Off-resonance light scattering from low-temperature Bose and Fermi gases

Juha Javanainen and Janne Ruostekoski
Department of Physics, University of Connecticut, Storrs, Connecticut 06269-3046
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We study interactions of light with a sample of two-level atoms, with full inclusion of angular momentum degeneracy, at temperatures and densities such that the quantum statistics of the atoms may have an effect. Coupled propagation equations are given for light and matter fields, and plausible general simplifications are enumerated. In particular, the motion of the atoms during the excited-state lifetime may often be ignored. The propagation equations of light and matter fields are decoupled within the assumption that the detuning of the driving light from atomic resonance is large, and the spectrum of scattered light is studied for ideal Bose and Fermi gases. In addition to the expected image of the velocity distribution, the spectra contain qualitatively distinct features that depend on the statistics of the atoms. This is because for bosons (fermions), those scattering events in which an atom recoils to an already occupied state are enhanced (inhibited).

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I. INTRODUCTION

Trapping and cooling of neutral atoms [1] is presently one of the mainstreams of atomic, molecular, and optical physics. Given the relentless progress toward higher densities and lower temperatures, in the end quantum statistics of the atoms will have measurable consequences. Indeed, Bose condensation of a weakly interacting gas remains the holy grail of cooling. Lasers are an integral part of many of these experiments, and the question of the optical response of the Bose condensate has come up [2–4]. The consequences of the dipole-dipole interactions between the atoms in a degenerate gas [5,6] are another closely related area of research. The main attraction here is the possibly nonlinear behavior of the atomic fields, "nonlinear atom optics" [5].

An extremely broad optical resonance has been identified as a prominent characteristic of a Bose condensate [3,4]. However, the prospects of using the linewidth for the detection of Bose condensation are clouded by the fact that broad resonances are commonplace in dense atomic samples, even for the classical Maxwell-Boltzmann gas. In search of a more unambiguous signature of Bose condensation, we have recently [7] pointed out that the spectrum of light scattered from a degenerate Bose gas contains distinct qualitative features associated with atom statistics. In a dilute gas, the spectrum mirrors the velocity distribution of the atoms [8], whereas in a degenerate sample photon recoil events that take an atom to an already occupied state are enhanced by the Bose-Einstein statistics. In the present paper we give a thorough derivation and discussion of this phenomenon. We also carry out the corresponding analysis for fermions, which turn out to display equally singular features in light scattering.

We begin in Sec. II with a formulation of the interaction of a (possibly) degenerate atomic gas with light. The emphasis is on the observation that one may under conceivably fairly common circumstances ignore collisions and center-of-mass (c.m.) motion affecting the excited atoms. Coupled evolution equations for the matter and light fields, each quite transparent, constitute the main result of our general theory.

Light scattering is treated in Sec. III. The main item is the assumption of sufficiently large detuning of the laser from atomic resonance. What should be meant by "large" is studied at length. A general expression for the spectrum of the scattered light is derived, complete with the dependence on atomic level scheme and on various light polarizations relevant in an experiment. Next, in Sec. IV we present two examples for an ideal (noninteracting) gas, one for bosons and one for fermions. In a degenerate gas, the statistics may cause dramatic qualitative changes of the spectrum, which are highlighted.

A few concluding remarks are made in Sec. V. In particular, we discuss the relation of our theory to existing neutron scattering methods, and ponder on the future paths opened up by our research.

II. GENERAL THEORY

A. Hamiltonian

We first generalize and extend our previous models [3,7] of many-atom systems interacting with light. The present discussion also draws from the detailed formulation of quantum field theory for atoms interacting with photons, as given by Lewenstein *et al.* [9].

In order to have near-resonant optical transitions, the atom has to have (at least) two internal energy levels. Since our aim is to study light scattering, including angular distribution and polarization, we also have to account fully for the angular-momentum degeneracy of the energy levels. Our complete set of internal-state kets therefore reads $\{|gm\rangle,|em'\rangle\}_{m,m'}$. Here g stands for the "ground" and e for the "excited" level, the corresponding angular momenta being j_g and j_e, and m, m' denote the z components of angular momentum. We use α as the generic energy level label, so that $\alpha = g$ or e. The frequency of the optical transition between the levels g and e is denoted by ω_0, and $\mathbf{d}$ stands for the dipole moment operator of the transition.

The atoms are assumed to move in an arbitrary conservative potential $V(\mathbf{r})$. The c.m. Hamiltonian for each atom reads, in the position representation,

$$H_{\text{c.m.}} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}), \tag{1}$$

where m is the mass. In principle, the c.m. Hamiltonian may be different for different internal states of the atom, but such cases will not be considered here. At any rate, we are going to ignore the c.m. motion of the excited atoms.

In order to manage the many-body aspects of the theory, we introduce the conventional field operators for each internal state, $\psi_{\alpha m}(\mathbf{r})$. It is occasionally convenient to utilize the relationship between the field operators and atom creation and annihilation operators. Thus, given an arbitrary orthonormal basis of states in three-dimensional space, $\{u_k(\mathbf{r})\}_k$, and the annihilation operators for atoms with energy level α, c.m. state k, and magnetic quantum number m, $b_{\alpha k m}$, we have

$$\psi_{\alpha m}(\mathbf{r}) = \sum_k u_k(\mathbf{r}) b_{\alpha k m}, \quad b_{\alpha k m} = \int d^3r\, u_k^*(\mathbf{r})\psi_{\alpha m}(\mathbf{r}). \tag{2}$$

Until otherwise noted, the results of our general development do not depend explicitly on the statistics of the atoms. However, anyone wishing to specialize to bosons or fermions may do so at any stage by introducing the appropriate commutator or anticommutator relations. These may be formulated equivalently either for the atomic annihilation and creation operators or for the fields,

$$[b_{\alpha k m}, b_{\alpha' k' m'}]_\pm = [b^\dagger_{\alpha k m}, b^\dagger_{\alpha' k' m'}]_\pm = 0,$$
$$[b_{\alpha k m}, b^\dagger_{\alpha' k' m'}]_\pm = \delta_{\alpha k m, \alpha' k' m'}, \tag{3}$$

or

$$[\psi_{\alpha m}(\mathbf{r}), \psi_{\alpha' m'}(\mathbf{r}')]_\pm = [\psi^\dagger_{\alpha m}(\mathbf{r}), \psi^\dagger_{\alpha' m'}(\mathbf{r}')]_\pm = 0,$$
$$[\psi_{\alpha m}(\mathbf{r}), \psi^\dagger_{\alpha' m'}(\mathbf{r}')]_\pm = \delta_{\alpha m, \alpha' m'}\delta(\mathbf{r}-\mathbf{r}'). \tag{4}$$

The subscript $+$ denotes the anticommutator, as appropriate for fermions; the subscript $-$, the default, stands for the commutator, and applies to bosons.

Angular momenta of the energy levels in a given atomic species are all integers, or all half-integers. No contradiction with the familiar spin-statistics theorem arises if one attributes either the boson or the fermion character to the entire atom. Optical processes in which a Bose atom turns into a Fermi atom or vise versa, such as ionization, will not be discussed here.

While dealing with light, we have found it convenient to retain the plane wave representation for photons. The mode index q incorporates both the wave vector $\mathbf{q}$ and the transverse polarization $\hat{\mathbf{e}}_q$, the mode frequencies are denoted by ω_q, and the photon annihilation operator is a_q. We write the electric field operator as the sum of positive and negative frequency components $\mathbf{E}^+(\mathbf{r})$ and $\mathbf{E}^-(\mathbf{r}) = [\mathbf{E}^+(\mathbf{r})]^\dagger$ as $\mathbf{E}(\mathbf{r}) = \mathbf{E}^+(\mathbf{r}) + \mathbf{E}^-(\mathbf{r})$, with

$$\mathbf{E}^+(\mathbf{r}) = \sum_q \xi_q \hat{\mathbf{e}}_q a_q e^{i\mathbf{q}\cdot\mathbf{r}}, \quad \xi_q = \sqrt{\frac{\hbar\omega_q}{2\epsilon_0 V}}. \tag{5}$$

Here V is the quantization volume. Without restricting the generality, we assume that all vectors $\hat{\mathbf{e}}_q$ are real.

