigner function. This property is, however, unsuitable for defining the NP-Wigner function because of the asymmetry between the unbounded discrete photon-number and bounded continuous-phase spectra. Rather, we define the NP-Wigner function by another criterion in Sec. II B and then check *a posteriori* the presence of this property.

Collecting our results we find from Eqs. (1), (7), and (18) that

$$\hat S_{\rm NP}(n,\theta)=\frac{1}{2\pi}\left\{\sum_{p=0}^{\infty}\Lambda_{p-n}(0)|p\rangle\langle p|+\left[\sum_{p=0}^{\infty}\sum_{q=0}^{p-1}\Lambda_{p-n}(p-q)\times e^{i(p-q)\theta}|p\rangle\langle q|+\text{ H.c.}\right]\right\} \ , \tag{24}$$

where $\{\mathbf{\Lambda}(n)\}_{n=0,1,2,\ldots}$ is a set of vectors with the following properties. The elements of $\mathbf{\Lambda}(n)$ are given by $\Lambda_m(n)$ for integer m and, according to Eqs. (10), (11), and (16) with Eq. (18), satisfy

$$\Lambda_{p-n}(0)=\delta_{p,n} \ , \tag{25}$$

$$\sum_{m=0}^{\infty}\Lambda_{p-m}(r)=1 \ , \tag{26}$$

$$\Lambda_{p-n}(r)=\Lambda_{p-n}(r)^* \ , \tag{27}$$

respectively, for non-negative integers n and p and for $0\leqslant r\leqslant p$. The elements also satisfy Eq. (23), which can be rewritten as

$$\delta_{p,q}=\sum_{n=0}^{\infty}\Lambda_{p-n}(r)\Lambda_{q-n}(r) \tag{28}$$

for non-negative p and q and for $0 \leqslant r \leqslant p,q$.

It is worthwhile to trace the origins of these expressions if only to keep track of the analysis so far: the Hermitian form of Eq. (24) arises from property (i); the factorization of the θ dependence in Eq. (24) is due to the invariance to shifts in phase (iii) and the classical equation of motion (v); Eqs. (25) and (26) arise from the marginal distribution requirements (ii) for the photon-number and phase probability distributions, respectively; Eq. (27) is due to the invariance to time reflections (iv) and Eq. (28) arises from the overlap (vi) and trace (vii) requirements. The invariance to shifts in photon number, property (iii), is responsible for the translational property of Eq. (12), which allows $\Omega_{p,q}(n)$ to be written in terms of the vectors $\mathbf{\Lambda}(r)$ as given by Eq. (18).

Any set of vectors $\{\mathbf{\Lambda}(n)\}$ satisfying these equations will give rise to a NP-Wigner operator, which has all seven of the properties considered. Let us examine what sort of vectors they are. Equation (25) specifies uniquely the vector $\mathbf{\Lambda}(0)$; it has only one nonzero element i.e., $\Lambda_0(0)=1$. Setting $p=r,r+1,r+2,\ldots$ successively in Eq. (26) reveals that

$$\Lambda_m(r)=0 \quad \text{for} \quad m>r \tag{29}$$

for all vectors $\mathbf{\Lambda}(r)$ and so Eq. (26) implies that the sum of all the elements of any given vector is unity. We now show that only one element of each vector is nonzero. Choose any vector except $\mathbf{\Lambda}(0)$, say, $\mathbf{\Lambda}(r')$, with $r'>0$, and for clarity relabel its elements as $d_n=\Lambda_{r'-n}(r')$ for $n=0,1,2,\ldots,$ where d_n are the elements of the vector $\boldsymbol{d}$. The remaining elements of $\mathbf{\Lambda}(r')$ are zero according to Equation (29). Equation (27) implies that the d_n are real and Eq. (28) with $p=q=r=r'$ shows that the vector $\boldsymbol{d}$ is normalized as $\Sigma_{n=0}^{\infty}(d_n)^2=1$. The periodic function $f(\theta)$ defined as

$$f(\theta)=\frac{1}{2\pi}\left|\sum_{n=0}^{\infty} d_n e^{in\theta}\right|^2$$

has Fourier coefficients f_m, which are given by

$$f_m=\int_{2\pi} d\theta \; f(\theta)e^{im\theta}$$

$$=\sum_{n=|m|}^{\infty} d_n d_{n-|m|}=\sum_{n=|m|}^{\infty} \Lambda_{r'-n}(r')\Lambda_{r'+|m|-n}(r')=\delta_{m,0}$$

for an integer m. We arrived at the last two lines by making use of Eq. (28) with $p=r'$, $q=r'+|m|$, and $r=r'$ and noting that $\Lambda_{r'+|m|-n}(r')=0$ for $n=0,1,2,\ldots,|m|-1$ according to Eq. (29). Reconstructing $f(\theta)$ from its Fourier components yields

$$f(\theta)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} f_m e^{-im\theta}=\frac{1}{2\pi} \; ,$$

which is independent of θ. Hence *only one element* of $\boldsymbol{d}$, and thus of $\mathbf{\Lambda}(r')$, is nonzero and, from Eq. (26), the nonzero value is unity. But there are no further restrictions on the vectors $\mathbf{\Lambda}(r)$. Apart from satisfying Eqs. (25) and (29), the unit values can otherwise occur at any position in the vector $\mathbf{\Lambda}(r)$ and so $\hat{S}_{\mathrm{NP}}$ is not uniquely specified at this stage. Thus, even though only five of the properties listed by Hillery *et al.* [27,29,30] are sufficient to uniquely define the PM-Wigner function $W(x,p)$, all seven of the analogous properties considered here are not sufficient to define the NP-Wigner function uniquely.

B. An additional property

We need an additional property to further restrict the vectors $\mathbf{\Lambda}(n)$. Which extra property of the PM-Wigner function $W(x,p)$ should we use? The interference fringes [36–38] displayed by $W(x,y)$ have been a valuable tool in the study of coherent superpositions (e.g. Schrödinger cat states [39–41]) and the loss of coherence in noisy environments [42]. Giving S_{NP} an analogous property would be a great advantage. The interference fringes in $W(x,p)$ arise as a result of the wave function interfering with itself; that is, $W(x,p)$ is the Fourier transform of the "interfering" product $f(x-y)f^*(x+y)$, where $f(x)$ is the wave function. This property is expressed in terms of the PM-Wigner operator matrix elements as a "skew diagonal" form, i.e.,

$$\langle \mu|\hat{W}(x,p)|\nu\rangle=\frac{1}{\pi}\int_{-\infty}^{\infty} dy \; e^{2ipy}\langle \mu|x+y\rangle\langle x-y|\nu\rangle$$

$$=\frac{1}{\pi}e^{2ip(\mu-x)}\delta(\mu+\nu-2x) \; , \tag{30}$$

where the bras and kets in this expression are position eigenstates and $\delta(x)$ is the Dirac distribution. $W(x,p)$ is given by the expectation value of $\hat{W}(x,p)$.

