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Approach to a generalized Jaynes-Cummings model and the geometric phase

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(Received 12 December 1994)

Adopting the recently introduced inverses of boson creation and annihilation operators, we construct a nonunitary transformation to solve exactly a generalized Jaynes-Cummings (JC) model. The method reveals that the JC model in a new frame can be viewed simply as a system of a spin-$\frac{1}{2}$ atom interacting with a magnetic field dependent on the photon number. For the initial state with fixed photon number l, the JC model becomes a cyclic evolution, and exhibits a geometric phase γ_g. We discuss the properties of γ_g in the large l limit.

PACS number(s): 03.65.−w, 42.50.−p

The Jaynes-Cummings (JC) model [1] as a basis of the fully quantum description of radiation-matter interaction has extensive applications in quantum optics, quantum electronics, etc. Various modifications and generalizations to the original JC model have been made to approach quantum effects such as quantum collapses and revivals of atomic coherence [2], squeezing phenomena [3], and so on. Among them, the JC coherent states are used to evaluate the dynamical quantities.

Recently, geometrical properties of the JC Hamiltonian have acquired much interest, since the supersymmetry structure was found to embed in the JC model. It was pointed out that the JC Hamiltonian may be interpreted as an element of a superalgebra [4]. Exact solvability of various JC models is argued to attribute to some common structures among them: Using a deformed oscillator algebra, Bonatsos *et al.* [5] gave a unified solvable formulation of various JC Hamiltonians; Yu *et al.* [6] pointed out further that there exists a fundamental SU(2) structure within these JC Hamiltonians. Others are still attempting to develop new techniques to evaluate nonperturbative expansions of the JC Hamiltonian and search for new physical manifestations.

In this paper, we introduce an alternative method—nonunitary transformation to approach these JC models. The method is shown to be simple and useful. As is well known, the unitary transformation in quantum theory preserves the physical quantities and reality of probabilities, but not the nonunitary transformation generally. Here we discuss the applicability of a set of nonunitary transformations to quantum systems through a theorem. Then using the recently introduced inverses of boson creation and annihilation operators [7], we construct a nonunitary transformation, covered by the theorem, to solve exactly a generalized JC Hamiltonian with multiphoton interactions and a quartic anharmonic Kerr. The method reveals that in the transformed frame the generalized JC model simply describes the motion of a spin-$\frac{1}{2}$ atom in a magnetic field dependent on photon number. When the initial state is specially chosen, the JC model shows the property of cyclic evolution, and exhibits a geometric phase which has been neglected previously. The geometric phase is in fact the Aharonov and Anandan (AA) phase [8,9], which is related to the initial state and a certain structure of the Hamiltonian.

Let us look at a Hermitian operator Λ as a physical quantity with a discrete spectrum

$$\Lambda|\psi_n\rangle=\lambda_n|\psi_n\rangle, \quad n=1,2,\ldots,\infty, \tag{1}$$

the eigenkets $|\psi_n\rangle$ are orthogonal mutually. Consider an operator U which transforms Λ into another frame Λ'. If Λ' is still a Hermitian quantity and has the same spectrum as Λ, U is called *applicable* to Λ. The form $\Lambda'=U\Lambda U^\dagger$ is generally chosen to preserve the Hermitian of Λ', and $UU^\dagger=I$ is required to maintain the unity: $\Lambda=\Lambda'=1$. In the infinite dimensional Hilbert space, we know that $UU^\dagger=I$ does not mean that U is unitary. In sum, the nonunitary U we are interested is in the following form:

$$\begin{cases} UU^\dagger=I, \\ U^\dagger U\neq I. \end{cases} \tag{2}$$

We note that some nonunitary transformations, such as *time reversal* T, are not included in Eq. (2), since T is an antilinear operator without a proper Hermitian conjugate. In order to ensure that $\Lambda'=U\Lambda U^\dagger$ has the same spectrum as Λ, U is found to satisfy the conditions described by the following theorem.