The Hamiltonian for the atom-field system finally emerges as the integral over the quantization volume of the Hamiltonian density $\mathscr{H}(\mathbf{r})$:

$$H = \int d^3r\, \mathscr{H}(\mathbf{r}), \tag{6}$$

$$\mathscr{H} = \sum_m \psi^\dagger_{gm} H_{\text{c.m.}} \psi_{gm} + \sum_{m'} \psi^\dagger_{em'}(H_{\text{c.m.}} + \hbar\omega_0)\psi_{em'} - \sum_{mm'} (\langle gm|\mathbf{d}\cdot\mathbf{E}|em'\rangle \psi^\dagger_{gm}\psi_{em'} + \langle em'|\mathbf{d}\cdot\mathbf{E}|gm\rangle \psi^\dagger_{em'}\psi_{gm}) + \mathscr{H}_F + \mathscr{H}_{gg} + \mathscr{H}_{ee} + \mathscr{H}_{eg}. \tag{7}$$

While writing Eq. (7), we have suppressed the dependence on the position $\mathbf{r}$ in our notation. The first two terms reflect the energies, internal and center-of-mass, of the atoms in the absence of electromagnetic fields. The c.m. Hamiltonian $H_{\text{c.m.}}$ as in Eq. (1) acts on the position argument of the field to the right of it. The following two terms are for the atom-light dipole interaction. It should be noted that the domain of an index specifying the z component of the angular momentum (here m and m') may be different depending on whether the index refers to the ground state or the excited state. Next, $\mathscr{H}_F$ is the Hamiltonian density for the free electromagnetic field. We naturally have the corresponding Hamiltonian

$$H_F = \int d^3r\, \mathscr{H}_F(\mathbf{r}) = \hbar\sum_q \omega_q a_q^\dagger a_q. \tag{8}$$

Finally, $\mathscr{H}_{gg}$, $\mathscr{H}_{ee}$, and $\mathscr{H}_{eg}$ govern those interactions of an atom with the electromagnetic field and other atoms that cannot be accounted for when the atoms are modeled as point dipoles, e.g., effects of multipoles other than electric dipole and electron exchange. These terms stand, respectively, for the interactions of ground state atoms, interactions of excited atoms, and interactions that involve ground state atoms and excited atoms. Some of the prominent atom-atom interactions, such as the van der Waals force, are already encompassed by the dipole coupling. Nonetheless, as long as care is taken to avoid double counting, van der Waals interactions may be viewed phenomenologically as part of $\mathscr{H}_{gg}$. In this paper, $\mathscr{H}_{gg}$ stands for any and all collisional interactions of ground state atoms.

B. Quantum optics considerations

The Hamiltonian (6) covers a wide range of phenomena including collisions between the atoms, formation and properties of an interacting Bose condensate, and interactions of the gas with light. In this paper the focus is on optical physics, so we will have little to say about the collision terms $\mathscr{H}_{gg}$, $\mathscr{H}_{ee}$, and $\mathscr{H}_{eg}$. It should be noted [9–11], though, that resonant dipole-dipole interactions, often the dominant

mechanism of collisions between a ground state atom and an excited atom, are accounted for by the dipole interaction terms shown explicitly in Eq. (7).

Our studies of the optical interactions of the condensate are rooted in the established theory of quantum optics, as summarized for instance in Ref. [12]. First and foremost, we always assume that there is a dominant frequency in the light field Ω, which is nearly resonant with the atomic transition frequency ω_0. The corresponding characteristic wave number k and wavelength λ are defined as $k=\Omega/c\simeq\omega_0/c$ and $\lambda=2\pi/k$.

Along the way we will make the rotating-wave approximation (RWA), in fact in two different versions. We prepare for the RWA by writing the dipole operator as the sum of raising and lowering parts $\mathbf{d}^+$ and $\mathbf{d}^- = [\mathbf{d}^+]^\dagger$ as $\mathbf{d}=\mathbf{d}^+ + \mathbf{d}^-$, with

$$\mathbf{d}^+ = \sum_{mm'} |em'\rangle \mathbf{d}^+_{m'm}\langle gm|,\ \mathbf{d}^+_{m'm}=\langle em'|\mathbf{d}|gm\rangle. \quad (9)$$

The goal is to find equations of motion for the tilded operators

$$\tilde{\psi}_{em'}=e^{i\Omega t}\psi_{em'},\quad \tilde{\mathbf{E}}^+=e^{i\Omega t}\mathbf{E}^+,\ \ldots, \quad (10)$$

that vary "slowly" in the Heisenberg picture.

In quantum optics, radiative damping of excited atoms and the associated Lamb shifts are important considerations. The well known expression for the linewidth of a transition with the dipole moment matrix element $\mathscr{D}$ is

$$\gamma=\frac{\mathscr{D}^2k^3}{6\pi\hbar\epsilon_0}. \quad (11)$$

On the other hand, studies of a Bose condensate with $N\gg 1$ atoms in a region of the size $\ell\gg\lambda$ suggest [3,4] that, instead of the one-atom linewidth γ, a collective linewidth estimated as

$$\Gamma=\frac{3}{2}N\left(\frac{\lambda}{2\pi\ell}\right)^2\gamma \quad (12)$$

applies. In this paper we circumvent the QED issues of linewidths and line shifts to a large extent, but we will generally use Eq. (12) when a linewidth is needed for an excited atom.

C. Evolution of matter fields

We have encountered no problems when applying the RWA to the equations of motion of the matter fields. For the time being, we therefore implement the RWA by retaining in the dipole interaction only those terms in which a photon is absorbed and an atom is excited, and the reverse processes. Terms with a simultaneous emission of a photon and excitation of an atom, and the reverses of such processes, are ignored.

The RWA dipole interaction is the integral of the density

$$\mathscr{H}_D^{\mathrm{RWA}}=-\sum_{mm'}(\mathbf{d}^-_{mm'}\cdot\mathbf{E}^-\psi^\dagger_{gm}\psi_{em'}+\mathbf{d}^+_{m'm}\cdot\mathbf{E}^+\psi^\dagger_{em'}\psi_{gm}). \quad (13)$$

It is easy to integrate the Hamiltonian density to obtain the Hamiltonian, then use the quantization rules (4) to find the Heisenberg equations of motion for the matter fields. For instance, we have

$$\dot{\psi}_{em'}=-i\left(\omega_0+\frac{H_{\mathrm{c.m.}}}{\hbar}\right)\psi_{em'}+\frac{i}{\hbar}\sum_m \mathbf{d}^+_{m'm}\cdot\mathbf{E}^+\psi_{gm} + \left.\frac{d}{dt}\right|_C\psi_{em'}. \quad (14)$$

Here the last term stands for collisions, whatever follows from $\mathscr{H}_{ee}$ and $\mathscr{H}_{eg}$.

We make the following approximations in the equations of motion of the excited state fields $\psi_{em'}$:

(i) We ignore the collision terms.

(ii) We ignore the c.m. Hamiltonian.

We have in mind situations in which the density of the excited atoms is low, so they do not collide among themselves. Also, resonant dipole-dipole interactions are already incorporated in the dipole interaction with the field.

To justify omitting the c.m. motion, we first note that if an excited atom only moves a small fraction of the wavelength during its spontaneous lifetime, the motion has little effect on the optical response. Thermal motion in a homogeneously broadened gas, the usual outcome of modern atom cooling methods, satisfies this condition even with respect to the one-atom linewidth (11), let alone the larger collective linewidth (12).