A first attempt at translating this form to one suitable for the number-phase Wigner function might be to require that $\langle p|\hat{S}_{\mathrm{NP}}(n,\theta)|q\rangle$ be proportional to the Kronecker delta $\delta_{p+q,2n}$. However, we would then find that the resulting vector $\mathbf{\Lambda}(p-q)$, for $p\geqslant q$, would not satisfy Eq. (26) since the left-hand side of Eq. (26) would be zero for odd values of $(p+q)$. Similarly, requiring $\langle p|\hat{S}_{\mathrm{NP}}(n,\theta)|q\rangle$ to be proportional to $\delta_{p+q,2n-1}$ results in a vector $\mathbf{\Lambda}(p-q)$, for $p\geqslant q$, which gives a zero on the left-hand side of Eq. (26) for even $(p+q)$. The solution to this problem is to require $\langle p|\hat{S}_{\mathrm{NP}}(n,\theta)|q\rangle$ to be proportional to $\delta_{p+q,2n}+\delta_{p+q,2n-1}$, i.e.,

$$\Lambda_{p-n}(p-q)=\delta_{p+q,2n}+\delta_{p+q,2n-1} \tag{31}$$

for $p\geqslant q$. It is easily checked that this expression for $\mathbf{\Lambda}(p-q)$ satisfies all the requirements made above. Thus we *define* the special number-phase Wigner function in the Fock basis as [28]

$$\hat{S}_{\mathrm{NP}}(n,\theta)\equiv\frac{1}{2\pi}\sum_{p,q=0}^{\infty} e^{i(p-q)\theta}(\delta_{p+q,2n}+\delta_{p+q,2n-1})|p\rangle\langle q| \; , \tag{32}$$

which can be rearranged as

$$\hat{S}_{\rm NP}(n,\theta)=\frac{1}{2\pi}\left(\sum_{p=-n}^{n} e^{i2p\theta}|n+p\rangle\langle n-p| + \sum_{p=-n}^{n-1} e^{i(2p+1)\theta}|n+p\rangle\langle n-p-1|\right) \quad (33)$$

provided we take the second sum as being zero for $n=0$. $S_{\rm NP}$ can be expressed in terms of the phase state basis as

$$\hat{S}_{\rm NP}(n,\theta)=\frac{1}{2\pi}\int_{2\pi} d\phi\ e^{-2in\phi}(1+e^{i\phi})|\theta+\phi\rangle\langle\theta-\phi| \ . \quad (34)$$

One can check this last step by evaluating the Fock state matrix elements $\langle p|\hat{S}_{\rm NP}(n,\theta)|q\rangle$ from Eq. (34) and comparing them with the corresponding elements in Eq. (32) or (33).

C. Comparison with other Wigner functions

It is interesting to compare $S_{\rm NP}$ with the PM-Wigner function

$$W(x,p)=\frac{1}{\pi}\left\langle \int_{-\infty}^{\infty} dy\ e^{2ipy}|x+y\rangle\langle x-y|\right\rangle \ ,$$

where the bras and kets are position eigenstates, or, equivalently,

$$W(x,p)=\frac{1}{\pi}\left\langle \int_{-\infty}^{\infty} dy\ e^{-2ixy}|p+y\rangle\langle p-y|\right\rangle, \quad (35)$$

where the bras and kets are momentum eigenstates. The expressions on the right-hand sides of Eqs. (34) and (35) share a high degree of similarity. The main formal difference between the operators in Eqs. (35) and (34), apart from the different limits of integration, is the extra factor $\frac{1}{2}[1+\exp(i\phi)]$ in Eq. (34). This factor is a direct result of the *two* Kronecker δ's on the right-hand side of Eq. (31) whose presence can be traced to the discrete nature of the photon-number spectrum as follows. If the photon number had a continuous spectrum, then the variables p,q,n in Eq. (31) would be continuous and there would exist solutions to $p+q=2n$ for every value of p and q; thus we would need only one term on the right-hand side of Eq. (31) and so the factor $\frac{1}{2}[1+\exp(i\phi)]$ in Eq. (34) would be replaced with 1. The upshot of this hypothetical continuous n case is that Eq. (34) would then be in exact formal agreement with Eq. (35). Hence, we conclude the main formal difference between $W(x,p)$ and $S_{\rm NP}(n,\theta)$ is due to the discrete nature of the photon-number spectrum.

It is also interesting to compare $S_{\rm NP}$ with previous definitions of number-phase Wigner functions. We consider first the discrete number-phase Wigner function $W_{N\phi}(n,\theta)$ for physical states. We find that

$$S_{\rm NP}(n,\theta)=\lim_{s\to\infty}\frac{s+1}{2\pi}[W_{N\phi}(n,\theta)+W_{N\phi}(n+s/2,\theta)] \ ,$$

where $W_{N\phi}(n,\theta)$ is defined on a $(s+1)$-dimensional Hilbert space [see Ref. [25], Eq. (4.4)]. This shows how $S_{\rm NP}$ takes care of the approximate revival in the n dependence of $W_{N\phi}$ for $n>s/2$. For example, in the case of a coherent state the second peak in $W_{N\phi}$ is simply superimposed on the first peak. Comparing $S_{\rm NP}$ now with the function $W_{n\varphi}$ defined by Lukš and Peřinová [26], we find that at the half-odd photon-number values of n, $W_{n\varphi}$ corresponds to the expectation value of the second term in Eq. (33). In fact,

$$S_{\rm NP}(n,\theta)=W_{n\varphi}(n,\theta)+W_{n\varphi}(n-1/2,\theta)$$

for $n=0,1,2,\ldots$, where each term on the right-hand side is equal to the expectation value of the corresponding operator on the right-hand side of Eq. (33). Hence, whereas Lukš and Peřinová take the values of $W_{N\phi}(n+s/2,\theta)$ as the half-odd photon-number values of their function, here we avoid the introduction of unphysical half-odd photon numbers altogether by, in effect, simply adding $W_{N\phi}(n+s/2,\theta)$ to $W_{N\phi}(n,\theta)$.

D. Approximate symmetry property

We now consider the symmetry (or lack of it) between the expressions for $S_{\rm NP}$ in the Fock and phase state bases; this symmetry is analogous to the eighth property of the PM-Wigner function listed by Hillery *et al.* [27]. Let us rewrite Eq. (32) in the following way:

$$\hat{S}_{\rm NP}(n,\theta)=\frac{1}{2\pi}\sum_{p,q=0}^{\infty} e^{2i(n-q)\theta} \times(\delta_{p+q,2n}+e^{-i\theta}\delta_{p+q,2n-1})|p\rangle\langle q| .$$

We wish to interchange n with θ and swap the Fock state with the phase state, Kronecker's delta $\delta_{k,j}$ with the periodic Dirac delta $\delta(k-j)=\sum_{n=-\infty}^{\infty}\exp[in(k-j)]/(2\pi)$, the photon-number variables p,q with the continuous phase variables ϕ,φ, and the sums $\sum_{p=0}^{\infty}$, $\sum_{q=0}^{\infty}$ with the integrals $\int_{2\pi}d\phi,\int_{2\pi}d\varphi$, respectively, in the expression on the right-hand side. To be consistent with the symmetry of $W(x,y)$ we must also swap c numbers with their complex conjugates. We note that the two Kronecker deltas $\delta_{p+q,2n}$ and $\delta_{p+q,2n-1}$ differ by a single step in the photon-number variable. If, in this process of interchanging and swapping, we adopt the principle that a single step in the photon number corresponds to an infinitesimally small (or zero) step in phase, then both Kronecker δ's become the same periodic Dirac δ distribution and we obtain

$$\frac{1}{2\pi}\int_{2\pi}d\phi\int_{2\pi}d\varphi\ e^{-2i(\theta-\varphi)n} \times\delta(\phi+\varphi-2\theta)(1+e^{in})|\phi\rangle\langle\varphi| \ .$$

Performing the integral over φ gives

$$\frac{1}{2\pi}\int_{2\pi}d\phi\ e^{-2in\phi}(1+e^{in})|\theta+\phi\rangle\langle\theta-\phi|, \quad (36)$$

which is quite similar to the right-hand side of Eq. (34). The only difference is that Eq. (34) contains the factor $\frac{1}{2}[1+\exp(i\phi)]$, whereas Eq. (36) contains the factor $\frac{1}{2}[1+\exp(in)]$. Hence this shows that there is an approximate

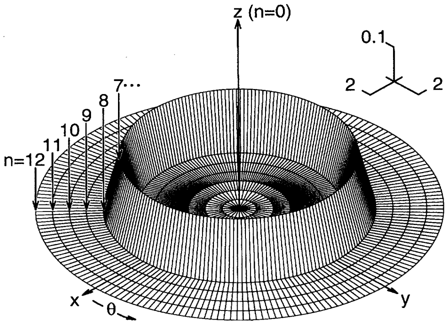

FIG. 1. Three-dimensional polar diagram illustrating S_{NP} for the single Fock state $|7\rangle$. The function $S_{\mathrm{NP}}(n,\theta)$ is depicted as an interpolated surface at a height of $z=S_{\mathrm{NP}}(n,\theta)$ above the point $(n\cos\theta, n\sin\theta)$ in the x-y plane, i.e., above the point (n,θ) in polar coordinates, for $n=0,1,2,\ldots$ and θ in $(0,2\pi)$. The surface is interpolated linearly between the integral values of n. On the surface radial lines indicate lines of constant θ and circles indicate lines of constant n. The triad in the top right corner gives the units for each axis. The raised ring at $n=7$ illustrates the uncertain phase and sharp photon number of a Fock state.

