Bistability in a quantum nonlinear oscillator

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The response of a nonlinear oscillator to an external electromagnetic field is studied in the quantum regime. It is demonstrated that bistability occurs when the shift of the resonance frequency, induced by the external field and modified by fluctuations (quantum and thermal), surpasses the natural linewidth. This theory can be applied to the study of the nonlinear response, near resonance, of a single electron, trapped in a magnetic field, to an external electromagnetic field. It is shown that, even at low quantum levels of the oscillator, bistability can be observed due to the interplay between the small relativistic nonlinearity, the external electromagnetic field, and the heat bath.

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I. INTRODUCTION

The study of *nonlinear dynamics* of quantum systems is attractive for its fundamental aspect as well as for its interesting applications [1–11]. A particular system, which has been widely studied is the *nonlinear oscillator* (NO), as it serves as an instructive prototype for a large class of phenomena, i.e., driven tunneling [4], quantum chaos [5], bistable resonance [6–8,10], etc.

Even classically, straightforward methods of perturbation are not applicable in solving a large class of nonlinear equations of motion [12,13]. A successful approximation procedure, which overcomes some of the obstacles, is the Krylov-Bogoliubov (KB) method of averaging [13]. According to the KB method of averaging, *bistable resonance* is classically expected when the NO is driven by an external force and the system is affected by its environment. Due to nonlinearity, the frequency of natural oscillations becomes dependent upon the amplitude of the oscillations and the frequency-response curve exhibits hysteresis when the amplitude of the driving force exceeds a threshold value. When the frequency of the external force is swept through resonance, a jump occurs from high to low excitation level. At a different external frequency, a jump occurs from low to high excitation level when the sweep through resonance is in the opposite direction.

An individual electron in a magnetic field can display bistability under the action of an electromagnetic (e.m.) field. The small relativistic mass increase of the electron causes the driven motion to be highly nonlinear. This phenomenon, which constitutes a microscopic realization of nonlinear dynamics, was predicted theoretically by Kaplan [14,15], who indicated that the motion of the electron is of a forced nonlinear oscillator. This phenomenon was verified experimentally by Gabrielse *et al.* [16,17] by probing the cyclotron motion of one electron in a Penning trap. The *intrinsic bistability* of this *microscopic* system has been analyzed *classically*. However, a *quantum-mechanical* description is required in order to understand the nature of the system at low levels of excitation.

The subject of bistable resonance of quantum systems was approached by Drummond and Walls, who modeled a nonlinear dispersive medium by a NO and analyzed its response to a driving field in limiting cases [7]. They used a Fokker-Planck equation in a generalized P representation to claim that in the limit of large quantum noise no bistability is obtained, whereas the system exhibits hysteresis in the semiclassical limit. Savage and Carmichael have considered a single atom in a cavity and showed that absorbtive optical bistability can exist within a quantum-mechanical theory, in the good-cavity limit [10]. In this limit the effect of quantum fluctuations is relatively small. Another work, by Horsthemke and McCarty, examines the effect of noise upon a nonlinear system [11]. They analyzed a system that exhibits nonequilibrium transitions and bistability when critical values are reached. The presence of noise modifies the critical points, which become shifted by a noise-dependent term. Although this result was derived for a very different system —an autocatalytic photochemical reaction with incident light upon it — we will show that similar results come out of the discussions of this paper. In the analysis of bistable response, our approach finds that the effect of fluctuations upon the steady-state response is to shift the resonance frequency, as found in Ref. [11]. In the present paper we do not deal with the effect of fluctuations upon the stability of the two stable states, that is, its influence upon the time scale of the stability.

The purpose of the present paper is to investigate the nature of nonlinear resonance in the quantum regime. This would provide an understanding of the bistable resonance in low levels of excitation, where its feasibility was questioned [7,14–17]. The model to be used is of the harmonic oscillator with a small anharmonic term due to relativistic effects. The nonlinear oscillator is coupled to a heat bath, and we analyze the effect of an externally applied e.m. field of frequency Ω on this system. We formulate the problem with a master equation that is transformed into an evolution equation in terms of the Wigner quasiprobability distribution function. In order to analyze the interplay between the nonlinearity, the e.m. wave, and the bath, we employ two schemes of approximation: first, by assuming that the Wigner quasidistribution function is a Gaussian, centered around the *expectation values* of the dynamical variables, and second, by truncating the hierarchy of *moments' equations* extracted from the evolution equation for the Wigner function. We find that the two schemes lead to the same results in the long-time

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limit. We show that the effect of the interactions is to transform the relativistic nonlinearity into its semiclassical form and to provide for a relativistic frequency shift that depends upon fluctuations, quantum and thermal. As a result, the response becomes highly nonlinear. When the analysis is applied to the experimental conditions, which occurred in the single-electron cyclotron resonance, the response should exhibit hysteresis when the relativistic frequency shift exceeds the width of the resonance line.