The second obvious condition is that the resonant dipole-dipole interactions should not accelerate an excited atom over a distance larger than about $\lambda/2\pi$ during the lifetime [13]. Let us consider two aligned dipoles with the dipole moment $\mathscr{D}$ at the distance r from one another and denote the direction between the dipoles by $\hat{\mathbf{n}}$; then the interaction energy is

$$E_D=\frac{3(\hat{\mathbf{n}}\cdot\mathscr{D})^2-\mathscr{D}^2}{4\pi\epsilon_0 r^3}. \quad (15)$$

The maximum force between two such dipoles is $F=3\mathscr{D}^2/2\pi\epsilon_0 r^4$. It is instructive to compare the conditions that the displacement due to this maximum force be less than $\lambda/2\pi$ using both the one-atom lifetime γ^{-1} and the collective lifetime Γ^{-1}. Except for numerical factors close to 1, we find

$$r>\frac{\lambda}{2\pi}\left(\frac{\epsilon_R}{\gamma}\right)^{1/4},\quad r>\frac{\ell}{\sqrt{N}}\left(\frac{\epsilon_R}{\gamma}\right)^{1/4}, \quad (16)$$

respectively, where

$$\epsilon_R=\frac{\hbar k^2}{2m} \tag{17}$$

is the frequency associated with photon recoil. For typical dipole-allowed optical transitions $\epsilon_R\sim 10^{-2}\gamma$, so the first condition from (16) permits about $(2\pi)^3$ atoms per cubic wavelength. The second condition allows for higher densities. Perhaps paradoxically, the second inequality is satisfied with an increasing margin when the density is increased for a fixed-size sample.

We proceed with the stated approximations, and introduce the slowly varying quantities as in Eq. (10). The equations of motion for the matter fields are

$$\dot{\tilde{\psi}}_{em'}=i\delta\tilde{\psi}_{em'}+\frac{i}{\hbar}\sum_m \mathbf{d}^+_{m'm}\cdot\mathbf{E}^+\psi_{gm}, \tag{18a}$$

$$\dot{\psi}_{gm}=\frac{i}{\hbar}\sum_{m'} \mathbf{E}^-\cdot\mathbf{d}^-_{mm'}\tilde{\psi}_{em'}-i\frac{H_{\text{c.m.}}}{\hbar}\psi_{gm}+\frac{d}{dt}\bigg|_C\psi_{gm}. \tag{18b}$$

In Eq. (18a),

$$\delta=\Omega-\omega_0 \tag{19}$$

is the detuning of the light frequency from the atomic resonance. While only the detuning needs to be kept in the equations for the excited state fields, there is no *a priori* justification for dropping the c.m. Hamiltonian or $\mathscr{H}_{gg}$ collisions from the equations of motion of the ground state fields.

In sum, we have analyzed the evolution of matter fields under the assumptions that the density of the excited atoms is low, and that the c.m. motion of an excited atom may be ignored during the spontaneous lifetime. The familiar structure of the equations of motion for the probability amplitudes in the standard two-state approach to quantum optics comes through in the results, Eqs. (18). For the ground state atomic fields, however, we have complications: both the c.m. motion and the collisions are included.

D. Evolution of the light field

During our analysis we have noticed that when the electromagnetic fields are considered, the straightforward RWA leads to mathematical and physical problems. In this section we therefore address the evolution of the electromagnetic fields starting from the full Hamiltonian (6). For future reference we document the analysis in somewhat more detail than is necessary for the immediate purposes of the present paper.

From Eqs. (5)–(8) one easily finds the Heisenberg equations of motion for the annihilation operators of the electromagnetic fields,

$$\dot{a}_q(t)=-i\omega_q a_q(t)+\frac{i\xi_q}{\hbar}\hat{\mathbf{e}}_q\cdot\sum_{mm'}\int d^3\mathbf{r}'e^{-i\mathbf{q}\cdot\mathbf{r}'}[\mathbf{d}^-_{mm'}\psi^\dagger_{gm}(\mathbf{r}')\psi_{em'}(\mathbf{r}')+\mathbf{d}^+_{m'm}\psi^\dagger_{em'}(\mathbf{r}')\psi_{gm}(\mathbf{r}')]. \tag{20}$$

It is now straightforward to integrate (20) and its Hermitian conjugate, and insert the results into the definition of the field, c.f. (5). This gives

$$\mathbf{E}(\mathbf{r}t)=\sum_q \xi_q\hat{\mathbf{e}}_q e^{i(\mathbf{q}\cdot\mathbf{r}-\omega_q t)}a_q+\sum_{qmm'}\hat{\mathbf{e}}_q\frac{i\xi_q^2\hat{\mathbf{e}}_q\cdot\mathbf{d}^-_{mm'}}{\hbar}\times\int d^3r'\int_{-\infty}^{t}dt'e^{-\eta(t-t')}\{e^{i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}-e^{-i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}\}\psi^\dagger_{gm}(\mathbf{r}'t')\psi_{em'}(\mathbf{r}'t')+\text{H.c.}, \tag{21}$$

where we have introduced a convergence factor η such that $\eta\to 0+$ at the end of the calculations. The exponential preceded by the minus sign would have been absent within the RWA.

In order to simplify Eq. (21), we study the subexpression

$$S=\sum_q\hat{\mathbf{e}}_q\frac{i\xi_q^2\hat{\mathbf{e}}_q\cdot\mathbf{d}^-_{mm'}}{\hbar}\{e^{i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}-e^{-i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}\}. \tag{22}$$

The standard tricks for doing the implicit sum over the polarization vectors $\hat{\mathbf{e}}_q$ and converting the sum over the wave vectors $\mathbf{q}$ into an integral lead in to the following manipulations:

$$\begin{aligned}S&=\frac{-ic}{16\pi^3\epsilon_0}\int d^3q\,q\,\frac{\mathbf{q}}{q}\times\left(\frac{\mathbf{q}}{q}\times\mathbf{d}^-_{mm'}\right)\{e^{i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}-e^{-i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}\}\\&=\frac{ic}{16\pi^3\epsilon_0}[(\mathbf{d}^-_{mm'}\times\nabla)\times\nabla]\int d^3q\,\frac{1}{q}\{e^{i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}-e^{-i[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')-\omega_q(t-t')]}\}\\&=\frac{c}{4\pi\epsilon_0}[(\mathbf{d}^-_{mm'}\times\nabla)\times\nabla]\frac{\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))-\delta(|\mathbf{r}-\mathbf{r}'|+c(t-t'))}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned} \tag{23}$$

The value of *not* having made the RWA emerges from Eq. (23). Had we implemented the RWA, we would have to contend with additional principal value terms. Carrying out the indicated gradients would lead, e.g., to a term of the form

$$\delta(\mathbf{r}-\mathbf{r}')\,\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t')).$$

When this is inserted into (21), integral over t' gives a result proportional to $\delta(\mathbf{r}-\mathbf{r}')$. Further integration over $\mathbf{r}'$ produces a contact term, which entails that the electric field at $\mathbf{r}$ has a contribution proportional to the matter fields at the same point $\mathbf{r}$. Now, for the correct form of (23), such a contact term precisely cancels. All told, because we did not make the run, we arrive at a simple and transparent result. From now on we write

$$S=\frac{c}{4\pi\epsilon_0}(\mathbf{d}^-_{mm'}\times\nabla)\times\nabla\frac{\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))}{|\mathbf{r}-\mathbf{r}'|}, \quad (24)$$

with the tacit understanding that the δ function singularity resulting from the second derivatives of $1/|\mathbf{r}-\mathbf{r}'|$ is omitted.

Let us now resume the development of Eq. (21). All terms that have been written explicitly on the right-hand side have the dominant time dependence $e^{-i\Omega t}$. We thus *define* the displayed terms as making the positive-frequency part of the electromagnetic field. The notation of Eqs. (10) and (24) then give

$$\tilde{\mathbf{E}}^+(\mathbf{r}t)=\tilde{\mathbf{E}}^+_F(\mathbf{r}t)+\frac{1}{4\pi\epsilon_0}\sum_{mm'}(\mathbf{d}^-_{mm'}\times\nabla)\times\nabla\int d^3r'\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,\psi^\dagger_{gm}(\mathbf{r}'t_d)\,\tilde{\psi}_{em'}(\mathbf{r}'t_d), \quad (25)$$

where

$$\tilde{\mathbf{E}}^+_F(\mathbf{r}t)=\sum_q \xi_q\hat{\mathbf{e}}_q e^{i[\mathbf{q}\cdot\mathbf{r}-(\omega_q-\Omega)t]}a_q \quad (26)$$

is the free field that would prevail in the absence of matter, and

$$t_d=t-\frac{|\mathbf{r}-\mathbf{r}'|}{c} \quad (27)$$

is the retarded time.