Theorem. If U has eigenvalue being zero, i.e.,

$$U|\Phi_l\rangle=0,\ l=1,2,\ldots,s, \text{ and the set } \{|\Phi_l\rangle,\ l=1,2,\ldots,s\} \text{ is linearly isomorphic with the set } \{|\psi_l\rangle,\ l=1,2,\ldots,s\}, \tag{3}$$

U is applicable to Λ. Here $\{|\psi_l\rangle,\ l=1,2,\ldots,s\}$ is a subset of the total eigenkets $\{|\psi_n\rangle,\ n=1,2,\ldots,\infty\}$ in Eq. (1).

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Proof. Equation (4) yields that $|\psi_l\rangle=\Sigma_{i=1}^{s}d_{li}|\Phi_i\rangle$, $l\leq s$, and

$$\begin{cases} U|\psi_l\rangle=0, & l=1,2,\ldots,s, \\ U|\psi_m\rangle\neq 0, & m=s+1,\ldots,\infty. \end{cases} \tag{4}$$

Since $\{|\psi_l\rangle\}$ can be determined by both U and Λ, we treat only the remaining subspace $\{|\psi_m\rangle\}$. Let $|\psi_m'\rangle=U^\dagger U|\psi_m\rangle$, by Eq. (2), so we have $U(|\psi_m'\rangle-|\psi_m\rangle)=0$. Equation (4) implies $|\psi_m'\rangle=|\psi_m\rangle+\Sigma_{l=1}^{s}c_{ml}|\psi_l\rangle$. Taking $|\psi_m\rangle$ into Eq. (1), we obtain $c_{ml}=0$, namely, $|\psi_m'\rangle=|\psi_m\rangle$. Therefore, in the subspace $\{|\psi_m\rangle\}$, U acts as a unitary operator. Then it is easily proved that $\Lambda'=U\Lambda U^\dagger$ has the same spectrum as Λ, its eigenkets $|\Psi_n\rangle=U|\psi_n\rangle$ are normalized, and $U^\dagger|\Psi_n\rangle=|\psi_n\rangle$.

In order to construct a nonunitary transformation to approach the JC model, we adopt a recent discovery by Mehta *et al.* [7] that the boson creation operator $a^\dagger$ has a left universe b and the annihilation operator a has a right inverse $b^\dagger$,

$$ba^\dagger=ab^\dagger=1. \tag{5}$$

We know in the scheme of second quantization, an arbitrary operator can be expressed by a and $a^\dagger$, namely, a and $a^\dagger$ form a *complete operator set*, by citing the definition of the completeness of wave functions. One infers further that b and $b^\dagger$ can be expressed by a and $a^\dagger$. For this purpose, we turn to the technique of coherent state and normal ordered product. For the coherent state $|z\rangle$,

$$\begin{cases} |z\rangle=\exp(a^\dagger z-az^*)|0\rangle, \\ \int |z\rangle\langle z|\,\frac{dz^2}{\pi}=1, \end{cases} \tag{6}$$

we have

$$b=b\int |z\rangle\langle z|\,\frac{dz^2}{\pi}=\int e^{-|z|^2}be^{a^\dagger z}|0\rangle\langle 0|e^{az^*}\frac{dz^2}{\pi}. \tag{7}$$

Using the normal ordered expression of the vacuum projection operator [10]: $|0\rangle\langle 0|=:e^{-a^\dagger a}:$, we obtain explicitly

$$b=:e^{-a^\dagger a}\sum_{n=1}^{\infty}\frac{(a^\dagger)^{n-1}a^n}{n!}:. \tag{8}$$

By the same process, we have

$$b^\dagger=:e^{-a^\dagger a}\sum_{n=1}^{\infty}\frac{(a^\dagger)^{n}a^{n-1}}{n!}:. \tag{9}$$

These two expressions can be simplified into the following compact forms, which are provable by the same technique of coherent state as in Eq. (7),

$$b=\left(\frac{1}{aa^\dagger}\right)a,\quad b^\dagger=a^\dagger\left(\frac{1}{aa^\dagger}\right). \tag{10}$$

By Eq. (10), one obtains that a and $a^\dagger$ can be expressed by b and $b^\dagger$ too,

$$a=\left(\frac{1}{bb^\dagger}\right)b,\quad a^\dagger=b^\dagger\left(\frac{1}{bb^\dagger}\right). \tag{11}$$

Equation (11) implies that any operator can be expressed in terms of b and $b^\dagger$. Therefore, b and $b^\dagger$ form a *complete operator set* too, which is connected to a and $a^\dagger$ by a pair of nonlinear transformations (10) and (11). These results will be used in the later calculations.