II. MODEL FOR A NONLINEAR OSCILLATOR

Consider a one-dimensional (1D) nonlinear charged oscillator, interacting with a bath, under the action of an external e.m. field. As a concrete case, to be identified as *cyclotron oscillator* (CO), imagine one electron trapped in a Penning cage. The restoring force on the electron is provided by a constant magnetic field in the z direction. The external e.m. field, near resonance with the cyclotron frequency, propagates along the z axis with circular polarization. The oscillator is weakly anharmonic due to a *weak* relativistic correction, its Hamiltonian being

$$H_{\rm NO}=H_{\rm osc}+H_{\rm rel}=\left[\frac{p^2}{2m}+\frac{m\omega^2q^2}{2}\right]+\left[-\frac{p^4}{8m^3c^2}\right], \quad (1)$$

where p is the oscillator's momentum operator, q is the coordinate operator, ω is the natural frequency, m is the mass, and c is the speed of light. The oscillator is weakly coupled to a bath having many degrees of freedom. For example, the bath can consist of a large number of $1D$ harmonic oscillators i, of frequency ω_i. In the CO case, the bath is, e.g., the radiation background. Then the bath Hamiltonian is given by

$$H_{\rm bath}=\sum_i \hbar\omega_i\left(a_i^\dagger a_i+\frac{1}{2}\right), \quad (2)$$

where $a_i^\dagger$ and a_i are the creation and annihilation operators of the bath's ith oscillator. We can also assume that initially the bath is at thermal equilibrium at a temperature T, in which case the bath has a Bose-Einstein distribution. We also introduce, for our harmonic oscillator, creation and annihilation operators $b^\dagger$ and b in terms of q and p, namely,

$$b=q\left(\frac{m\omega}{2\hbar}\right)^{1/2}+ip\left(\frac{1}{2m\hbar\omega}\right)^{1/2},$$

$$b^\dagger=q\left(\frac{m\omega}{2\hbar}\right)^{1/2}-ip\left(\frac{1}{2m\hbar\omega}\right)^{1/2}, \quad (3)$$

and the oscillator's Hamiltonian of Eq. (1) is then

$$H_{\rm NO}=\hbar\omega\left[b^\dagger b+\frac{1}{2}-\frac{3}{16}\left(\frac{\hbar\omega}{mc^2}\right)\{b^\dagger b b^\dagger b\}_{\rm sym}\right], \quad (4)$$

where the symmetrical product of creation and annihilation operators is the average of all possibilities of ordering the operators.

Given the model for the bath, it is convenient to describe the coupling hamiltonian between the NO and the bath in the form

$$H_{\rm int}=\sum_i (g_i b^\dagger a_i+g_i^* b a_i^\dagger), \quad (5)$$

where g_i is the coupling between the NO and the ith oscillator of the bath. By assuming this type of coupling Hamiltonian with the bath, we have neglected processes that do not conserve energy and thus do not contribute to the damping of the oscillator [20]. This approximation is referred to as the *rotating-wave approximation* (RWA), which we will maintain throughout the paper.

A classical e.m. wave of frequency Ω drives the NO. The interaction between the NO and the e.m. wave is well described by the dipole approximation, provided the electromagnetic wavelength is much larger than the amplitude of oscillation of the NO. The Hamiltonian of interaction with the field is then

$$H_{\rm ext}=eE(t)q=eE(t)\sqrt{\frac{\hbar}{2\omega m}}(b+b^\dagger), \quad (6)$$

where $E(t)$ is the electric field and e is the charge, and we have used Eq. (3).

III. DYNAMICS

The interplay between the small $(p\ll mc)$ relativistic term $H_{\rm rel}$, and the EM field term, $H_{\rm ext}$ is expected to be the cause of a *dynamical nonlinearity* in the system's behavior. This kind of behavior is known to exist in the classical regime, and we wish to investigate how the classically nonlinear dynamics are modified when one considers the corresponding *quantum* regime. This would provide an understanding of the bistable resonance in low levels of excitation, where its feasibility was questioned.

The state of the whole system, the NO interacting with the bath and subject to a classical driving field, may be described by the density operator $\rho(t)$, which obeys the evolution equation

$$\frac{d}{dt}\rho(t)=\frac{1}{i\hbar}[H_{\rm tot},\rho(t)], \quad (7)$$

where

$$H_{\rm tot}=H_{\rm osc}+H_{\rm rel}+H_{\rm bath}+H_{\rm int}+H_{\rm ext} \quad (8)$$

is given by Eqs. (2) and (4)–(6). Since we are interested in the evolution of the NO only, one may derive the master equation for the reduced density operator $\sigma(t)$, which we obtain by tracing $\rho(t)$ over the bath variables, using standard methods [20,21].