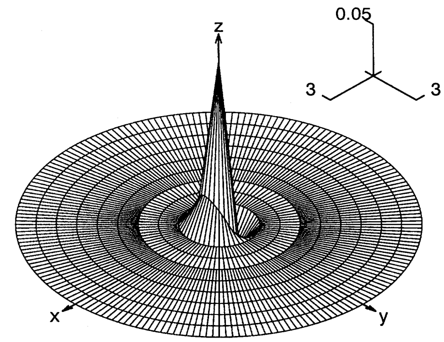

FIG. 2. S_{NP} for the weak coherent state $|\alpha\rangle$ with $\alpha=0.1$. The largest value of n is 10. The raised point of the vacuum state has broadened slightly.

symmetry between the two forms of S_{NP} in Eqs. (33) and (34), which is analogous to the symmetry of the PM-Wigner function under the transformation from the position to the momentum representation and vice versa. We noted earlier that the reason the factor $\frac{1}{2}[1+\exp(i\phi)]$ in Eq. (34) differs from unity can be attributed to the discrete nature of the photon-number spectrum. This suggests that the lack of an *exact* symmetry is due to the asymmetry between the discrete photon-number and continuous phase variables.

III. S_{NP} FOR VARIOUS STATES

The Fock and Glauber coherent states are two of the most familiar states in quantum optics. The S_{NP} representation of the general Fock state $|m\rangle$

$$S_{\mathrm{NP}}(n,\theta)=\frac{1}{2\pi}\delta_{n,m} \tag{37}$$

is a "raised ring" of radius m as illustrated in Fig. 1. The phase is completely uncertain whereas the photon number is sharp.

For the coherent state $|\alpha\rangle$ we find [28]

$$S_{\mathrm{NP}}(n,\theta)=\frac{e^{-|\alpha|^2}}{2\pi}\left[|\alpha|^{2n}\sum_{p=-n}^{n}\frac{e^{i2p(\theta-\varphi)}}{\sqrt{(n+p)!(n-p)!}}+|\alpha|^{2n-1}\sum_{p=-n}^{n-1}\frac{e^{i(2p+1)(\theta-\varphi)}}{\sqrt{(n+p)!(n-p-1)!}}\right], \tag{38}$$

where $\alpha=|\alpha|\exp(i\varphi)$. Let us look first at the weak-field regime. In the limit as $\alpha\to 0$ the coherent state $|\alpha\rangle$ becomes the vacuum state, which is represented by a raised point (i.e., a raised ring of zero radius) at the origin. For small, but nonvanishing, values of α we find that $S_{\mathrm{NP}}(0,\theta)\approx 1/(2\pi)$, $S_{\mathrm{NP}}(1,\theta)\approx|\alpha|\cos(\theta-\varphi)/\pi$, and $S_{\mathrm{NP}}(n,\theta)\approx 0$ for $n>1$ to first order in $|\alpha|$. Thus the raised point of the vacuum state broadens as $|\alpha|$ increases. This is illustrated in Fig. 2 for $\alpha=0.1$. It is perhaps a little surprising that $S_{\mathrm{NP}}(1,\theta)$ is negative for $\theta\leqslant\varphi-\pi/2$ and $\theta\geqslant\varphi+\pi/2$ in view of the fact that the PM-Wigner function is positive for all coherent states. Nevertheless, the phase probability distribution $P(\theta)$ obtained as a marginal of $S_{\mathrm{NP}}(n,\theta)$ is always positive as expected, i.e., $P(\theta)\approx[1+2|\alpha|\cos(\theta-\varphi)]/(2\pi)$ to first order in $|\alpha|$. For a slightly higher intensity with $\alpha\approx 0.64$ the picture shown in Fig. 3 is similar. The highest point is now situated on the $n=1$ curve along the $\theta=0$ direction.

For relatively intense coherent states we can approximate the Poisson photon-number distribution of the coherent state with a Gaussian of the same mean and variance. We find eventually that

$$S_{\mathrm{NP}}(n,\theta)\approx\frac{1}{2\pi}\{G_P(\theta)[G_N(n)+G_N(n-1/2)]+G_P(\theta+\pi)\times[G_N(n)-G_N(n-1/2)]\}\ , \tag{39}$$

where $G_N(n)\equiv\exp[-(n-\bar{n})^2/(2\bar{n})]$ and $G_P(\theta)=\exp[-2\bar{n}(\theta-\varphi)^2]$ for $\bar{n}=|\alpha|^2\gg 1$. In this intense-field regime the fluctuations in number and phase decouple and both the number and phase dependence of S_{NP} become approximately Gaussian. These features are evident even in the plot in Fig. 4 of

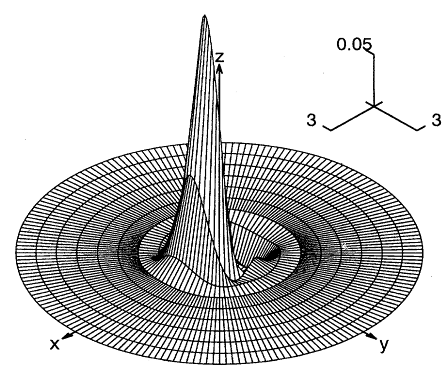

FIG. 3. S_{NP} for a coherent state $|\alpha\rangle$ of a slightly higher intensity with $\alpha=0.64$. The largest value of n is 10. The peak in Fig. 2 has moved to a point above the curve $n=1$ in the $\theta=0$ direction.

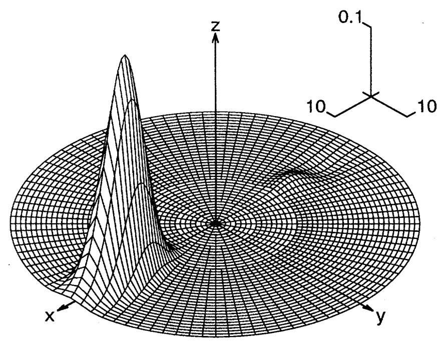

FIG. 4. S_{NP} for the coherent state $|\alpha\rangle$ with $\alpha=5$. The largest value of n is 40. The hill is centered on the point $n=25, \theta=0$.

S_{NP} for the coherent state $|\alpha\rangle$ with $\alpha=5$ where a relatively "smooth" hill shape is centered on the point $n=25$, $\theta=0$. Interestingly, there is also a small "wave" along $\theta=\pi$ in Fig. 4. This feature, however, has little effect on the phase distribution, which is obtained by summing S_{NP} radially. It arises from the relatively small difference between the two Gaussians $G_N(n)$ and $G_N(n-1/2)$ in Eq. (39). This difference vanishes as $\bar{n}$ increases and so in the limit of large $\bar{n}$ we obtain

$$S_{\mathrm{NP}}(n,\theta)\approx\frac{1}{\pi}G_P(\theta)G_N(n) \ , \tag{40}$$

which is a two-dimensional Gaussian in the polar coordinates n,θ centered on the point $n=|\alpha|^2$, $\theta=\phi$.