In all physical situations, the sample of matter interacting with light has a finite size. We again denote the characteristic linear dimension of the sample by ℓ. On the other hand, the slowly varying matter fields have a characteristic evolution time scale Δt. Henceforth we assume that

$$\ell\ll c\Delta t. \quad (28)$$

Electromagnetic signals then travel across the sample in a time much shorter than it takes the slowly varying properties of the sample to change. We may therefore take the slowly varying fields inside the integral at the same time, say $t_d=t-r/c$. Next, we ignore the trivial overall propagation delay from the source to the field point, r/c , and simply write t in lieu of t_d on the right-hand side of Eq. (25). Finally we carry out the indicated derivatives, keeping in mind that the ensuing δ function divergence is to be ignored.

A bit of straightforward algebra gives

$$\tilde{\mathbf{E}}^+(\mathbf{r})=\tilde{\mathbf{E}}^+_F(\mathbf{r})+\sum_{mm'}\int d^3r'\,\mathbf{K}(\mathbf{d}^-_{mm'};\mathbf{r},\mathbf{r}')\,\psi^\dagger_{gm}(\mathbf{r}')\,\tilde{\psi}_{em'}(\mathbf{r}'). \quad (29)$$

Since our assumptions have rendered the result instantaneous in time, we have omitted the common time argument t everywhere in (29). The kernel $\mathbf{K}(\mathbf{D};\mathbf{r},\mathbf{r}')$ is nothing but the familiar expression [14] of the positive-frequency component of the electric field from a monochromatic dipole with the complex amplitude $\mathbf{D}$, given that the dipole resides at $\mathbf{r}'$ and the field is observed at $\mathbf{r}$:

$$\mathbf{K}(\mathbf{D};\mathbf{r},\mathbf{r}')=\frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{D})\times\hat{\mathbf{n}}\,\frac{e^{ikR}}{R}+[3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{D})-\mathbf{D}]\left(\frac{1}{R^3}-\frac{ik}{R^2}\right)e^{ikR}\right\}, \quad (30)$$

with

$$\mathbf{R}=\mathbf{r}-\mathbf{r}', \quad R=|\mathbf{R}|, \quad \hat{\mathbf{n}}=\frac{\mathbf{R}}{R}. \quad (31)$$

In particular, R is the distance between the source point and the field point, and $\hat{\mathbf{n}}$ is a unit vector pointing from the source toward the field point. As before, we write $k=\Omega/c$.

All told, we have arrived at the integral equation (29) for light propagation in a sample made of a (possibly degenerate) atomic gas. The main assumption (28) essentially states that phase coherence of light prevails over the entire sample. We could have done perfectly well without this assumption, but since we anticipate that it is usually valid and since it simplifies the notation, we keep it. The final result is intuitively obvious. We could have obtained it simply by plugging the dipole moment density (polarization) operator

$$\mathbf{P}(\mathbf{r}t)=\sum_{mm'}\mathbf{d}^-_{mm'}\psi^\dagger_{gm}(\mathbf{r}t)\,\psi_{em'}(\mathbf{r}t)+\text{H.c.} \quad (32)$$

into classical Maxwell's equations.

E. Concluding remark on theory

While the evolution of the matter fields and the light field is governed by preeminently physical relations (18) and (29), the problem remains that these are coupled equations: matter affects light, and light affects matter. The entire coupled Maxwell-Bloch equations, some c.m. motion, and a good deal of collision physics are still embodied in our innocuous-looking formalism, not to mention QED line shifts and linewidths. Further progress is contingent upon finding further approximations that break the theory into manageable pieces.

III. LIGHT SCATTERED FROM A DEGENERATE GAS

A. Assumption of large detuning

In the present paper we decouple atomic and electromagnetic fields with the assumption that the atom-field detuning is large. While considering what "large" should be, we have to discuss a number of physical features of a degenerate gas at least qualitatively.

1. QED effects

An excited atom radiates a field that falls both on the emitter itself, and on the other atoms. The self-field of the emitter causes radiative damping and Lamb shifts; neighboring atoms are subject to super-radiant decay and resonant dipole-dipole interactions.

As already noted, a plausible estimate exists [3,4] for the collective linewidth of a Bose condensate, Eq. (12). It may be shown easily that if the spatial density profiles were the same, the collective linewidth of a homogeneously broadened gas of classical Maxwell-Boltzmann atoms would be just the same as the linewidth of a Bose condensate [15]. In the absence of any better alternatives, we surmise that the same applies to fermions, with two caveats. First, although this is not an issue for a low-density gas, even a low-temperature Fermi gas need not be homogeneously broadened. Second, given the c.m. potential $V(\mathbf{r})$, temperature T, and particle number N, the spatial distribution of the atoms and the size of the sample ℓ may be quite different for bosons and fermions.

The line shift due to neighboring atoms qualitatively equals the potential energy of the dipole-dipole interactions. Here we offer two estimates.

We have already written down the near-field dipole-dipole interaction energy (15), which could also be derived easily from the field expression (30). Consider the potential of a continuous spatial distribution of aligned dipoles on a similarly aligned test dipole located at the origin, assuming the number density of the dipoles $n(\mathbf{r})$. The average over $\hat{\mathbf{n}}$ in (15) shows that the potential in fact vanishes for an arbitrary spherically symmetric spatial distribution, including a distribution with constant density.

Two ways to break the zero are evident, leading to our two estimates. First, one might argue that the nonzero dipole-dipole interaction energy depends in an essential way on the discreteness of the dipoles. To the order of magnitude the interaction energy of the test dipole is then estimated from (15), where the nearest-dipole distance r related to the particle density n as $r\sim n^{-1/3}$ is to be used. This gives

$$E_D^{(1)}\sim \frac{n\mathscr{D}^2}{4\pi\epsilon_0}. \tag{33}$$

Alternatively, the dipole-dipole interactions retain the $1/r^3$ form up to a distance of the order λ before retardation sets in. We therefore only consider the dipoles in a sphere of radius λ around the test dipole, and expand the density $n(\mathbf{r})$ as a function of position around the test dipole. It transpires that even a density gradient does not lead to a nonzero averaged interaction energy, but one has to go to second-order variations of density. Let us call the common direction of the dipoles z and write a model with $n(\mathbf{r})=n[1+(z/\ell)^2]$, ℓ again being the characteristic size of the sample. Integral over the dipoles in a sphere of radius λ gives

$$E_D^{(2)}\sim\left(\frac{\lambda}{\ell}\right)^2\frac{n\mathscr{D}^2}{4\pi\epsilon_0}. \tag{34}$$

Recalling Eqs. (11) and (12), and once more dropping dimensionless constants of the order unity, we have the comparisons between the dipole shifts and the collective decay rate

$$\frac{E_D^{(1)}}{\hbar\Gamma}\sim\frac{\lambda}{\ell},\quad \frac{E_D^{(2)}}{\hbar\Gamma}\sim\left(\frac{\lambda}{\ell}\right)^3. \tag{35}$$

If either one of our arguments is correct, for samples much larger than a wavelength the dipole shifts are negligible in comparison with the collective linewidth.

2. Optical thickness

For increasing detunings, the effective atom-field coupling gets progressively weaker. This means that multiple scattering of light in the sample is increasingly improbable. Rudimentary estimates with the Bose condensate suggest that the optical thickness at line center is of the order of unity [3]. To have a genuinely optically thin sample, one has to detune somewhat beyond the collective linewidth Γ.

3. Saturation

One of the key assumptions of our theory is that the excited state fraction of the atoms is small. Existing estimates on this count again help [3]. For the present purposes these may, perhaps, most profitably be recited as follows. Call the saturation intensity of one atom I_s [12], then for a given light intensity the fraction of saturated atoms at exact atom-field resonance is of the order

$$p_e\sim\frac{I}{I_s}\frac{1}{N}\left(\frac{\ell}{\lambda}\right)^2. \tag{36}$$

For ordinary spectroscopic continuous-wave laser intensities, $I\sim I_s$, saturation of the sample is expected to be insignificant, whether the laser is on or off the atomic resonance.

4. Closing remarks about large detuning

For a sample substantially larger than a wavelength, $\ell\gg\lambda$, we may ignore QED linewidths and lineshifts as well as multiple scattering of light whenever the detuning of the laser from the atomic resonance is at least comparable to the collective linewidth Γ:

$$|\delta|\geq\Gamma. \tag{37}$$

Under such conditions saturation of the sample is typically insignificant at ordinary spectroscopic laser intensities, and even far above.