A. Wigner distribution

We are concerned with the nonlinear response of the oscillator to the applied e.m. field, which we are able to study by means of the master equation. In order to get insight concerning the resemblences and differences between the *classical* and *quantum* descriptions we formulate the problem in terms of the Wigner quasiprobability distribution function $P_W(q,p,t)$ defined as [21,22]

$$P_W(q,p,t)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}dp'\langle p-p'|\sigma(t)|p+p'\rangle e^{-2iqp'/\hbar}, \tag{9}$$

where $\sigma(t)$ is the reduced density operator defined above and $\langle p-p'|\sigma(t)|p+p'\rangle$ is a matrix element of $\sigma(t)$ in terms of the momentum eigenstates of the unperturbed NO, such that $P_W(q,p,t)$ behaves like a probability density in q and p, namely,

$$\int_{-\infty}^{\infty}dp P_W(q,p,t)=\langle q|\sigma(t)|q\rangle,$$

$$\int_{-\infty}^{\infty}dq P_W(q,p,t)=\langle p|\sigma(t)|p\rangle.$$

$P_W(q,p,t)$ is a phase-space quasidistribution, i.e., a real function that satisfies

$$\int dq\int dp P_W(q,p,t)=1.$$

Rather than using Wigner's original formulation of quasidistribution in terms of q and p, we will consider an equivalent form that is more convenient in evaluating expectation values of products of creation and annihilation operators b and $b^\dagger$. Introducing the parameter α, defined as

$$\alpha=q\left(\frac{m\omega}{2\hbar}\right)^{1/2}+ip\left(\frac{1}{2m\hbar\omega}\right)^{1/2}$$

in terms of the c numbers q and p, one can obtain by standard techniques [20,21] the evolution equation for the Wigner quasidistribution function

$$\begin{aligned}\frac{\partial}{\partial t}P_W(\alpha,\alpha^*,t)=&-i(\omega+\delta\omega)\left[\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}\right]P_W(\alpha,\alpha^*,t)+i\mu\left[|\alpha|^2\left(\alpha\frac{\partial}{\partial\alpha}-\alpha^*\frac{\partial}{\partial\alpha^*}\right)\right.\\&\left.-\frac{1}{4}\left(\alpha\frac{\partial}{\partial\alpha}-\alpha^*\frac{\partial}{\partial\alpha^*}\right)\frac{\partial^2}{\partial\alpha^*\partial\alpha}\right]P_W(\alpha,\alpha^*,t)\\&+\frac{\gamma}{2}\left[\frac{\partial^2}{\partial\alpha^*\partial\alpha}+\alpha^*\frac{\partial}{\partial\alpha^*}+\alpha\frac{\partial}{\partial\alpha}+2\right]P_W(\alpha,\alpha^*,t)+\gamma\bar{n}\left[\frac{\partial^2}{\partial\alpha^*\partial\alpha}\right]P_W(\alpha,\alpha^*,t)\\&+iE'(t)\left[\frac{\partial}{\partial\alpha^*}-\frac{\partial}{\partial\alpha}\right]P_W(\alpha,\alpha^*,t).\end{aligned} \tag{10}$$

Here $E'(t)=-eE(t)(1/8m\hbar\omega)^{1/2}$, $\bar{n}=(\exp\hbar\omega/kT-1)^{-1}$, γ represents the damping constant, $\mu=\omega(3\hbar\omega/8mc^2)$ plays the role of a *relativistic frequency* scale, and the natural frequency ω is supplemented by the small shift due to the coupling to the heat bath $\delta\omega$, which we will not, from now on, write explicitly.

We are concerned with the response of the system (NO) to the e.m. field near resonance, i.e., $\omega\sim\Omega$, in the RWA. More specifically, we concentrate on the rate of energy change, which can be derived using the evolution equation for the quasidistribution function and the relation

$$\langle\{b^s(b^\dagger)^r\}_{\text{sym}}\rangle=\int d^2\alpha\ \alpha^s(\alpha^*)^r P_W(\alpha,\alpha^*,t). \tag{11}$$

Denoting by $B=\{b^\dagger b\}_{\text{sym}}$, the rate of energy change takes the form

$$\frac{d}{dt}\langle B\rangle=-\gamma\langle B\rangle+\gamma\left(\bar{n}+\frac{1}{2}\right)+iE'(t)(\langle b^\dagger\rangle-\langle b\rangle). \tag{12}$$

This equation is a statement of conservation of energy: the energy absorbed from the e.m. field is translated into the rate of change of energy from its equilibrium value $\bar{n}+\frac{1}{2}$. Similarly, the equation for expectation value of the annihilation operator reads

$$\frac{d}{dt}\langle b\rangle=-\left(i\omega+\frac{\gamma}{2}\right)\langle b\rangle+i\mu\langle\{b^\dagger b^2\}_{\text{sym}}\rangle+iE'(t). \tag{13}$$

Equations (12) and (13) determine the dynamical behavior of the system. Notice that in Eq. (13) the *nonlinear* term on the right-hand side, i.e., the small relativistic correction, is depicted by the $\langle\{b^\dagger b^2\}_{\text{sym}}\rangle$ expression. One should remark that near resonance, i.e., when $\omega\sim\Omega$, a straightforward perturbation treatment is not applicable in calculating this expectation value. In order to examine the interplay between H_{rel}, H_{ext} and the heat bath, we have to resort to nonperturbative methods.