To compare this result with the PM-Wigner representation $W(x,p)$ of coherent states we need to change coordinates. First note that $S_{\mathrm{NP}}(n,\theta)$ is vanishingly small unless n and θ differ by relatively small amounts from their mean values $\bar{n}$ and φ, respectively. The transformation from the Cartesian coordinates (x,p) to the polar coordinates (r,θ), where $x=r\cos\theta$ and $p=r\sin\theta$, is characterized by the metric $dx^2+dp^2=dr^2+r^2d\theta^2$. Here the radial coordinate r corresponds to amplitude in the $W(x,p)$ phase space whereas n in Eq. (40) represents intensity. Transforming to an intensity coordinate n with $r=\sqrt{2n}$ gives $dx^2+dp^2=dn^2/(2n)+2nd\theta^2$. [The $\sqrt{2}$ factor arises from the fact that the mean of x^2+p^2 for $W(x,p)$ is $\langle 2\hat{N}+1\rangle$.] Replacing differentials with small deviations from mean values gives

$$(x-\bar{x})^2+(p-\bar{p})^2\approx(n-\bar{n})^2/(2\bar{n})+2\bar{n}(\theta-\varphi)^2 \ ,$$

where $x=\sqrt{2n}\cos(\theta)$, $p=\sqrt{2n}\sin(\theta)$. Hence the expression for $S_{\mathrm{NP}}(n,\theta)$ in Eq. (40) becomes approximately

$$\frac{1}{\pi}\exp[-(x-\bar{x})^2-(p-\bar{p})^2] \ ,$$

which is the PM-Wigner function $W(x,p)$ for the same coherent state where $\bar{x}=\sqrt{2\bar{n}}\cos(\varphi)$, $\bar{p}=\sqrt{2\bar{n}}\sin(\varphi)$ with $\bar{n}=|\alpha|^2$ and $\alpha=|\alpha|\exp(i\varphi)$. Thus $S_{\mathrm{NP}}(n,\theta)$, under the change of variables $n,\theta\mapsto x,p$, converges asymptotically to $W(x,p)$ for large $\bar{n}$.

There has been quite a deal of attention given recently to the study of Schrödinger cats states [39]. These states may be defined generally as superpositions of macroscopically distinguishable states [40,41]. The position-momentum Wigner function has been found to exhibit interference fringes for such states [36–38,42]. In [28] we gave the S_{NP} representation of the simplest "Fock cat," which is an equally weighted superposition of two Fock states. Here we shall give the S_{NP} representation of the most general two-state Fock cat $\cos(\eta)|m\rangle+e^{i\phi}\sin(\eta)|m+r\rangle$. This state has a S_{NP} representation given by

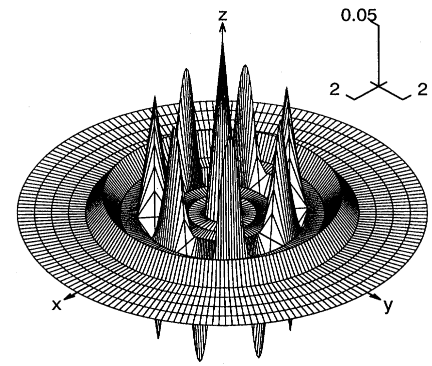

FIG. 5. S_{NP} for the Fock cat $\cos(\eta)|0\rangle+\sin(\eta)|7\rangle$ with $\eta=\pi/10$. The largest value of n is 12. The interference ring lies halfway between the raised point at the origin and the raised ring at $n=7$.

$$S_{\mathrm{NP}}(n,\theta)=\frac{1}{2\pi}[\delta_{n,m}\cos^2\eta+\delta_{n,m+r}\sin^2\eta + \delta_{n,m+k}\sin(2\eta)\cos(r\theta-\varphi)] \ ,$$

where k is the largest integer not exceeding $(r+1)/2$. A plot for $m=0$, $r=7$, $\phi=0$, and $\eta=\pi/10$ is given in Fig. 5. $S_{\mathrm{NP}}(n,\theta)$ consists of two raised rings at radii $n=m$ and $n=m+r$, corresponding to the individual Fock states, and an interference ring at $n=m+k$. The relative height of each ring corresponds to the relative probability of finding the Fock cat in each Fock state. The number of oscillations in the interference ring depends on the distance r between the other two rings and the orientation of the interference ring depends on the phase factor ϕ. A similar dependence is evident in the interference displayed for Schrödinger cats by the PM-Wigner function. We note also that the interference ring is not present for the S_{NP} function of the corresponding mixture of two Fock states given by the density operator $\cos^2(\eta)|m\rangle\langle m|+\sin^2(\eta)|m+r\rangle\langle m+r|$.

It is now well known that Schrödinger cat states involving a discrete superposition of coherent states can be produced by a Kerr medium [40], by transferring atomic coherences to a cavity field [36] and by the atom-cavity interaction at the half atomic-inversion revival time in the Jaynes-Cummings model [43]. Let us look at the so-called odd and even coherent states as typical examples. We give a plot of the even coherent state $c(|\alpha\rangle+|-\alpha\rangle)$ for $\alpha=0.83$, where c is a normalization constant, in Fig. 6. This value of α gives approximately the same mean photon number as the coherent state in Fig. 3. In Fig. 6 there are ridges along the $\theta=0$ and $\theta=\pi$ directions where we would expect to see the hill shape

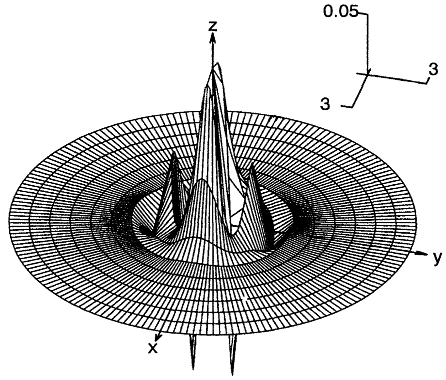

FIG. 6. S_{NP} for the even coherent state $c(|\alpha\rangle+|-\alpha\rangle)$ with $\alpha=0.83$. The largest value of n is 10. The mean photon number for this state $\langle\hat{N}\rangle\approx0.41$ is the same as for the coherent state in Fig. 3.

of Fig. 3 and its image after a rotation of π rad corresponding to the individual coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, respectively. The figure also shows interference fringes along the directions $\theta=-\pi/2$ and $\theta=\pi/2$.

There is a striking difference between the odd and even coherent states in the limit as $\alpha\rightarrow0$ as they become the Fock states $|1\rangle$ and $|0\rangle$, respectively. To see how this difference is manifested in S_{NP} we have also plotted the S_{NP} representation of the odd coherent state $d(|\alpha\rangle+|-\alpha\rangle)$ for $\alpha=1.14$, where d is a normalization constant, in Fig. 7. The mean photon number of this state is approximately one photon more than the state represented in Fig. 6. The main features of the plot in Fig. 6 also appear in Fig. 7. However, there is one quite noticeable difference: the S_{NP} representation of the odd coherent state is zero at the origin. Perhaps this should be expected since the overlap of the odd coherent state with the vacuum is zero. Another interesting feature of Fig. 7 is the absence of oscillations in the curve above $n=1$; evidently the first Fock state in the odd coherent state plays a similar role to that of the vacuum state in the even coherent state.

Figure 8 is the S_{NP} representation of an odd coherent state with $\alpha=4$. The two hills lying on the radial lines $\theta=0$ and

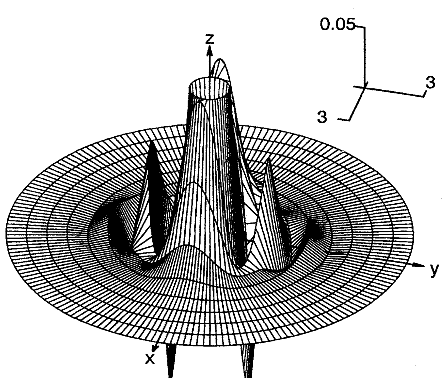

FIG. 7. S_{NP} for the odd coherent state $d(|\alpha\rangle-|-\alpha\rangle)$ with $\alpha=1.14$. The largest value of n is 10. The mean photon number for this state $\langle\hat{N}\rangle\approx1.41$.