Of course, to reach our conclusions we have repeatedly used semi-quantitative estimates made especially for the Bose condensate. A purist's viewpoint should be that while the neglect of QED effects, etc., clearly are valid at large enough detunings, the condition (37) applies specifically to a

1Bose condensate. However, in view of the equality of the collective linewidth for the Bose condensate and the classical gas mentioned above, we are not overly concerned about this issue.

B. Scattering at large detunings

It is an easy task to analyze the scattering of light in the limit of large detunings, as specified in (37). For concreteness, we assume an incoming classical plane wave with the frequency Ω, wave vector $\boldsymbol{\kappa}$, complex amplitude $\mathscr{E}$, and complex polarization $\hat{\mathbf{e}}_i$, so that

$$\tilde{\mathbf{E}}_F^+(\mathbf{r}) = \tfrac{1}{2}\mathscr{E}\hat{\mathbf{e}}_i e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}. \tag{38}$$

For large detunings multiple scattering of light and QED fields are negligible; in other words, the incident field dominates inside the sample. Also, because the input is assumed monochromatic (the same argument applies if the bandwidth of the input is much smaller than $|\delta|$), we may obtain the steady state excited atom fields by simply solving Eq. (18a) adiabatically. We have

$$\tilde{\psi}_{em'}(\mathbf{r}t) = -\sum_m \frac{\mathscr{E}\hat{\mathbf{e}}_i\cdot\mathbf{d}^+_{m'm}}{2\hbar\delta} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\psi_{gm}(\mathbf{r}t). \tag{39}$$

When discussing scattering, we only retain the $1/R$ far field of the dipole radiation in Eq. (30). Besides, we make the standard $1/r$ expansion,

$$R = |\mathbf{r}-\mathbf{r}'| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}', \tag{40}$$

where r is the distance from the detector to a representative point of origin in the sample, and everywhere except in explicit phase factors retain only the leading r term. With these provisos, we obtain the first Born approximation for the scattered field from (29),

$$\mathbf{E}_S^+(\mathbf{r}t) = \frac{\mathscr{E}k^2 e^{ikr}}{8\pi\epsilon_0\hbar\delta r}\sum_{mm'm''}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{d}^-_{mm'})\hat{\mathbf{e}}_i\cdot\mathbf{d}^+_{m'm''} \times\int d^3r' e^{-i\Delta\boldsymbol{\kappa}\cdot\mathbf{r}'}\psi^\dagger_{gm}(\mathbf{r}'t)\psi_{gm''}(\mathbf{r}'t). \tag{41}$$

Here

$$\Delta\boldsymbol{\kappa} = k\hat{\mathbf{n}} - \boldsymbol{\kappa} \tag{42}$$

is the change of the wave vector of light upon scattering.

The scattered field may be analyzed in many ways. One may use lenses, polarizers, and other optical elements to transform the field. Subsequently, one may set up photoelectric devices to measure either the spectral properties or the photon statistics of the light. Henceforth we concentrate on the spectrum of light. We assume that enough time has elapsed since the turn-on of the light to make the scattered light field stationary. For completeness, we also allow for the possibility that the measurement of the spectrum is preceded by a selection of the polarization. The spectrum may then be calculated [12] from the Fourier transformation of a two-time field correlation function as follows:

$$\mathscr{S}(\omega) = \hat{\mathbf{e}}\cdot\mathsf{S}(\omega)\cdot\hat{\mathbf{e}}^*, \tag{43a}$$

$$\mathsf{S}_{ij}(\omega) = K\int dt\, e^{i\omega t}\langle\mathbf{E}_i^-(0)\mathbf{E}_j^+(t)\rangle. \tag{43b}$$

Here $\hat{\mathbf{e}}$ is a complex unit vector characterizing the polarization accepted by the detector, the indices i and j stand for the (say, Cartesian) components of the tensor S, and K is a normalization factor. The brackets denote the expectation value over the quantum state of the field.

It now proves expedient to discuss the dipole moment matrix elements a little further. Alongside with the Cartesian unit vectors $\hat{\mathbf{e}}_{1,2,3}$, we adopt the standard spherical unit vectors

$$\hat{\mathbf{e}}_{-1} = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2), \quad \hat{\mathbf{e}}_0 = \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_{+1} = -\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2). \tag{44}$$

The matrix elements may be written

$$\mathbf{d}^+_{m'm} = \mathscr{D}\sum_{\sigma=-1,0,+1}\langle m'|\sigma m\rangle\hat{\mathbf{e}}^*_\sigma. \tag{45}$$

We have introduced

$$\langle m'|\sigma m\rangle \equiv \langle m\sigma|m'\rangle \equiv \langle j_g 1 j_e m'|j_g m 1\sigma\rangle \tag{46}$$

as shorthands for the appropriate Clebsch-Gordan coefficients, and $\mathscr{D}$ is the reduced dipole moment matrix element for the transition $g\to e$; the matrix element that would apply for a transition with unit Clebsch-Gordan coefficient. The reason for the complex conjugate in Eq. (45) is that we want a light field with the polarization $\hat{\mathbf{e}}_{+1}$ to drive a transition with $m'-m=+1$.

When analyzing the scattered radiation, one usually does not stick the detector directly in the incident field. Correspondingly, one only considers the scattered fields in Eq. (43b). Some tedious but straightforward algebra remains to sort out the inter-relations between the internal states of the atom, the incident and the exit polarizations, and the observation direction. Using Eqs. (41) and (45), we obtain the tensor S in the form

$$\mathsf{S}(\mathbf{r}\omega) = \frac{1}{2\pi}I(r)\frac{\mathscr{R}^2}{\delta^2}\sum_{\substack{m_1m_2\\m_3m_4}}\mathsf{M}^{m_4m_3}_{m_2m_1} \times\int dt\int d^3r_1 d^3r_2\, e^{i\omega t + i\Delta\boldsymbol{\kappa}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \times\langle\psi^\dagger_{gm_4}(\mathbf{r}_1 0)\psi_{gm_3}(\mathbf{r}_1 0)\psi^\dagger_{gm_2}(\mathbf{r}_2 t)\psi_{gm_1}(\mathbf{r}_2 t)\rangle. \tag{47}$$

The numerical factor $1/2\pi$ reflects our normalization; we have chosen the constant K in (43b) in such a way that the integral of the spectrum over all frequencies and polarizations gives the total radiation intensity. Next,

$$I(r)=\frac{\mathscr{D}^2\Omega^4}{32\pi^2\epsilon_0 c^3 r^2} \tag{48}$$

is the intensity scale, equal to the radiation intensity from a linearly polarized dipole of magnitude $\mathscr{D}$ at the distance r in the plane that contains the dipole and is oriented perpendicular to the dipole. We have defined the Rabi frequency as

$$\mathscr{R}=\frac{\mathscr{D}\mathscr{E}}{2\hbar}, \tag{49}$$

and the factor $\mathscr{R}^2/\delta^2$ is basically the excitation probability for each atom. Finally, the tensors M,

$$\mathsf{M}^{m_4m_3}_{m_2m_1}(\hat{\mathbf{e}}_i,\hat{\mathbf{n}})=\sum_{\substack{M_1M_2\\ \sigma_1\sigma_2\\ \sigma_3\sigma_4}}(\hat{\mathbf{e}}_i^*\cdot\hat{\mathbf{e}}_{\sigma_4})\langle m_4\sigma_4|M_2\rangle\langle M_2|\sigma_3 m_3\rangle \times[\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\hat{\mathbf{e}}^*_{\sigma_3})][(\hat{\mathbf{e}}_{\sigma_2}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}] \times\langle m_2\sigma_2|M_1\rangle\langle M_1|\sigma_1 m_1\rangle(\hat{\mathbf{e}}^*_{\sigma_1}\cdot\hat{\mathbf{e}}_i), \tag{50}$$

encompass the dependencies on the level structure of the atom and on various directions of the experiment.