B. Gaussian wave packet

As a first attempt, we make an assumption concerning the form of the Wigner function. We look for a solution of the evolution equation Eq. (10) in the form of a *Gaussian wave packet*. Given that this choice would be exact in the nonrelativistic limit, i.e., $\mu\to 0$, we expect that for a *weak* relativistic correction, a Gaussian wave packet would be a good approximate solution of the evolution equation for the quasidistribution. Heller has studied extensively over the years the semiclassical dynamics of chaotic and integrable systems by means of wave-packet dynamics [18,19]. According to Ref. [18], the assumption of a Gaussian form for $P_W(\alpha,\alpha^*,t)$ suggests that the distribution is well localized in the phase space defined by α and α^*. Denoting as α_t and α_t^* the center of the Wigner function in phase space, the evolution of $P_W(\alpha,\alpha^*,t)$ turns to be affected only by the *local* behavior of the potential in the neighborhood of the Wigner function's center (the "potential" term that needs to be approximated in our case refers to the small relativistic Hamiltonian $H_{\rm rel}$). Therefore, in the Taylor expansion of the potential about the wave packet's center, only terms up to quadratic are assumed to be significant. This approximation yields what is commonly referred to as the *semiclassical approximation*, namely, the factorization of the relativistic term, $\langle\{b^\dagger b^2\}_{\rm sym}\rangle\rightarrow\langle b^\dagger\rangle\langle b\rangle^2$. That means that the *local* approximation of the potential does not account for the *quantum* effects that may arise due to the interaction with the heat bath or with the e.m. field.

We shall follow a different procedure, where the Gaussian wave packet is determined by including a self-consistent contribution due to the nonlinear relativistic term. We assume a Wigner function of the form

$$P_W(\alpha,\alpha^*,t)=\exp[K(t)]$$

where

$$K(t)=-\frac{1}{\xi(t)}(\alpha-\alpha_t)(\alpha^*-\alpha_t^*)+\ln\nu(t). \tag{14}$$

Here the parameters α_t and α_t^* represent the center of the Wigner function and according to Eq. (11) satisfy the relations

$$\langle b\rangle=\alpha_t,\quad \langle b^\dagger\rangle=\alpha_t^*, \tag{15}$$

i.e., the Wigner function is centered around the *expectation values* of the dynamical variables. The evolution of the center, e.g., the evolution of α_t, is given by Eq. (13), with α_t^* its complex conjugate.

The parameters $\xi(t)$ and $\nu(t)$ of Eq. (14) are to be determined by substituting $P_W(\alpha,\alpha^*,t)$ into the evolution equation Eq. (10) and comparing coefficients of like powers of $(\alpha-\alpha_t)(\alpha^*-\alpha_t^*)$. This yields the equations

$$\frac{d}{dt}\,\xi(t)=\frac{\gamma}{2}[1-2\xi(t)]+\gamma\bar{n},$$

$$\frac{1}{\nu(t)}\frac{d}{dt}\,\nu(t)=\frac{1}{\xi(t)}\frac{d}{dt}\,\xi(t), \tag{16}$$

with the solutions

$$\xi(t)=\xi(0)e^{-\gamma t}+\left(\bar{n}+\frac{1}{2}\right)[1-e^{-\gamma t}],$$

$$\nu(t)=\frac{1}{\pi\xi(t)}\,. \tag{17}$$

Our next major assumption deals with the nonlinear relativistic term of Eq. (13), $\langle\{b^\dagger b^2\}_{\rm sym}\rangle$. We evaluate this expectation by means of the Gaussian wave packet of Eq. (14). This yields

$$\langle\{b^\dagger b^2\}_{\rm sym}\rangle=2\alpha_t\xi(t)+\alpha_t|\alpha_t|^2, \tag{18}$$

where $\xi(t)$ is given by Eq. (17). This result, when inserted back into Eq. (13), yields a simpler form of a nonlinear equation, namely,

$$\frac{d}{dt}\,\alpha_t=-\left[i\omega+\frac{\gamma}{2}-i\mu(|\alpha_t|^2+2\xi(t))\right]\alpha_t+iE'(t). \tag{19}$$

Equations (12), (17), and (19) determine the evolution in time of the parameters of the Gaussian Wigner function and they include the *nonlinear dynamic relativistic term* and a *relativistic quantum frequency shift*. Obviously, this constitutes the nonlinear response problem at hand, including the effect of the heat bath and of the e.m. field.