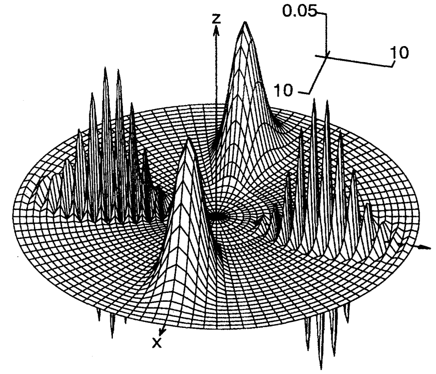

FIG. 8. S_{NP} for an odd coherent state with $\alpha=4$. The largest value of n is 30. The hills along the directions $\theta=0$ and $\theta=\pi$ correspond to the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, respectively. There are also pronounced interference fringes along the directions $\theta=\pm\pi/2$, which indicate that the field is in a coherent superposition of the two coherent states.

$\theta=\pi$ represent the two coherent states of the cat whereas the interference at $\theta=-\pi/2$ and $\theta=\pi/2$ indicates that the cat is in a superposition of the two coherent states. This figure is very similar to that of the even coherent state with the same value of α in Ref. [28]. The subtle difference between the S_{NP} representations of these two states is that the interference fringes in Fig. 8 are negative or positive for n even or odd, respectively, whereas for the corresponding even coherent state (given in [28]) the sign of the interference fringes is reversed. Since the photon-number probability distribution is given by the integral of $S_{NP}(n,\theta)$ over θ according to Eq. (4) these figures give *a different picture of how the odd and even nature of the associated photon-number distributions of these states arises.*

Finally, this section would not be complete without mentioning the S_{NP} representation of squeezed states [44–46]. The PM-Wigner representation of these states is a two-dimensional "squeezed" Gaussian with elliptical equal-height contours. These contours characterize the quantum statistics of the field quadrature amplitudes. From the perspective of phase, however, the picture is quite different. For example, the squeezed vacuum has a bimodal phase probability distribution [47–49] and being a pure state suggests that the squeezed vacuum is a superposition of two states with macroscopically distinguishable phase properties, in other words, a phase cat. One would expect from this to see evidence of quantum interference. Indeed it is well known that the squeezed vacuum contains only even numbered Fock states with the odd numbered Fock states lost to destructive interference. This interference is not exhibited by the PM-Wigner function. It is, however, displayed by the NP-Wigner function. Figure 9 illustrates the S_{NP} representation of the squeezed vacuum

$$|0,\xi\rangle=\sum_{n=0}^{\infty}\frac{[-(\xi/|\xi|)\tanh|\xi|]^n}{2^n n!}\sqrt{\frac{(2n)!}{\cosh|\xi|}}|2n\rangle$$

for $\xi=-0.6$. $S_{NP}(n,\theta)$ exhibits a great deal of oscillations and negativity especially along $\theta=-\pi/2$ and $\theta=\pi/2$. View-

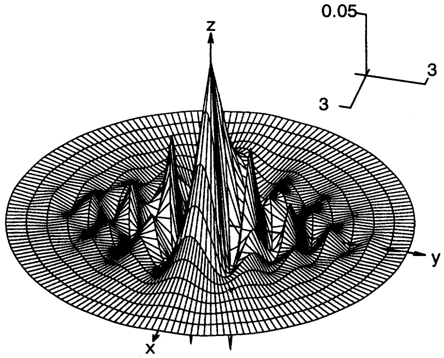

FIG. 9. $S_{\rm NP}$ for the squeezed vacuum $|0,\xi\rangle$ with $\xi=-0.6$. The largest value of n is 10. There is a ridge along the $\theta=0$ and $\theta=\pi$ directions. Interference fringes occur in between the ridges and reach a maximum along the $\theta=-\pi/2$ and $\theta=\pi/2$ directions. The relatively large photon-number spread of this state is the reason why the interference fringes are relatively spread out. The mean photon number of this state is approximately the same as for the even coherent state in Fig. 6.

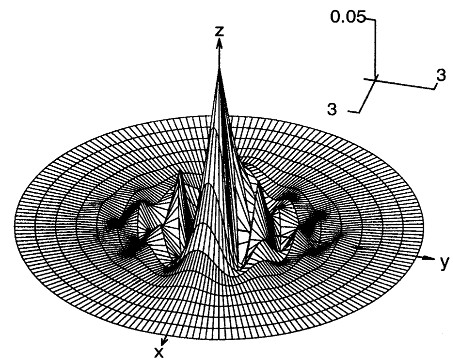

FIG. 10. $S_{\rm NP}$ for the squeezed vacuum of Fig. 9 after having interacted with a noisy environment and suffered a loss of 15% of its energy. The largest value of n is 10. The interference fringes are reduced only slightly compared to the fringes in Fig. 9. This shows that the interference fringes associated with the squeezed vacuum are relatively robust.

ing the squeezed vacuum as a phase cat can help explain the features of the figure. The ridge oriented along $\theta=0$ and $\theta=\pi$ in Fig. 9 corresponds to the two states of the phase cat, one with a mean phase of 0 and the other π, whereas the oscillations along $\theta=-\pi/2$ and $\theta=\pi/2$ are the interference fringes associated with the superposition of the two states of the cat.

It is interesting to compare Fig. 9 with the $S_{\rm NP}$ representation of the even coherent state in Fig. 6. The states in both figures have a mean photon number of approximately 0.41. Although the main features of the even coherent state are confined to a much smaller region, nevertheless, the interference fringes in both figures do share common features such as oscillations along $\theta=-\pi/2$ and $\theta=\pi/2$.

The picture of the squeezed vacuum given by $S_{\rm NP}$ in Fig. 9 contrasts markedly with the corresponding PM-Wigner representation, which is a smooth Gaussian and positive everywhere. For this reason the squeezed vacuum is perhaps the most interesting state considered in this section. Moreover, it appears that the $S_{\rm NP}$ representation of the squeezed vacuum will give a *picture of quantum interference in phase space* for an experimentally determined quantum state. Indeed, the experiments of optical quantum-state determination by Raymer's group [50,51] involved the squeezed vacuum and just recently the PM-Wigner representation of a state with approximately the same degree of squeezing as the field represented in Fig. 9 was determined experimentally [52]. We note that the $S_{\rm NP}$ representation can be determined from any representation of the quantum state, including the PM-Wigner representation, as the expectation value of $\hat{S}_{\rm NP}(n,\theta)$ using, for example, Eq. (33). Furthermore, the squeezing referred to in the latter case is the squeezing detected with nonideal photodetectors; the actual squeezing in the field is much greater [52] and, correspondingly, the field itself would have a $S_{\rm NP}$ representation with more pronounced interference fringes.

It is well known that Schrödinger cat states can quickly decohere in noisy environments [40]. One might wonder then how robust the interference fringes in Fig. 9 are. To see the effects of a noisy environment (or detection losses) on the fringes, consider the following. Let the squeezed vacuum represented in Fig. 9 be the initial state of the field and let the field interact linearly with the environment. We can model this interaction by imagining that the field is mixed with the vacuum at a beam splitter that has a nonunity transmittance [40]. Let the transmittance be some reasonable value, say, 0.85. (This transmittance value, in fact, models the detection losses in the first optical quantum-state determination experiment by Raymer's group [50].) The state of the field at one of the output ports of the beam splitter corresponds to the mixed state of the field after it has interacted with the noisy environment and suffered losses. Figure 10 shows the $S_{\rm NP}$ representation of the field at this point. The interference fringes are clearly still evident. We conclude that the interference fringes associated with *the squeezed vacuum are relatively robust* in this case and it is quite likely that they will be seen in experimental data.