Now we consider a generalized JC model with a density $\rho(N)$-dependent multiphoton (n) interaction and a nonlinear Kerr cavity $\beta a^{\dagger 2}a^2$,

$$H=\omega a^\dagger a+\tfrac{1}{2}\omega_0\sigma_3+\beta a^{\dagger 2}a^2 +\lambda[a^{\dagger n}\rho(N)\sigma_-+\rho(N)a^n\sigma_+], \tag{12}$$

where $N=a^\dagger a$, ω and ω_0 are the field and atomic transition frequencies, respectively, and λ is the real atom-field coupling constant [11]. The Kerr cavity is used to modify the photon statistics of the micromaser field towards the state with a *low* number of photons [12], and also applied to reduce the amplitude noise. The detuning $\Delta=\omega_0-n\omega$ should satisfy that $|\Delta|\ll\omega_0,\omega$ in order to maintain the reliability of the rotating-wave approximation. We mention that various JC Hamiltonians in the literature are covered in this general one, Eq. (12). The λ term is the main object of our treatment. For this purpose, we construct a 2×2 operator matrix U as

$$U=\begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{(N+n)!}{N!}\right)^{1/2} b^n \end{pmatrix}, \tag{13}$$

where $(N+n)!/N!=:(N+n)(N+n-1)\cdots(N+1)=(b^nb^{\dagger n})^{-1}$. U has the properties

$$\begin{cases} UU^\dagger=I, \\ U^\dagger U=\begin{pmatrix} 1 & 0 \\ 0 & 1-\sum_{k=0}^{n-1}|k\rangle\langle k| \end{pmatrix}. \end{cases} \tag{14}$$

That is, U is a special nonunitary operator belong to Eq. (2). Further, U has an eigenvalue of zero, namely, $U|\psi_l\rangle=0$, where

$$|\psi_l\rangle=\begin{pmatrix} 0 \\ |l\rangle \end{pmatrix},\quad l=0,1,\ldots,n-1. \tag{15}$$

One can check that these $\{|\psi_l\rangle\}$ are exactly the eigenkets of H with eigenvalues $E_l=\omega l+\beta l(l-1)-\omega_0/2$. These results indicate that the U is covered by the theorem, and the U can be applied to the remaining subspace of H besides Eq. (15). Denote $H_1=UHU^\dagger$, we obtain

$$H_1=\begin{pmatrix} f(N)+\frac{1}{2}\omega_0 & g_n(N) \\ g_n(N) & f(N+n)-\frac{1}{2}\omega_0 \end{pmatrix}, \tag{16}$$

where $f(N)=\omega N+\beta N(N-1)$, $g_n(N)=\lambda\rho(N)[(N+n)!/N!]^{1/2}$. H_1 is further rearranged into the following compact form:

$$H_1 = B_0 - \vec{\mu}\cdot\vec{B}, \tag{17}$$

where $B_0 = \frac{1}{2}[f(N)+f(N+n)]$, $B_1 = g_n(N)$, $B_2 = 0$, $B_3 = \frac{1}{2}[f(N)-f(N+n)+\omega_0]$, $\vec{\mu} = -\vec{\sigma}$. Since H_1 is the function of N, its eigenvalues are directly obtained as

$$E_l^{\pm} = B_0(l) \pm \sqrt{B_1^2(l)+B_3^2(l)}, \quad l=0,1,2,\ldots, \tag{18}$$

the corresponding eigenstates are

$$|E_l^+\rangle' = \begin{pmatrix} \cos\frac{\theta_l}{2}|l\rangle \\ \sin\frac{\theta_l}{2}|l\rangle \end{pmatrix}, \quad |E_l^-\rangle' = \begin{pmatrix} -\sin\frac{\theta_l}{2}|l\rangle \\ \cos\frac{\theta_l}{2}|l\rangle \end{pmatrix}, \tag{19}$$