Before turning to an alternative method of approximation that will reinforce the obtained results, let us examine the effect of the heat bath on the NO. Apart from the usual effect of introducing a dissipative term into the evolution of the oscillator, the effect of the heat bath on the nonlinear part of the oscillator manifests itself through the destruction of correlations, i.e., the factorization of the expectation $\langle\{b^\dagger b^2\}_{\rm sym}\rangle$, and the appearance of a relativistic quantum frequency shift $2\mu\xi(t)$. When $t\rightarrow\infty$, i.e., when the transient motion decays, it is interesting to analyze the classical and quantum limits of this shift. The classical limit is obtained for $\hbar\omega\ll kT$ and $N\rightarrow\infty$, where we have identified $|\alpha_t|^2$ at steady state as the excitation level and denoted it by N. Then, the relativistic frequency shift is modified by thermal fluctuations, that is, $\xi_s\rightarrow kT/\hbar\omega$. At the limit $T\rightarrow 0$, i.e., in the quantum regime, $\xi_s\rightarrow 1/2$. This time the zero-point quantum fluctuations are responsible for the relativistic frequency shift.

C. Moments' equations

We now would like to try an alternative way to analyze the interplay between $H_{\rm rel}$, $H_{\rm ext}$, and the heat bath, in order to emphasize the results of the preceding subsection. We start with the equations for $\langle B\rangle$ and $\langle b\rangle$, Eqs. (12) and (13). Consider first the expectation $\langle\{b^\dagger b^2\}_{\rm sym}\rangle$ of Eq. (13). If we write, e.g., the annihilation operator b in terms of its expectation $\langle b\rangle$ as $b=\langle b\rangle+\delta$, where δ plays the role of *the fluctuation part* of the operator, we observe that

$$\langle\{b^\dagger b^2\}_{\rm sym}\rangle=\langle b^\dagger\rangle\langle b\rangle^2+\langle b^\dagger\rangle\langle\delta^2\rangle+2\langle b\rangle\langle\{\delta^\dagger\delta\}_{\rm sym}\rangle+\langle\{\delta^\dagger\delta^2\}_{\rm sym}\rangle. \tag{20}$$

We then notice that the contribution of the term $\mu\langle\{b^\dagger b^2\}_{\rm sym}\rangle$ of Eq. (13) is very small. We therefore make a further *approximation*, which consists in keeping terms only up to first order in the relativistic correction, i.e., first order in μ, in the equation for $\langle b\rangle$, in Eq. (13). Accordingly, we evaluate the terms $\langle\delta^2\rangle$, $\langle\{\delta^\dagger\delta\}_{\rm sym}\rangle$ and $\langle\{\delta^\dagger\delta^2\}_{\rm sym}\rangle$ of relation (20) in zeroth order in μ and then insert the resulting *approximate* form for $\langle\{b^\dagger b^2\}_{\rm sym}\rangle$ back into Eq. (13). The evolution of the expectations $\langle \delta^2 \rangle$ and $\langle \{\delta^\dagger \delta\}_{\rm sym}\rangle$ are readily obtained using the definitions $\langle \delta^2\rangle = \langle b^2\rangle - \langle b\rangle^2$ and $\langle\{\delta^\dagger\delta\}_{\rm sym}\rangle = \langle\{b^\dagger b\}_{\rm sym}\rangle - \langle b^\dagger\rangle\langle b\rangle$, and $\langle\{\delta^\dagger\delta^2\}_{\rm sym}\rangle$ is defined by Eq. (20). The equations for $\langle b\rangle$, $\langle\{b^\dagger b\}_{\rm sym}\rangle$, $\langle b^2\rangle$, and $\langle\{b^\dagger b^2\}_{\rm sym}\rangle$ needed to evaluate the fluctuation expectations in zeroth order in μ are obtained using Eqs. (10) and (11) with $\mu\to 0$ (the results are given in the Appendix). We end up with the set of approximate equations

$$\frac{d}{dt}\alpha_t = -\left(i\omega + \frac{\gamma}{2}\right)\alpha_t + i\mu[\alpha_t|\alpha_t|^2 + \alpha_t^*\langle\delta^2\rangle + 2\alpha_t\langle\{\delta^\dagger\delta\}_{\rm sym}\rangle + \langle\{\delta^+\delta^2\}_{\rm sym}\rangle] + iE'(t), \quad (21)$$

$$\frac{d}{dt}\langle\{\delta^\dagger\delta\}_{\rm sym}\rangle = -\gamma\langle\{\delta^\dagger\delta\}_{\rm sym}\rangle + \gamma\left(\bar n + \frac{1}{2}\right), \quad (22)$$

$$\frac{d}{dt}\langle\delta^2\rangle = -2\left(i\omega + \frac{\gamma}{2}\right)\langle\delta^2\rangle, \quad (23)$$

$$\frac{d}{dt}\langle\{\delta^\dagger\delta^2\}_{\rm sym}\rangle = -\left(i\omega + \frac{3\gamma}{2}\right)\langle\{\delta^\dagger\delta^2\}_{\rm sym}\rangle, \quad (24)$$

where we used the expression $\langle b\rangle = \alpha_t$ of Eq. (15).