IV. THE $S_{\rm NP}$-REPRESENTATION OF ARBITRARY OPERATORS

The NP-Wigner function also supplies a representation of operators. Each linear operator $\hat{A}$ on the Fock space has a corresponding unique Wigner representation $S_{\rm NP}^{(A)}(n,\theta)$ given by Eq. (20). The uniqueness can be proved by showing that every nonzero operator has a nonzero Wigner representation and then using the linear property of the operators and the Wigner representations to show that different operators have different representations. In particular, this means that the special number-phase Wigner function represents uniquely the number and phase properties of any quantum state $\hat{\rho}$.

Conversely, an operator $\hat{A}$ can be reconstructed from its NP-Wigner representation $S_{\rm NP}^{(A)}(n,\theta)$ by

$$\hat{A}=2\pi\sum_{n=0}^{\infty}\int_{2\pi}d\theta\;\hat{S}_{\rm NP}(n,\theta)S_{\rm NP}^{(A)}(n,\theta)\;. \qquad (41)$$

This can be proved easily by taking matrix elements in the Fock basis of both sides of Eq. (41). However, this mapping is not unique and there is more than one function $A(n,\theta)$, which maps to $\hat{A}$ according to

$$\hat{A}=2\pi\sum_{n=0}^{\infty}\int_{2\pi}d\theta\,\hat{S}_{\mathrm{NP}}(n,\theta)A(n,\theta)\ . \tag{42}$$

Let us call the set of functions $\{A_i(n,\theta)\}_i$ a set of $\hat{A}$-*equivalent functions* if all elements map to the same operator $\hat{A}$ by Eq. (42). For example, adding $e^{ik\theta}\delta_{m,n}$, where $|k|>2m$ and m is a non-negative integer, to $A(n,\theta)$ does not alter the left-hand side of Eq. (42); in other words, $A(n,\theta)$ and $A(n,\theta)+e^{ik\theta}\delta_{m,n}$ are $\hat{A}$-equivalent functions. The set of $\hat{A}$-equivalent functions for any given operator $\hat{A}$ contains a unique element $S_{\mathrm{NP}}^{(A)}(n,\theta)$, which is given by Eq. (20). We can always obtain this unique element by taking an arbitrary element of the set, say $A(n,\theta)$, determining the operator $\hat{A}$ by Eq. (42) and then using $\hat{A}$ to produce $S_{\mathrm{NP}}^{(A)}(n,\theta)$ via Eq. (20). This procedure can be represented as a single operation on $A(n,\theta)$ as

$$S_{\mathrm{NP}}^{(A)}(n,\theta)=\mathrm{tr}\left[\hat{S}_{\mathrm{NP}}(n,\theta)2\pi\sum_{m=0}^{\infty}\int_{2\pi}d\varphi\ \hat{S}_{\mathrm{NP}}(m,\varphi)A(m,\varphi)\right]$$
$$=\frac{1}{2\pi}\int_{2\pi}d\varphi\left[\sum_{p=-n}^{n}e^{i2p(\theta-\varphi)}+\sum_{p=-n}^{n-1}e^{i(2p+1)(\theta-\varphi)}\right]A(n,\varphi)\ .$$

Thus, given any function $A(n,\theta)$ one can calculate the unique S_{NP} representation of the associated operator $\hat{A}$ [53].

What then are the operators $\hat{A}$ associated with the functions $A(n,\theta)=(1/2\pi)\exp(im\theta)$? A simple calculation reveals that $\hat{A}$ is $(\widehat{e^{i\phi}})^m$ or $(\widehat{e^{-i\phi}})^{|m|}$ for positive and negative m, respectively, where $\widehat{e^{i\phi}}=\sum_{n=0}^{\infty}|n\rangle\langle n+1|\ =\widehat{e^{-i\phi}}^{\dagger}$ is the nonunitary Susskind-Glogower exponential phase operator. This result should not be unexpected for the following reasons. Our use of the phase probability distribution $P(\theta)$ and the infinite-dimensional Fock space implies that we are in the domain of the infinite-s limit of the Pegg-Barnett formalism and so it follows that the exponential phase operators represented by S_{NP} will be the corresponding Susskind-Glogower operators as these are the weak limits [34] of the Pegg-Barnett unitary phase operators. Moreover, the restriction of our analysis to the Fock space (instead of a larger space) implies that the phase operators of the Newton, Ban, and H_{sym} formalisms, which operate operate on larger spaces, will be represented here by their projection onto operators on the Fock space, that is, by the corresponding Susskind-Glogower operators.

Taking this one step further, the operator given by Eq. (42) with $A(n,\theta)=(1/2\pi)\cos^2(\theta)$ is not the square of the operator represented by $(1/2\pi)\cos\theta$,

$$\tfrac{1}{4}(\widehat{e^{i\phi}}+\widehat{e^{-\phi}})^2=\tfrac{1}{4}(\widehat{e^{i\phi2}}+\widehat{e^{-i\phi2}}+2-|0\rangle\langle 0|)\ ,$$

but rather

$${}^{*}_{*}\tfrac{1}{4}(\widehat{e^{i\phi}}+\widehat{e^{-i\phi}})^2{}^{*}_{*}=\tfrac{1}{4}(\widehat{e^{i\phi2}}+\widehat{e^{-i\phi2}}+2), \tag{43}$$

where ${}^{*}_{*}\cdots{}^{*}_{*}$ represents the antinormal ordering operation introduced by Lukš and Peřinová [54,55]. This operation places all positive powers of $\widehat{e^{i\phi}}$ to the left of all positive powers of $\widehat{e^{-i\phi}}$. Similarly the operator corresponding to $A(n,\theta)=(1/2\pi)\sin^2(\theta)$ is ${}^{*}_{*}-\tfrac{1}{4}(\widehat{e^{i\phi}}-\widehat{e^{-i\phi}})^2{}^{*}_{*}\ =\tfrac{1}{4}(-\widehat{e^{i\phi2}}-\widehat{e^{-i\phi2}}+2)$. Adding this to the right-hand side of Eq. (43) gives unity: that is, the S_{NP}-representation gives the operator equivalent of the mathematical identity $\cos^2(\theta)+\sin^2(\theta)=1$. Thus the Wigner-Weyl correspondence between Wigner functions and operators, embodied here by Eqs. (41) and (42), leads to a consistent set of phase operators provided we adopt Lukš and Peřinová's antinormal ordering.

Let us now look at the relationship between the photon-number and phase operators given by Eq. (42). Setting $A(n,\theta)$ in Eq. (42) alternatively to $(1/2\pi)n$ and $(1/2\pi)e^{in}$ gives the photon-number operators $\hat{N}$ and $e^{i\hat{N}}$, respectively. Using the fact that the phase operators are antinormally ordered we find that

$${}^{*}_{*}\widehat{e^{i\phi}}e^{i\hat{N}}\widehat{e^{-i\phi}}{}^{*}_{*}={}^{*}_{*}e^{i(\hat{N}+1)}\widehat{e^{i\phi}}\widehat{e^{-i\phi}}{}^{*}_{*}=e^{i(\hat{N}+1)}\widehat{e^{i\phi}}\widehat{e^{-i\phi}}$$
$$=e^{i}e^{i\hat{N}}\ ,$$
$${}^{*}_{*}\widehat{e^{-i\phi}}e^{i\hat{N}}\widehat{e^{i\phi}}{}^{*}_{*}={}^{*}_{*}e^{i(\hat{N}-1)}\widehat{e^{-i\phi}}\widehat{e^{i\phi}}{}^{*}_{*}=e^{i(\hat{N}-1)}\widehat{e^{i\phi}}\widehat{e^{-i\phi}}$$
$$=e^{-i}e^{i\hat{N}}\ ,$$

where we have used the fact that $\widehat{e^{i\phi}}e^{i\hat{N}}=e^{i(\hat{N}+1)}\widehat{e^{i\phi}}$, $\widehat{e^{-i\phi}}e^{i\hat{N}}\ =e^{i(\hat{N}-1)}\widehat{e^{-i\phi}}$ and $\widehat{e^{i\phi}}\widehat{e^{-i\phi}}=1$. This shows that the number and phase observables given by Eq. (42) are canonically conjugate in the Weyl sense [56].

