Comment on "Perturbation expansion of closed-time-path Green's functions"

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The "incoming" interaction picture found by Xu and Tsai [Phys. Rev. A **41**, 53 (1990)], used to develop a convenient formulation of the closed-time-path approach to Green's function theory, is shown to be a particular case of a more general set of solutions.

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Two standard methods have been developed to introduce the interaction picture. In the first one, the state vectors are traditionally defined to coincide with their Heisenberg counterparts at time $t=0$. The second method sets the coincidence time in the remote past, before the "interaction" part of the Hamiltonian has been turned on. In their article [1] (see also Refs. [2] and [3]), Xu and Tsai pointed out that the latter method suffered from a lack of completeness, since the entire set of transformations between the three pictures had, surprisingly, never been explicitly or correctly derived. Solving by mere inspection a self-consistent functional equation, they obtained an interaction picture which they called the "incoming" interaction picture by analogy with the incoming fields of quantum field theory. However, this work appears to be criticable in one aspect since it is there argued that the correct (generalized) interaction picture, prescribed by

$$|\psi_I(t)\rangle = u_I(t,-\infty)|\psi_H\rangle \ , \tag{1}$$

$$u_I(t,t') = T\exp\left\{\frac{1}{i\hbar}\int_{t'}^{t} V_I(t'')dt''\right\} \ , \tag{2}$$

is the one defined by $V_I(t) = e^{(i/\hbar)H_S t}U(0,-\infty)V_S U^{-1}(0,-\infty)e^{(-i/\hbar)H_S t}$, V_S being the interaction Hamiltonian in the Schrödinger picture and $U(t,t')$ the well-known evolution operator of the more traditional interaction picture [see the discussion after Eq. (7) below]. In fact, a more direct mathematical treatment shows that the solution described above is not unique, but a member of a group of solutions which may also generate other variants to the closed-time-path approach of Green's-function theory.

It is supposed that the Hamiltonian operator, in the Schrödinger picture, may be adequately split into two terms: $H_S(t) = H_{0S}(t) + V_S(t)$. The Heisenberg and interaction pictures are defined, respectively, by the following unitary transformations:

$$|\psi_S(t)\rangle = R_H(t)|\psi_H\rangle \ , \tag{3a}$$

$$|\psi_S(t)\rangle = R_I(t)|\psi_I(t)\rangle \ . \tag{3b}$$

By definition, these operators satisfy the customary equations

$$i\hbar\frac{dR_H(t)}{dt} = H_S(t)R_H(t) \ , \tag{4a}$$

$$i\hbar\frac{dR_I(t)}{dt} = H_{0S}(t)R_I(t) \ , \tag{4b}$$

whose general solutions may be cast in the form:

$$R_H(t) = T\exp\left\{\frac{1}{i\hbar}\int_{t_0}^{t} H_S(t')dt'\right\}R_H(t_0) = U_S(t,t_0)R_H(t_0) \ , \tag{5a}$$

$$R_I(t) = T\exp\left\{\frac{1}{i\hbar}\int_{t_0}^{t} H_{0S}(t')dt'\right\}R_I(t_0) = U_{0S}(t,t_0)R_I(t_0) \ , \tag{5b}$$

where t_0 is an arbitrary (finite) reference time and $U_S(t,t_0)$ is the Schrödinger evolution operator. Using the Schrödinger equation, the evolution operator for the interaction picture, defined by

$$u_I(t,t') = T\exp\left\{\frac{1}{i\hbar}\int_{t'}^{t} V_I(t'')dt''\right\} \tag{6}$$

may be expressed as

$$u_I(t,t') = R_I^{-1}(t)U_S(t,t')R_I(t') = R_I^{-1}(t_0)U(t,t')R_I(t_0) \ , \tag{7}$$

where the more usual evolution operator $U(t,t') = U_{0S}^{-1}(t,t_0)U_S(t,t')U_{0S}(t',t_0)$, which corresponds to $u_I(t,t')$ with the particular choice $R_I(t_0)=1$, has been introduced. The entire set of transformations may thus be written as

$$|\psi_S(t)\rangle = U_S(t,t_0)R_H(t_0)|\psi_H\rangle \ , \tag{8a}$$

$$|\psi_S(t)\rangle = U_{0S}(t,t_0)R_I(t_0)|\psi_I(t)\rangle \ , \tag{8b}$$

$$|\psi_I(t)\rangle = u_I(t,t_0)R_I^{-1}(t_0)R_H(t_0)|\psi_H\rangle = R_I^{-1}(t_0)U(t,t_0)R_H(t_0)|\psi_H\rangle \ . \tag{8c}$$

With no loss of generality, t_0 may now be set to zero. It is not difficult to see that the conditions $R_I(0)=R_H(0)=1$ lead to the well-known situation where the three pictures coincide at the origin of times, while the identity

$$R_H(0)R_I^{-1}(0)=U(0,t_C) \tag{9}$$

sets the coincidence of the interaction picture with the Heisenberg picture for $t=t_C$, t_C being possibly different from the reference time $t_0=0$. Since Eq. (9) must be supplemented with another condition on either $R_I(0)$ or $R_H(0)$, various sets of solutions may thus be defined.

Xu and Tsai considered the case of a time-independent Hamiltonian with a coincidence time t_C shifted to minus infinity; their solution is easily seen to be the one corresponding to the particular choice $R_H(0)=1$ and $R_I(0)=U_I^{-1}(0,-\infty)$ [4]. As an example, the initial conditions $R_I(0)=1$ and $R_H(0)=U(0,-\infty)$ yield the following transformations:

$$|\psi_S(t)\rangle=U_S(t,0)U(0,-\infty)|\psi_H\rangle\ , \tag{10a}$$

$$|\psi_S(t)\rangle=U_{0S}(t,0)|\psi_I(t)\rangle\ , \tag{10b}$$

$$|\psi_I(t)\rangle=U(t,-\infty)|\psi_H\rangle\ , \tag{10c}$$

which feature another interaction picture satisfying Eqs. (1) and (2).

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[1] H.-H. Xu and C.-H. Tsai, Phys. Rev. A **41**, 53 (1990).

[2] See, for example, H.-H. Wu, Phys. Rev. D **47**, 2622 (1993); see also T. S. Evans, I. Hardman, H. Umezawa, and Y. Yamanaka, J. Math. Phys. **33**, 370 (1992).

[3] K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969), pp. 108–109.

[4] Xu and Tsai used the relation $H_S U(0,-\infty)=U(0,-\infty)H_{0S}$, known from formal scattering theory, to alter the resulting Eqs. (8b) and (8c).