Let us first analyze Eqs. (22)–(24). These equations determine the evolution in time of the mean of the fluctuations to zeroth order in μ. In the long-time limit $t\to\infty$, the energy variance reduces to its equilibrium value $(\bar n + 1/2)$, while $\langle\delta^2\rangle$ and $\langle\{\delta^\dagger\delta^2\}_{\rm sym}\rangle$ decay to zero. Gathering these results and inserting them back into Eq. (21), the equation for α_t in the long-time limit assumes the same form as we have obtained in the previous approximation scheme, namely, with the Gaussian Wigner function. We have therefore shown that in the limit $t\to\infty$, the two approximation schemes yield the same results.

We note that throughout the paper, the nonlinearity is assumed to be small and all the results are valid within this regime. However, the assumptions made concerning the nonlinearity do not imply the same restrictions upon fluctuations, quantum or thermal. In fact, after having formulated the problem in terms of the Wigner quasiprobabilty function, we have not made any assumptions concerning fluctuations. The results, therefore, should be valid in the limit of large fluctuations.

IV. NONLINEAR RESPONSE

Having displayed the interplay between all the constituents of the interacting system, $H_{\rm rel}$, $H_{\rm ext}$, and the heat bath, we now turn to the study of the nonlinear response to an external e.m. field $E(t) = \mathscr{E}\cos(\Omega t)$, near resonance $(\omega\sim\Omega)$, in the RWA. We proceed with our analysis with either set of dynamical equations, Eqs. (12), (17), and (19) or Eqs. (12) and (21)–(24), with the fluctuations evaluated at steady state. Note that we are only interested in the steady- state response to the e.m. field. The fast time dependence of α_t can be extracted by the transformation $\alpha_t = \tilde\alpha_t \exp[-i(\Omega t - \tilde\varphi)]$, where $\tilde\alpha_t$ and $\tilde\varphi$ are slowly varying functions of time, on the natural fast time scale $1/\Omega$. In the same approximation, notice that the energy $\tilde B = \langle B\rangle$ is itself slowly varying in time.

The response to the e.m. field can now be expressed in terms of the differential equations for the slow variables

$$\frac{d}{dt}\tilde\alpha_t = -\frac{\gamma}{2}\tilde\alpha_t + \tilde{\mathscr{E}}'\sin\tilde\varphi, \quad (25)$$

$$\frac{d}{dt}\tilde\varphi = \Delta + \mu(\tilde\alpha_t^2 + 2\xi_s) + \tilde{\mathscr{E}}'\cos\tilde\varphi\,\frac{1}{\tilde\alpha_t}, \quad (26)$$

$$\frac{d}{dt}\tilde B = -\gamma\tilde B + \gamma\xi_s + 2\tilde{\mathscr{E}}'\tilde\alpha_t\sin\tilde\varphi. \quad (27)$$

Here $\Delta = (\Omega - \omega)$ is the detuning, which satisfies $\Delta \ll \omega$, and $\tilde{\mathscr{E}}' = -e\tilde{\mathscr{E}}(1/8m\hbar\omega)^{1/2}$ and we have also used the equilibrium value for the energy variance $\xi_s = (\bar n + 1/2)$. Equations (25)–(27) represent the equations governing the response of the NO, interacting with a heat bath, to the applied e.m. field. The effect of the small relativistic correction consists of two terms in Eq. (26): the nonlinear term $\mu|\tilde\alpha_t|^2$, which would arise from a semiclassical factorization of the term $\mu\langle\{b^\dagger b^2\}_{\rm sym}\rangle$, and the fluctuation-dependent frequency shift $2\mu\xi_s$.

In the present paper we are interested in the energy absorbed by the nonlinear oscillator in the presence of the e.m. field. More specifically, our concern is with the average power $Q = 2\tilde\alpha_t\tilde{\mathscr{E}}'\sin\varphi$ (in energy units), delivered to the system in the *steady state* Q_s, which can be experimentally measured near resonance. In the steady state, with $Q_s = \gamma(\tilde B - \xi_s)$, we obtain from Eqs. (25)–(27)

$$Q_s = (\sqrt{2}\tilde{\mathscr{E}}')^2 \frac{\gamma/2}{(\gamma/2)^2 + [\Delta + \mu(2\xi_s + \gamma^{-1}Q_s)]^2}. \quad (28)$$