Finally, an important point, which should be stressed, is that the presence here of the Susskind-Glogower operators does not mean that the description of phase is that given by Susskind and Glogower. For example, consider the vacuum state whose representation is simply $S_{\mathrm{NP}}(n,\theta)=(1/2\pi)\delta_{n,0}$. This representation yields a uniform phase probability distribution and hence attributes the vacuum state with a random phase in contrast to the nonrandom phase description of Susskind and Glogower.

V. DETERMINATION OF THE QUANTUM STATE VIA S_{NP}

An important feature of the PM-Wigner function $W(x,p)$ is that it can be determined experimentally [57,58,50,51]. Since $W(x,p)$ represents uniquely the density operator of the system this allows experimenters to determine (or, in a sense, to "measure") the quantum state. Can the quantum state be determined using the special number-phase Wigner function? The results of the preceding section show that the density operator $\hat{\rho}$ can be uniquely determined from knowledge of S_{NP}. We now consider whether S_{NP} can be determined experimentally. For this we require the expectation value of the Wigner operator $\hat{S}_{\mathrm{NP}}(n,\theta)$ for each value of n and θ. These expectation values can be calculated from the probabilities of finding the field mode in the eigenstates of the Hermitian operator $\hat{S}_{\mathrm{NP}}(n,\theta)$, i.e.,

$$\hat{S}_{\rm NP}(n,\theta)=\sum_i \lambda_i(n,\theta)|\psi_i(n,\theta)\rangle\langle\psi_i(n,\theta)| \ ,$$

where $\lambda_i(n,\theta)$ and $|\psi_i(n,\theta)\rangle$ is an eigenvalue and an eigenvector, respectively, of $\hat{S}_{\rm NP}(n,\theta)$. Thus the task reduces to finding the probabilities $\langle\psi_i(n,\theta)|\hat{\rho}|\psi_i(n,\theta)\rangle$, which can be determined experimentally, in principle. This procedure of determining the Wigner function by diagonalizing the Wigner operator is quite general and, in fact, can be applied to the PM-Wigner function.

It turns out, however, that the diagonalization operation in the present case can be simplified considerably by adopting a slightly modified procedure. Instead of diagonalizing $\hat{S}_{\rm NP}(n,\theta)$ we diagonalize the two terms $\hat{F}(n,\theta)$ and $\hat{G}(n,\theta)$ in Eq. (33) separately, where

$$\hat{F}(n,\theta)=\frac{1}{2\pi}\sum_{p=-n}^{n} e^{i2p\theta}|n+p\rangle\langle n-p| \ ,$$

$$\hat{G}(n,\theta)=\frac{1}{2\pi}\sum_{p=-n}^{n-1} e^{i(2p+1)\theta}|n+p\rangle\langle n-p-1| \ .$$

Solving the eigenvalue equation $\hat{F}(n,\theta)|f\rangle=\lambda|f\rangle$ shows that $\langle m|f\rangle=0$ for $m>2n$ and

$$e^{i2m\theta}\langle n-m|f\rangle=2\pi\lambda\langle n+m|f\rangle$$

for $0\leq m\leq n$. From this we find that for $n=0$ there is a single eigenvalue and eigenvector $\lambda=1/(2\pi)$ and $|f\rangle=|0\rangle$, whereas for $n>0$ there are just two degenerate eigenvalues $\lambda=\pm 1/(2\pi)$ and many different sets of eigenvectors. A particularly simple set of eigenvectors for $n>0$ is given in the Fock basis by

$$|f_m^+(n,\theta)\rangle=\frac{1}{\sqrt{2}}(|n-m\rangle+|n+m\rangle e^{i2m\theta}) \ ,$$

$$|f_m^-(n,\theta)\rangle=\frac{1}{\sqrt{2}}(|n-m\rangle-|n+m\rangle e^{i2m\theta}) \ ,$$

$$|f_n^+\rangle=|n\rangle$$

for $m=1,2,\ldots,n$, where the superscripts refer to the sign of the eigenvalue. Thus we now have

$$\hat{F}(n,\theta)=\frac{1}{2\pi}[|n\rangle\langle n|+\sum_{m=1}^{n}|f_m^+(n,\theta)\rangle\langle f_m^+(n,\theta)| - |f_m^-(n,\theta)\rangle\langle f_m^-(n,\theta)|]$$

with it being understood that the first sum is zero for the $n=0$ case. In a similar way we also find that $\hat{G}(n,\theta)$ can be diagonalized as

$$\hat{G}(n,\theta)=\frac{1}{2\pi}\sum_{m=1}^{n}|g_m^+(n,\theta)\rangle\langle g_m^+(n,\theta)|-|g_m^-(n,\theta)\rangle \times\langle g_m^-(n,\theta)| \ ,$$

where

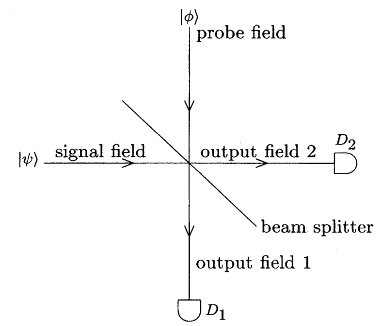

FIG. 11. Experimental setup for the determination of $S_{\rm NP}$. The signal and probe fields are mixed at a beam splitter and then directed onto the two photodetectors D_1 and D_2. The $S_{\rm NP}$-representation of the signal field is obtained from the joint photocount distributions for a given set of probe field states.

$$|g_m^+(n,\theta)\rangle=\frac{1}{\sqrt{2}}(|n-m\rangle+|n+m-1\rangle e^{i(2m-1)\theta}) \ ,$$

$$|g_m^-(n,\theta)\rangle=\frac{1}{\sqrt{2}}(|n-m\rangle-|n+m-1\rangle e^{i(2m-1)\theta})$$

for $m=1,2,\ldots,n$. Thus the expectation values of $\hat{F}(n,\theta)$ and $\hat{G}(n,\theta)$ can be calculated from knowledge of the probabilities of finding the field mode in the eigenstates $|f_m^\pm\rangle,|g_m^\pm\rangle$ for $m=1,2,\ldots,n$ and the Fock state $|n\rangle$. In other words, the special number-phase Wigner function can be determined from knowledge of the photon-number probability distribution and the probabilities of finding the field mode in the states $|m\rangle\pm|n\rangle\exp[i(n-m)\varphi]$ for all $n>m$ and all φ.