An extrapolation of the present experiments suggests that the first experiments on degenerate gases will be carried out with fairly small samples. However, these samples are still expected to be much larger than the wavelength of light. In our further development we therefore assume that the gas is effectively translationally invariant. The correlation functions such as those in (47) are then functions of $\mathbf{r}_1-\mathbf{r}_2$ only. One of the position integrals can be done and simply produces the volume of the sample. In what follows we always write the position integrals in Eq. (47) as

$$\int d^3r_1 d^3r_2\, e^{i\Delta\boldsymbol{\kappa}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\langle\psi^\dagger_{gm_4}(\mathbf{r}_1 0)\psi_{gm_3}(\mathbf{r}_1 0)\psi^\dagger_{gm_2}(\mathbf{r}_2 t)\psi_{gm_1}(\mathbf{r}_2 t)\rangle \to V\int d^3r\, e^{-i\Delta\boldsymbol{\kappa}\cdot\mathbf{r}}\langle\psi^\dagger_{gm_4}(00)\psi_{gm_3}(00)\psi^\dagger_{gm_2}(\mathbf{r}t)\psi_{gm_1}(\mathbf{r}t)\rangle. \tag{51}$$

We conclude with a comment on operator orderings, for brevity only for bosons. In Eq. (29), the fields $\psi^\dagger_{gm}$ and ψ_{em} commute, and may be put in any order. However, had we reversed the order from the one shown in (29), the noncommuting annihilation and creation fields in Eq. (47) would have got swapped. This could lead to quite dramatic changes in the final results.

The root of the dilemma lies in the adiabatic approximation of Eq. (39). As it often happens with approximations carried out with Heisenberg picture operators, this one does not preserve the commutators either. One way of putting the situation physically is that the adiabatic assumption is good for the components of the operators that evolve at frequencies close to Ω, but all frequency components are needed for the correct commutators. A possible way to rectify the problem is to introduce quantum noise operators to uphold the commutators, c.f. [16]. However, in view of the upcoming preeminently physical results of our formalism, we conclude that we have chosen the right operator ordering in Eq. (47). We have brought up the issue as a precaution, as it may surface in more pernicious ways in other settings.

IV. EXAMPLES

In this section we work out two examples of light scattering, one for the $j_g=0\to j_e=1$ and the other for the $j_g=\frac{1}{2}\to j_e=\frac{1}{2}$ transition. By virtue of the spin-statistics theorem, these gases must obey the Bose-Einstein and the Fermi-Dirac statistics, respectively. We assume a noninteracting, or *ideal* and spatially homogeneous gas. For completeness, we begin with a brief recap of the salient statistical mechanics of both Bose and Fermi gases [17].

A. Ideal quantum gas

In order to quantize we assume that the gas resides in a cubic box of volume V, and use periodic boundary conditions. The wave vector $\mathbf{k}$ is a good quantum number for the c.m. motion of an atom. The relevant one-atom wave functions are the familiar plane waves, $u_{\mathbf{k}}(\mathbf{r})=V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{r}}$. Since our final scattering formulas only refer to ground state atoms, we drop the subscript g. The Hamiltonian in the absence of light is

$$H_0=\hbar\sum_{\mathbf{k}m}\epsilon_{\mathbf{k}}b^\dagger_{\mathbf{k}m}b_{\mathbf{k}m}, \tag{52}$$

with the dispersion relation for massive atoms $\epsilon_{\mathbf{k}}=\hbar|\mathbf{k}|^2/2m$. For a theory this simple, the time dependence of the ground state fields is trivial,

$$\psi_m(\mathbf{r}t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\epsilon_{\mathbf{k}}t)}b_{\mathbf{k}m}, \tag{53}$$

where $b_{\mathbf{k}m}$ are operators at the initial time of the Heisenberg picture, defined here as $t=0$.

To describe the state of the gas, we use the grand canonical ensemble. Given the unperturbed Hamiltonian H_0, the density operator is a mixture of simultaneous eigenstates of all the number operators $n_{\mathbf{k}m}=b^\dagger_{\mathbf{k}m}b_{\mathbf{k}m}$. The expectation

values of these operators are determined by the temperature T (or $\beta=1/k_BT$) and the chemical potential μ; we have the familiar Bose-Einstein ($-$) and Fermi-Dirac ($+$) occupation numbers

$$\bar{n}_{\mathbf{k}m}=\frac{1}{e^{\beta(\hbar\epsilon_{\mathbf{k}}-\mu)}\pm 1}\,. \tag{54}$$

Strictly speaking the number of particles is not conserved in the grand canonical ensemble, but in the thermodynamic limit this may be ignored. We will equate the expectation number and the actual number of atoms, denoting both by N. The equilibrium density, of course, is $n=N/V$.

Computation of correlation functions such as those in Eq. (47) boils down to computation of thermal expectation values of operator products of the type

$$\langle b^{\dagger}_{\mathbf{k}_4m_4}b_{\mathbf{k}_3m_3}b^{\dagger}_{\mathbf{k}_2m_2}b_{\mathbf{k}_1m_1}\rangle.$$

Given the nature of the density operator, it is clear that a nonzero contribution may only emerge if the annihilation and creation operators pair up; either $(\mathbf{k}_1,m_1)=(\mathbf{k}_2,m_2)$, and $(\mathbf{k}_3,m_3)=(\mathbf{k}_4,m_4)$, or $(\mathbf{k}_1,m_1)=(\mathbf{k}_4,m_4)$ and $(\mathbf{k}_2,m_2)=(\mathbf{k}_3,m_3)$ [18]. In both cases the commutation or anticommutation relations may be used to reduce the surviving terms into expectation values of products of number operators, which may be deduced immediately with the aid of Eq. (54).

The final item is sums in statistical mechanics. One may normally make the continuum approximation for the $\mathbf{k}$ modes; whatever reasonably smooth function h, the replacement

$$\sum_{\mathbf{k}}\bar{n}_{\mathbf{k}m}h(\mathbf{k},m)\rightarrow\frac{V}{(2\pi)^3}\int d^3k\;\bar{n}_{\mathbf{k}m}h(\mathbf{k},m) \tag{55}$$

applies. Using $h=1$ as the function, one may express the density of the gas as a function of temperature and chemical potential,

$$n\lambda_D^3=(2j_g+1)g^{\pm}(z), \tag{56}$$

where

$$\lambda_D=\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2},\quad z=e^{\beta\mu},$$

$$g^{+}(z)=\frac{2}{\sqrt{\pi}}\int dx\;\frac{x^2ze^{-x^2}}{1+ze^{-x^2}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^n}{n^{3/2}},$$

$$g^{-}(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^{3/2}}, \tag{57}$$

are the thermal de Broglie wavelength, fugacity, and the proper functions for fermions ($+$) and bosons ($-$). Conversely, one may solve fugacity as a function of density and temperature from Eq. (56).

The significant exception to (55) occurs for bosons at low temperature or high density. To keep the occupation numbers (54) positive and finite, the fugacity is bounded: $z\in(0,1)$. Equation (56) then seems to imply that there is a maximum density for particles,

$$n_m=g^{-}(1)\frac{2j_g+1}{\lambda_D^3}\,. \tag{58}$$

Physically, all atoms in excess of the density n_m go to the ground state $\mathbf{k}=\mathbf{0}$ of the c.m. motion, which gives the $\mathbf{k}=\mathbf{0}$ density $n_c=n-n_m$. The continuum approximation must be amended to include explicitly the N_c atoms in the ground state,

$$\sum_{\mathbf{k}}\bar{n}_{\mathbf{k}m}h(\mathbf{k},m)\rightarrow\frac{V}{(2\pi)^3}\int d^3k\;\bar{n}_{\mathbf{k}m}h(\mathbf{k},m)+\frac{N_c}{2j_g+1}h(0,m). \tag{59}$$

It is these N_c $\mathbf{k}=\mathbf{0}$ atoms that make the Bose condensate.