Equation (28) is a cubic equation for Q_s. In the nonrelativistic limit $\mu\to 0$, Eq. (28) yields for Q_s a Lorentzian line shape near resonance. When the relativistic term becomes significant, the interplay between the nonlinear relativistic term and the field intensity dramatically modifies the absorbtion line shape. Introducing scaled variables $q = Q_s(2\mu/\gamma^2)$, $d = 2\Delta/\gamma$, and $\epsilon = \tilde{\mathscr{E}}'^2(8\mu/\gamma^3)$ (the variables are now dimensionless), we rewrite Eq. (28) in the form

$$q[(q + d + f)^2 + 1] = \epsilon, \quad (29)$$

where f is the dimensionless, fluctuation dependent, relativistic frequency shift $f = 4\xi_s\mu/\gamma$. Equation (29) is known in the theory of *nonlinear oscillators* to yield hysteretic behavior, when threshold conditions $d + f < -\sqrt{3}$ and $\epsilon > 8/(3\sqrt{3})$ are met [12,13]. When the e.m. field is above threshold, Eq. (29) yields two stable solutions for q and one unstable one and the response line shape is *bistable*. This nonlinear absorbtion results when the *relativistic shift* of the resonance frequency *modified by fluctuations* exceeds the *natural linewidth* of the oscillator. Figure 1 shows calculated curves for the rate of absorbed power at steady state Q_s as a function of the quantum mechanically modified resonant de-

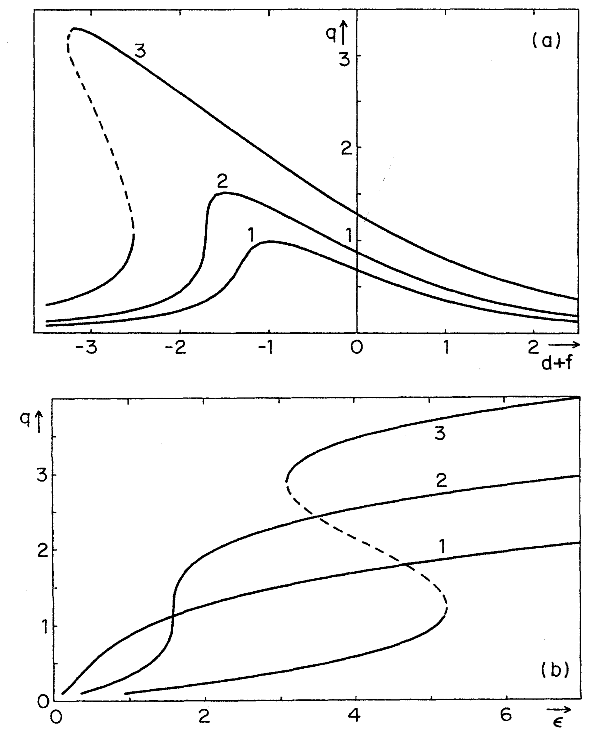

FIG. 1. Resonance curves for the scaled, dimensionless power $q=Q_s(2\mu/\gamma^2)$ as a function of (a) the dimensionless detuning modified by fluctuations $d+f$, where $d=2\Delta/\gamma$ and $f=4\xi_s\mu/\gamma$, for various intensities of the applied e.m. field: (1) $\epsilon=1$, (2) $\epsilon=8/(3\sqrt{3})$, and (3) $\epsilon=3$; (b) the scaled, dimensionless intensity of the e.m. field $\epsilon=\mathcal{E}'^2(8\mu/\gamma^3)$ for different detunings: (1) $d+f=-3$, (2) $d+f=-\sqrt{3}$, and (3) $d+f=-1$. The values for the parameters μ and ξ_s are taken from Refs. [14,15]. Accordingly, $\mu/\gamma\approx 320$ and $\xi\approx 11/18$.

tuning $\Delta+2\mu\xi_s$ and the scaled e.m. field intensity $\mathcal{E}'^2$ (in dimensionless units), with $\mu/\gamma\approx 320$ and $\xi_s\approx 11/18$.