where $\theta_l = \cos^{-1}[B_3(l)/\sqrt{B_1^2(l)+B_3^2(l)}]$. Then the eigenstates in the *original* frame H are obtained simply by $U^\dagger$'s acting on Eq. (19),

$$|E_l^+\rangle = \begin{pmatrix} \cos\frac{\theta_l}{2}|l\rangle \\ \sin\frac{\theta_l}{2}|l+n\rangle \end{pmatrix}, \quad |E_l^-\rangle = \begin{pmatrix} -\sin\frac{\theta_l}{2}|l\rangle \\ \cos\frac{\theta_l}{2}|l+n\rangle \end{pmatrix}. \tag{20}$$

The theorem tells us that $E_l^{\pm}$, Eq. (18), are exactly the eigenvalues of H; the corresponding eigenstates are $|E_l^{\pm}\rangle$. Here the eigenket set $\{|E_l^{\pm}\rangle\}$ forms a subspace of the Fock space, in which U acts as a unitary operator without changing the physics. Therefore, H_1 and H are essentially equivalent in this subspace.

The above method shows us a very simple version of the generalized JC model: In the *transformed* frame H_1, Eq. (17), the generalized JC method describes the motion of a spin-$\frac{1}{2}$ atom under an effective magnetic field $\vec{B}$ dependent on photon number, and there is no transition between the states with different photon numbers. In such a frame, the atom is taken as a neutral particle having only a constant magnetic moment $\vec{\mu} = -\vec{\sigma}$, and the magnetic field is diagonalized to take the pure photon number state as its eigenstate. For a pure ket $|l\rangle$, the Hamiltonian H_l turns out to be a simple two-level case $H_1(l)$. From these points of view, the JC model is actually a much simplified description of radiation-matter interaction, and therefore needs improving. In practice, this version makes it easier to study such properties as quantum collapses and revivals of the atomic inversion, statistics of photon number within this generalization JC model, and all the results can be directly transformed back to the original frame H by U. The above conclusions can be extended to other JC models without difficulty.

In this paper, we would like to point out a previously unobserved effect: the geometric phase in the generalized JC model. It has been shown [9] that the two-level system is generally a cyclic evolution, and exhibits a geometric phase. As mentioned above, when the photon number l is fixed in the transformed frame, H_1 is just a two-level case. One further deduces that in this case, H describes a two-level system too. Given an initial state of H,

$$|\Psi_l(0)\rangle = \begin{pmatrix} \cos\left(\frac{\phi_l}{2}\right)|l\rangle \\ \sin\left(\frac{\phi_l}{2}\right)|l+n\rangle \end{pmatrix}, \tag{21}$$

the evolution of the state with time is

$$|\Psi_l(t)\rangle = \cos\left(\frac{\phi_l-\theta_l}{2}\right)e^{-iE_l^+t}|E_l^+\rangle + \sin\left(\frac{\phi_l-\theta_l}{2}\right)e^{-iE_l^-t}|E_l^-\rangle, \tag{22}$$

where θ_l was introduced in Eq. (19). $|\Psi_l(t)\rangle$ yields

$$|\Psi_l(\tau)\rangle = e^{i\gamma}|\Psi_l(0)\rangle, \tag{23}$$

where $\tau = 2\pi/(E_l^+ - E_l^-)$, and $\gamma = -2\pi E_l^+/(E_l^+ - E_l^-)$. The result Eq. (23) indicates that for the initial state Eq. (21), the generalized JC model is in fact a *cyclic evolution* with the period τ and total phase γ. Here γ is the sum of geometric phase γ_g and dynamic phase γ_d, where $\gamma_d =: -\int_0^\tau \langle\Psi(t)|H|\Psi(t)\rangle dt$. The geometric phase is then obtained as

$$\gamma_g = \pi[1+\cos(\phi_l - \theta_l)]. \tag{24}$$

One can prove that the geometric phase in the transformed frame H_1 has the same form as Eq. (24), i.e., the geometric phase is invariant under U. This is easily understood: From physics, the geometric phase has an observable effect, therefore it does not change with a frame; from mathematics, γ_g here is in fact an Aharonov-Anandan (AA) phase [8], which reflects the topology properties of the JC Hamiltonian in parameter space, and is invariant under feasible transformations.