B. Bosons: $j_g=0\rightarrow j_e=1$

Our first example is atoms with a $j_g=0\rightarrow j_e=1$ internal transition. In this case, there are no magnetic substates to the ground state, so we temporarily drop the index m from our notation. The relevant polarization factor (50) may be calculated without difficulty. To simplify things further, we assume a nonpolarizing detector. Thus, we choose two orthogonal exit polarizations $\hat{\mathbf{e}}$ (also orthogonal to the propagation direction $\hat{\mathbf{n}}$), and sum the spectra. The result is

$$\mathscr{S}(\omega;r,\hat{\mathbf{n}})=\frac{1}{2\pi}I_1(r,\hat{\mathbf{n}})V\times\int d^3r'dt\;e^{i(\omega t-\Delta\boldsymbol{\kappa}\cdot\mathbf{r}')}\langle n(00)n(r't)\rangle. \tag{60}$$

Here

$$I_1(r,\hat{\mathbf{n}})=\frac{\mathscr{R}^2}{\delta^2}I(r)(1-|\hat{\mathbf{e}}_i\cdot\hat{\mathbf{n}}|^2) \tag{61}$$

is the light intensity that would be radiated by one atom, and

$$n(\mathbf{r}t)=\psi^{\dagger}(\mathbf{r}t)\psi(\mathbf{r}t) \tag{62}$$

is the particle density operator for lower-state atoms. The spectrum is the Fourier transformation of the density correlation function, $(\mathbf{r},t)\rightarrow(\Delta\boldsymbol{\kappa},\omega)$.

The result (60) reflects scattering processes in which the wave vector of the photon changes by $\Delta\boldsymbol{\kappa}$ and frequency by ω; our implementation of the RWA entails that $\omega=0$ experimentally refers to the laser frequency Ω. The fractional change of frequency upon scattering is normally negligible, $|\omega|\ll\Omega$, so that one finds the familiar relation between the magnitude of the change of the wave vector and the scattering angle θ, $\Delta\kappa=2\kappa\sin(\theta/2)$. The quantity $\Delta\kappa$ may be varied experimentally in the range $(0,2\kappa]$. The spectrum of scattered light may then be monitored for a fixed scattering geometry.

If an individual atom with wave vector $\mathbf{k}$ in the gas took up the photon recoil, the wave vector of the atom would change by $-\Delta\boldsymbol{\kappa}$ in a scattering event. Energy conservation then dictates a change in photon frequency equal to $\omega=\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\Delta\boldsymbol{\kappa}}$. To manage conservations of energy and mo-

mentum, we introduce an effective recoil frequency and an effective Doppler width corresponding to the change of the wave vector $\Delta\boldsymbol{\kappa}$,

$$\omega_R=\frac{\hbar(\Delta\kappa)^2}{2m}, \quad \omega_D=\sqrt{\frac{k_BT(\Delta\kappa)^2}{m}}. \tag{63}$$

These are the two most prominent frequency scales of our theory. It should be noted that ω_R and ω_D may be varied, to some extent independently, by varying the scattering angle and/or the temperature.

Given the notational preliminaries, for an ideal Bose gas the spectrum (60) may be calculated easily from the expression of the atomic field (53). Skirting so far the continuum approximation, we have

$$\mathcal{S}(\omega)=I_1\sum_{\mathbf{k}}\delta\left(\omega+\omega_R-\frac{\hbar\mathbf{k}\cdot\Delta\boldsymbol{\kappa}}{m}\right)[\bar{n}_{\mathbf{k}}+\bar{n}_{\mathbf{k}}\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}}]. \tag{64}$$

The result (64) is amenable to straightforward interpretation. If the effective recoil frequency ω_R and the product of two occupation numbers were omitted, the spectrum at ω would be proportional to the number of atoms with the velocity component in the direction of the vector $\Delta\boldsymbol{\kappa}$ equal to $v=\omega/\Delta\kappa$. This is precisely the underlying theory for Doppler velocimetry of atoms [8]. The presence of the quantity ω_R merely indicates that we are careful with the kinematics, taking into account the recoil shift of the atomic transition frequency.

The novelty of (64) lies in the product of the occupation numbers. The product indicates that the scattering $\mathbf{k}\to\mathbf{k}-\Delta\boldsymbol{\kappa}$ is *enhanced* if the final state is already occupied. This term is expected to be important whenever both the initial state $\mathbf{k}$ and the final state $\mathbf{k}-\Delta\boldsymbol{\kappa}$ may have a significant thermal population. It can be seen easily from the definitions (63) that this condition for significant degeneracy effects may be recast as

$$\omega_D\gtrsim\omega_R. \tag{65}$$

It remains to calculate the sums over $\mathbf{k}$. In the absence of a Bose condensate we use (55), and obtain

$$\mathcal{S}(\omega)=NI_1F(\omega;z), \tag{66}$$

with

$$F(\omega;z)=\frac{1}{\sqrt{2\pi}\omega_D g^-(z)}\left\{-\ln(1-x_+)+\frac{1}{x_+-x_-}\right.$$
$$\left.\times[x_+\ln(1-x_-)-x_-\ln(1-x_+)]\right\}, \tag{67a}$$

$$x_\pm=z\exp\left[-\frac{(\omega\pm\omega_R)^2}{2\omega_D^2}\right]. \tag{67b}$$

The normalization of $F(\omega;z)$ is such that the integral over ω would be unity if only the first term in the braces in (67a), the direct contribution from velocity distribution, were retained. On the other hand, if a fraction f of the gas is in the Bose condensate, (59) gives

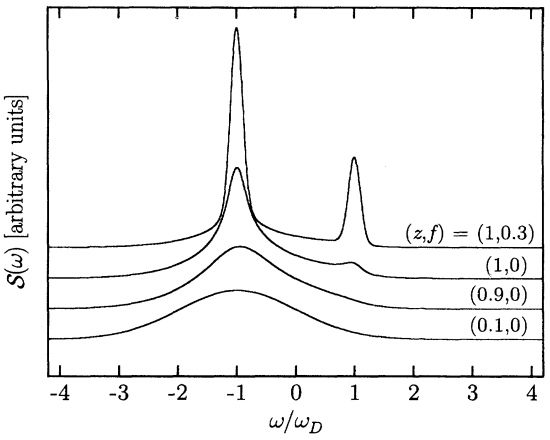

FIG. 1. Spectra of light scattered from a Bose gas with various degrees of degeneracy. Starting from the bottom curve, the fugacity and the condensate fraction are $(z,f)=(0.1, 0)$, $(0.9, 0)$, $(1, 0)$, and $(1, 0.3)$. The offset ω from the laser frequency is expressed in units of the effective Doppler width ω_D, and the effective recoil frequency is chosen as $\omega_R=\omega_D$. The spectra have been normalized to the same area, and vertical offsets have been added for easier comparison. To regularize integrable divergences in the spectra emerging at $z=1$, all computed spectra are convoluted with a Gaussian with the root-mean-square width $0.1\omega_D$.

$$\mathcal{S}(\omega)=NI_1\{f[\delta(\omega+\omega_R)(1+\bar{n}_{\Delta\boldsymbol{\kappa}})+\delta(\omega-\omega_R)\bar{n}_{\Delta\boldsymbol{\kappa}}]$$
$$+(1-f)F(\omega;1)\}. \tag{68}$$

We give representative spectra of the scattered radiation from Eqs. (66) and (68) in Fig. 1. The natural frequency scale is the effective Doppler width ω_D, so we use ω/ω_D as the horizontal axis. We choose the temperature and the scattering angle in such a way that $\omega_R=\omega_D$. At $z=1$, whether there is a condensate present or not, the spectra develop integrable singularities. We have removed these by convoluting all spectra with an "experimental resolution," a Gaussian whose root-mean-square width was arbitrarily chosen as $0.1\omega_D$.

At $z=0.1$ ($n=0.04\,n_m$) the spectrum is a Gaussian mirroring the velocity distribution, although shifted by $-\omega_R$. At $z=0.9$ ($n=0.62\,n_m$) small deformations from the Gaussian, a sharper peak and a rising right shoulder, may be detected. For $z=1$, $f=0$ ($n=n_m$) the peak at $\omega=-\omega_R$ has grown quite sharp, and a new peak has emerged at $\omega=+\omega_R$. At $z=1, f=0.3$ ($n=1.42\,n_m$) a Bose condensate is present. The spectrum has two dominant peaks at $\omega=\pm\omega_R$.