These probabilities are, in principle, measurable quantities. Indeed, probabilities of this type can be determined experimentally via an application of homodyne detection [59]. We now describe briefly an alternate scheme based directly on photodetection. Let the field in question, which we shall call the signal field, be in the state $|\psi\rangle$. For brevity we only treat the pure state case here; the extension to the more-general mixed state case is, however, straightforward. The signal field is mixed with a probe field, which is in the specially prepared state $|\phi\rangle$, at an ideal beam splitter as shown in Fig. 11. The output fields of the beam splitter are directed onto photodetectors of known quantum efficiency ϵ. We assume that the signal and probe fields are pulsed synchronously and that the detectors, field intensities, and pulse duration are chosen such that they allow the counting of the individual photoelectron events produced in the detectors in a manner similar to the operational-phase measurements of Mandel *et al.* [60]. The joint probability distribution giving the probability of finding p and q photons in the output fields 1 and 2, respectively, is given by

$$P(p,q)=\left|\sum_{k=0}^{p+q}\psi_k\phi_{p+q-k}C_p(k,p+q-k)\right|^2 \ , \quad (44)$$

where ψ_n and ϕ_m are the Fock state coefficients $\psi_n=\langle n|\psi\rangle$ and $\phi_m=\langle m|\phi\rangle$. The coefficients $C_k(n,m)$ are the complex numbers [61,62]

$$C_k(n,m)=(-1)^n\sqrt{\frac{k!(n+m-k)!}{n!m!}}e^{i\varphi_\tau(n-k)}e^{i\varphi_\rho(m-k)}$$
$$\times\sum_{p=0}^{n}\sum_{q=0}^{m}(-1)^p\binom{n}{p}\binom{m}{q}$$
$$\times\sqrt{\tau^{m+p-q}\rho^{n-p+q}}\delta_{n+m-k,p+q} ,$$

where $\delta_{n,m}$ is the Kronecker delta, $\binom{n}{m}$ is the binomial coefficient $n!/m!(n-m)!$, τ, and ρ are the transmittance and reflectance, respectively, of the beam splitter, and φ_τ and φ_ρ are phase factors as defined by Campos *et al.* [62]. It follows from Eq. (44) that if the probe field is prepared in the state

$$|\phi\rangle=c(a|n\rangle+b|m\rangle e^{i\eta}) , \tag{45}$$

where $c=1/\sqrt{[a|]^2+[b|]^2}$ is a normalization constant, $a=C_{p'}(p'+q'-n,n)^{-1}$, $b=C_{p'}(p'+q'-m,m)^{-1}$, and η is a phase parameter, then the particular value of $P(p,q)$ at $p=p'$ and $q=q'$ is

$$P(p',q')=2|c|^2|\langle\psi|\Theta\rangle|^2$$

for $p'+q'\geq n,m$, where the state $|\Theta\rangle=1/\sqrt{2}(|p'+q'-n\rangle+|p'+q'-m\rangle e^{-i\eta})$ is in the form of the eigenstates $|f_m^\pm\rangle$ and $|g_m^\pm\rangle$ of $\hat{F}$ and $\hat{G}$. Thus, by choosing appropriate values of p', q', m, n, and η for the probe field state $|\phi\rangle$ in Eq. (45) we can determine the required probabilities of finding the signal field in the eigenstates $|f_m^\pm\rangle$ and $|g_m^\pm\rangle$. Using these probabilities and the corresponding eigenvalues we can determine the expectation value of $\hat{S}_{\rm NP}(n,\theta)=\hat{F}(n,\theta)+\hat{G}(n,\theta)$, which is the $S_{\rm NP}$ representation of the signal field $S_{\rm NP}(n,\theta)$.

Of course, real photodetectors have quantum efficiencies ϵ that are less than unity and so the relative frequency $M(p,q)$ of counting p and q photoelectron events at detectors 1 and 2, respectively, is not equal to the joint photon-number probability distribution $P(p,q)$. By modeling the loss in each detector as the loss from a beam splitter of transmittance ϵ placed in front of an ideal detector it is not difficult to show that $M(p,q)$ and $P(p,q)$ are related by

$$M(m,n)=\sum_{p=n}^{\infty}\binom{p}{n}\epsilon_n(1-\epsilon)^{p-n}$$
$$\times\sum_{q=m}^{\infty}\binom{q}{m}\epsilon_m(1-\epsilon)^{q-m}P(p,q) .$$

This expression is in the form of a double Bernoulli transform which can be inverted [63] to give

$$P(p,q)=\sum_{n=p}^{\infty}\binom{n}{p}\left(1-\frac{1}{\epsilon}\right)^{n-p}\epsilon^{-p}$$
$$\times\sum_{m=q}^{\infty}\binom{m}{q}\left(1-\frac{1}{\epsilon}\right)^{m-q}\epsilon^{-q}M(n,m) .$$

As the values of n and m increase, the moduli of the factors

$$\binom{n}{p}\left(1-\frac{1}{\epsilon}\right)^n , \binom{m}{q}\left(1-\frac{1}{\epsilon}\right)^m$$

decay exponentially for $\frac{1}{2}<\epsilon<1$ and diverge otherwise. Thus the experimental determination of the photon-number probability distribution $P(p,q)$, and hence the determination of the $S_{\rm NP}$ representation of the signal field, can be obtained using realistic detectors provided the quantum efficiency is greater than $\frac{1}{2}$.

Experimentally the most difficult part of the scheme is the preparation of the probe field in the states given by Eq. (45) for a continuous range of η values and all two-state superpositions of Fock states. Nevertheless, it may be possible in the near future [64] to produce such states containing Fock state components near the vacuum and so this scheme may find a use in determining the $S_{\rm NP}$ representation of very weak fields [65]. We conclude that the special number-phase Wigner function $S_{\rm NP}(n,\theta)$ *can be determined from quantities that are, in principle, measurable*. The quantum state of the field mode can then be found by evaluating the density operator $\hat{\rho}$ according to

$$\hat{\rho}=2\pi\sum_{n=0}^{\infty}\int_{2\pi}d\theta\,\hat{S}_{\rm NP}(n,\theta)S_{\rm NP}(n,\theta) .$$

VI. SUMMARY AND CONCLUSION

We have analyzed the problem of defining a Wigner function $S_{\rm NP}(n,\theta)$ associated with the photon-number and phase observables on the infinite-dimensional Fock space, which is conventionally used to represent the state of a single-mode field. We began by requiring $S_{\rm NP}(n,\theta)$ to be a bilinear functional of the wave function and then we specified several properties that the function must exhibit. For this we used the properties of Wigner's original function for position and momentum observables, as listed by Hillery *et al.* [27], as a basis. These properties are sufficient to define the position-momentum Wigner function uniquely. Translated into analogous properties for the photon-number and phase observables, they require that the number-phase Wigner function be real, give the discrete photon-number and continuous phase probability distributions as marginals, be Galilei invariant under phase and photon-number shifts (and thus automatically satisfy the classical equation of motion), be invariant under time reflections, have the overlap property, and give the trace property for the associated representation of arbitrary operators. We found, however, that these properties are not sufficient to define the number-phase Wigner function uniquely. An extra property of the position-momentum Wigner function was used to define the special number-phase Wigner function $S_{\rm NP}(n,\theta)$. This property, which is a skew diagonal form of the position-momentum Wigner function, generates the quantum interferences fringes that are characteristic of Schrödinger cat states.

The outcome of this procedure is that $S_{\rm NP}(n,\theta)$ is a quasiprobability distribution that has properties analogous to those of the Wigner's original function but with the distinctive feature in that it gives a representation of states that displays their underlying photon-number and phase properties. For Schrödinger cat states, in particular, it exhibits interference fringes that reveal the coherent superpositions of states with either different mean photon number or different mean phase. Also, every operator (including every density operator) has a unique S_{NP} representation. $S_{\mathrm{NP}}(n,\theta)$ can be determined from quantities that are, in principle, measurable. We conclude that $S_{\mathrm{NP}}(n,\theta)$ is a bona fide Wigner function, which should be useful in the study of the phase and photon-number properties of quantum optical systems.

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$$\hat{\mathscr{A}}(n,\theta)\equiv\frac{1}{2\pi}\left[|n\rangle\langle n|+\sum_{p=n+1}^{\infty}e^{i(p-n)\theta}|p\rangle\times\langle n|+e^{-i(p-n)\theta}|n\rangle\langle p|\right].$$

However, the price for this is the loss of the skew-diagonal property described in Sec. II B with the result that $\mathscr{A}(n,\theta)$ displays quantum interferences in quite a different manner to $W(x,p)$ and $S_{\rm NP}(n,\theta)$.
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