Explicitly, θ_l is written as

$$\theta_l = \cos^{-1}\left\{\frac{\Delta - \beta n(n+2l-1)}{\sqrt{4\lambda^2\rho^2(l)(l+n)!/l! + [\Delta - \beta n(n+2l-1)]^2}}\right\}, \tag{25}$$

which indicates that γ_g is sensitive to n, l, and some other parameters. An appropriate design of the Kerr cavity (β) can possibly make $\theta_l = \pi/2$ for a certain state l, and then γ_g relates only to the initial state. Setting $\rho(l) = 1$, $\beta = 0$, $n = 1$, one obtains the expression of γ_g for the original JC model [1]. In the following, we discuss the properties of γ_g by different n in the large l limit. Without loss of generality, we still assume $\rho(l) = 1$.

(i) For the case of single photon interaction, i.e., $n = 1$, γ_g becomes

$$\gamma_g = \pi\left\{1 + \cos\left[\phi_l - \cos^{-1}\left(\frac{\Delta - 2l\beta}{\sqrt{4\lambda^2(l+1) + (\Delta - 2l\beta)^2}}\right)\right]\right\}. \tag{26}$$

For a sufficiently large l, if the Kerr cavity is so designed that $\beta<0$, the term $4\lambda^2(l+1)$ in Eq. (26) becomes negligible, γ_g will relate to the initial state only: $\gamma_g\approx\pi(1+\cos\phi_l)$; if $\beta=0$, the term $4\lambda^2(l+1)$ turns out to dominate, and then $\gamma_g\approx\pi(1+\sin\phi_l)$; if $\beta>0$, one obtains $\gamma_g\approx\pi(1-\cos\phi_l)$. These results show that γ_g varies sharply with the sign of β in the large l limit, e.g., for the initial state $\phi_l=0$, $\gamma_g(\beta>0$ or $\beta<0)\approx 0$, however, $\gamma_g(\beta=0)\approx\pi$.

(ii) $n=2$, the two-photon interaction case, when l is large enough, and $\beta\neq 0$, we obtain

$$\gamma_g\approx\pi\left\{1+\cos\left[\phi_l-\cos^{-1}\left(\frac{-2\beta}{\sqrt{\lambda^2+4\beta^2}}\right)\right]\right\}, \quad (27)$$

in this case γ_g is determined by both ϕ_l and β,λ, which is quite different from case $n=1$.

(iii) $n\geqslant 3$, for a large l, $\theta_l\approx\pi/2$. Hence, $\gamma_g\approx\pi(1+\sin\phi_l)$. This case is similar to that of $n=1$ while $\beta=0$.

The above initial discussions of γ_g are based on the state Eq. (21). This kind of state is the pure quantum object, and how to prepare such a state is still a current topic in experiment. As a quantum holonomy in a line bundle over the parameter space, γ_g shows essentially the geometrical aspect of the JC Hamiltonian. The approach of γ_g would be useful to the deep understanding of various JC models and amounts to a proposal for a practical experiment.

In conclusion, we have proposed a method—nonunitary transformation—to approach a generalized JC model. The method presents a simple version of the JC model: in the transformed frame, the JC model describes the motion a spin-$\frac{1}{2}$ atom under an effective magnetic field dependent on photon number. This version evidently simplifies the study of the JC model in both physics and geometry. By choosing a special initial state, the JC model shows the property of cyclic evolution and exhibits a geometric phase γ_g, which is the well-known AA phase. We discuss the properties of γ_g in the large photon number limit, and hope these effects will be manifested by experiments.

The author is deeply grateful to Professor D. Finkelstein for discussions. The material is based upon research supported in part by the M&H Ferst Foundation, and by the NSF, Grant No. PHY9211036.

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