The results we have obtained indicate that bistability should be obtained also in the quantum regime. The quantum-mechanical treatment of the interactions between the NO, the heat bath, and the external e.m. field does not seem to modify the essential character of the response. Our analysis does not lead to the conclusion obtained in Ref. 7, which states that in the limit of large quantum noise bistability should be destroyed. In fact, according to Eq. (29), the effect of quantum noise is to shift the line shape. In order to further illustrate these effects, we return to the experiment that we have mentioned before, that is, the response of a slightly relativistic cyclotronic electron to an applied e.m. field, which was classically analyzed [14,15] and experimentally verified [16,17]. The experiment perfomed by Gabrielse *et al.* was the following: A single electron was trapped in a Penning cage and cooled to liquid-helium temperature 4.2 K. The cyclotron motion in an axial magnetic field $B=60$ kG (the corresponding cyclotron frequency is $\omega\approx 164$ GHz) was observed. The corresponding time constant was $\gamma^{-1}\approx 0.3$ sec. Under these conditions, the dimensionless relativistic frequency was $\mu/\gamma\approx 320$ and the strength of the parameter that reflects quantum fluctuations was $\xi_s\approx 11/18$. A rf field was then applied and the nonlinear cyclotron resonance was measured indirectly, relying on the relativistic shift of the axial frequency. Note that the electron is trapped by the magnetic field in the radial plane, where its motion is partially cyclotronic, and is confined in the axial direction by electrostatic field and therefore also oscillates harmonically along the axis. In the experiment, changes in axial frequencies of $\Delta\omega_z/2\pi=1$ Hz out of $\omega_z/2\pi=62$ MHz could be resolved. The ratio $\Delta\omega_z/\omega_z$ was related to the cyclotronic excitation and consequently determined the level of cyclotronic excitation that could be observed. This put the limitation on the observable cycltronic state $N\geqslant 30$, noting that we have identified N as the steady-state excitation level. Obviously, classical analysis was adequate in that stage.

We can estimate the effect of the term f on the line shape for this level of excitation. For this purpose one can look at the value of the detuning for which the power absorbed is maximal, noting that N is related to the absorbed power at steady state by $Q_s=\gamma N$. When the effect of fluctuations is taken into account we find that the maximal power absorption occurs for the dimensionless detuning value $d_{(\max\ q)}\sim -9900$, whereas without it $d_{(\max\ q)}\sim -9510$. Obviously in the classical regime the effect of fluctuations is negligible, e.g., about 4% for $N\sim 30$. However, in the quantum level of excitations the shift of the detuning as a result of fluctuations becomes significant: for example, for $N\sim 3$ (which corresponds to the dimensionless e.m. wave intensity $\epsilon\sim 950$, which is well above the threshold for bistability) d assumes the value ~ -950 when f is not considered, whereas when f is included d becomes ~ -1340. This amounts to a shift $\sim 40\%$ larger. Therefore, an experiment performed at low excitation levels should reveal pronounced fluctuation effects, although without masking the principal features of the line shape, namely, the bistable effect.

Given the typical experimental data and the threshold conditions for bistability, we can summarize our estimations concerning the possibility of observing hysterectic response at low levels of excitation. The level of cyclotronic excitation N at threshold satisfies $N\leqslant 1$. Accordingly, the excitation to $N\approx 1$ should already bring the system to the highly nonlinear regime. At this low level of excitation, the effect of quantum fluctuations is significant, i.e., the frequency-response curve is shifted, namely, the maximal response occurs for a higher value of the dimensionless detuning, d. Nevertheless, our estimations do not show that at any stage bistability is destroyed by quantum fluctuations. We therefore believe that bistability should be observed also for low-lying cyclotron excitations, provided this can be experimentally detected.

V. CONCLUSIONS

We have presented a quantum-mechanical study of the bistable resonance of a slightly relativistic oscillator. We showed that as a result of the interaction between the nonlinear oscillator, the external e.m. field, and the heat bath, the response to the e.m. field can become hysterectic. The effect of the interactions is to transform the relativistic nonlinearity in the system's evolution into its semiclassical form and to introduce a relativistic frequency shift that includes the effect of fluctuations, quantum and thermal. We indicated that the bistable resonance of a trapped, single cyclotronic electron should be observed for low-lying excitations.

APPENDIX

In this appendix we derive the equations for $\langle b\rangle$, $\langle\{b^\dagger b\}_{\rm sym}\rangle$, $\langle b^2\rangle$, and $\langle\{b^\dagger b^2\}_{\rm sym}\rangle$ to zeroth order in μ, using the equation for the quasidistribution function Eq. (10) and the relation (11), with $\mu\to 0$:

$$\frac{d}{dt}\,\alpha_t=-\left(i\omega+\frac{\gamma}{2}\right)\alpha_t+iE'(t), \tag{A1}$$

$$\frac{d}{dt}\,\langle B\rangle=-\gamma\langle B\rangle+\gamma\left(\bar{n}+\frac{1}{2}\right)+iE'(t)(\alpha_t^*-\alpha_t), \tag{A2}$$

$$\frac{d}{dt}\,\langle b^2\rangle=-2\left(i\omega+\frac{\gamma}{2}\right)\langle b^2\rangle+2iE'(t), \tag{A3}$$

$$\frac{d}{dt}\,\langle\{b^\dagger b^2\}_{\rm sym}\rangle=-\left(i\omega+\frac{3\gamma}{2}\right)\langle\{b^\dagger b^2\}_{\rm sym}\rangle+2\gamma\left(\bar{n}+\frac{1}{2}\right)\langle b\rangle+iE'(t)(2\langle\{b^\dagger b\}_{\rm sym}\rangle-\langle b^2\rangle). \tag{A4}$$

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