The feature at $\omega=+\omega_R$ reflects the Bose-Einstein statistics, in that a photon recoil taking an atom into an already occupied state is favored. Two somewhat contradictory conditions must be satisfied to observe such statistics effects. First, the gas must be degenerate, i.e., the temperature must be low enough and/or the density high enough. However, the temperature must also be high enough that both the initial and the final state of scattering are occupied, see (65). These two aspects combine into a condition on atomic density,

$$n \gtrsim \left(\frac{\Delta\kappa}{2\pi}\right)^3 . \tag{69}$$

In other words, the distance between two typical neighboring atoms must be smaller than the effective wavelength $\lambda' = 2\pi/\Delta\kappa$.

C. Fermions

Particles with half-integer total angular momenta are fermions. A nontrivial degeneracy of the ground level is inevitable. We present results for the simplest example in atomic systems, the level scheme $j_g = 1/2 \rightarrow j_e = 1/2$.

Any combination of the polarizations of the incoming light and of the polarizers in front of the detector may be analyzed straightforwardly using Eqs. (47) and (50). However, for the sake of brevity we only take up the case when the incoming light is linearly polarized, $\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_x$, and the detector is insensitive to polarization of the scattered light. We denote the propagation direction of incoming light by $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_z$, so that the angle θ in the conventional polar representation

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_1 \sin\theta\cos\phi + \hat{\mathbf{e}}_2 \sin\theta\sin\phi + \hat{\mathbf{e}}_1 \cos\theta \tag{70}$$

coincides with the scattering angle. The ground state atom has two possible magnetic quantum states $m = 1/2$ and $m = -1/2$, which we refer to as "spin up" and "spin down"; $\uparrow$ and $\downarrow$.

The spectrum of the scattered light reads

$$\mathcal{S}(\omega;r,\hat{\mathbf{n}}) = I(r)\frac{\mathscr{R}^2}{\delta^2}\sum_{\mathbf{k}} \delta\left(\omega + \omega_R - \frac{\hbar\mathbf{k}\cdot\Delta\boldsymbol{\kappa}}{m}\right) G_{\mathbf{k}}(\hat{\mathbf{n}}); \tag{71a}$$

$$\begin{aligned} G_{\mathbf{k}}(\hat{\mathbf{n}}) = \tfrac{1}{9}[&\bar{n}_{\mathbf{k}\uparrow}(1-\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}\uparrow})(1+\cos^2\theta) \\ &+ \bar{n}_{\mathbf{k}\downarrow}(1-\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}\downarrow})(1+\cos^2\theta) \\ &+ \bar{n}_{\mathbf{k}\downarrow}(1-\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}\uparrow})\sin^2\theta \\ &+ \bar{n}_{\mathbf{k}\uparrow}(1-\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}\downarrow})\sin^2\theta]. \end{aligned} \tag{71b}$$

The minus signs in $G_{\mathbf{k}}$ reflect Pauli's exclusion principle, which *forbids* transitions to an already occupied state. Another difference from the $j_g = 0 \rightarrow j_e = 1$ case is the presence of several sublevels, hence the possibility of inelastic transitions between the internal states. The first term in (71b) characterizes processes in which the c.m. motion of an atom goes as $\mathbf{k} \rightarrow \mathbf{k} - \Delta\boldsymbol{\kappa}$ and the spin stays $\uparrow$, while in the fourth term the spin also flips : $\uparrow \rightarrow \downarrow$. One might hope that such Raman processes could bring with them an angular dependence of scattering on the degeneracy of the atom, but at least in thermal equilibrium at zero magnetic field such hopes are dashed. When the occupation numbers are independent of angular momentum, $\bar{n}_{\mathbf{k}\uparrow} = \bar{n}_{\mathbf{k}\downarrow} \equiv \bar{n}_{\mathbf{k}}$, we have

$$\begin{aligned} \mathcal{S}(\omega;r,\hat{\mathbf{n}}) = &\frac{4}{9} I(r)\frac{\mathscr{R}^2}{\delta^2} \\ &\times \sum_{\mathbf{k}} \delta\left(\omega + \omega_R - \frac{\hbar\mathbf{k}\cdot\Delta\boldsymbol{\kappa}}{m}\right) \bar{n}_{\mathbf{k}}(1-\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}}). \end{aligned} \tag{72}$$

For this particular level scheme it so happens that scattering is isotropic.

In the continuum approximation, the result is again of the form (66), except that the one-atom intensity is now

$$I_1(r,\hat{\mathbf{n}}) = \frac{4}{9}\frac{\mathscr{R}^2}{\delta^2} I(r), \tag{73}$$

and the new function F is given by

$$\begin{aligned} F(\omega;z) = &\frac{1}{\sqrt{2\pi}\omega_D g^+(z)}\left\{\ln(1+x_+)\right. \\ &\left. - \frac{1}{x_+ - x_-}[x_+\ln(1+x_-) - x_-\ln(1+x_+)]\right\}. \end{aligned} \tag{74}$$

For the Fermi-Dirac statistics, the fugacity does not have an upper bound. When the temperature goes to zero (or the density becomes infinite), the fugacity tends to infinity. The result is the Fermi sphere with the radius equal to the Fermi wave number $k_F = (6\pi^2 N/V)^{1/3}$: $\bar{n}_{\mathbf{k}} = \theta(k_F - k)$. The resulting spectrum may either be derived as a limiting case from Eq. (74), or directly for the Fermi sphere. The critical consideration is the comparison between the Fermi energy (in frequency units) $\epsilon_F = \hbar k_F^2/2m$ and the effective recoil frequency ω_R. For $\omega_R > 4\epsilon_F$, every recoil event takes an atom out of the Fermi sea. The spectrum of scattered light reads

$$\mathcal{S}(\omega) = \frac{3NI_1}{8\sqrt{\epsilon_F\omega_R}} \times \begin{cases} 1 - \dfrac{\omega_R + 2\omega}{4\epsilon_F} - \dfrac{\omega^2}{4\epsilon_F\omega_R}, & -2\sqrt{\epsilon_F\omega_R} \le \omega + \omega_R \le 2\sqrt{\epsilon_F\omega_R} \\ 0, & \text{otherwise.} \end{cases} \tag{75}$$

This is a shifted image of the velocity distribution. On the other hand, for $\omega_R \le 4\epsilon_F$ some recoil events would lead to an already filled state in the Fermi sea, and are forbidden. The spectrum is

$$\mathscr{S}(\omega)=\frac{3NI_1}{8\sqrt{\epsilon_F\omega_R}}\times\begin{cases}\dfrac{|\omega|}{\epsilon_F}, & \omega_R-2\sqrt{\epsilon_F\omega_R}\leqslant\omega\leqslant 0\\ 1-\dfrac{\omega_R+2\omega}{4\epsilon_F}-\dfrac{\omega^2}{4\epsilon_F\omega_R}, & -\omega_R\leqslant\omega+2\sqrt{\epsilon_F\omega_R}\leqslant\omega_R\\ 0, & \text{otherwise.}\end{cases}\tag{76}$$

In the extreme case $\omega_R\ll\epsilon_F$ the spectrum turns into a triangle that rises linearly from 0 when ω decreases from 0, and then cuts off abruptly at $-2\sqrt{\epsilon_F\omega_R}$. These kind of line shapes are once more a qualitative signature of atom statistics.

An example is given in Fig. 2. Here we choose the atom density and scattering parameters so that $\epsilon_F=\omega_R$, and vary the temperature to give fugacity the values $z=0.1$ ($k_BT=3.93\ \hbar\epsilon_F$; dotted line), $z=0.9$ ($k_BT=1.05\ \hbar\epsilon_F$; dashed line), and $z=\infty$ ($T=0$; solid line). As the temperature is decreased, the high-temperature Gaussian shape evolves into a wedge consisting of a linear and a quadratic piece.

Just as for bosons, when the gas is degenerate, the velocity distribution of the atoms is modified from the Gaussian. The change in the velocity distribution is reflected in the spectrum of the scattered radiation. Moreover, if both the initial and the final state for a recoiling atom may be occupied, or $\epsilon_F\gtrsim\omega_R$, the Fermi statistics also directly influences the spectrum through Pauli's exclusion principle. Interestingly enough, the qualitative condition for such modifications is the same as for bosons: a density such that, on the average, at least one atom resides in a cube of an effective wavelength.

D. Bosons and fermions: Joint considerations

If the occupation numbers are independent of the magnetic substates, one may, in fact, derive from Eq. (47) a general expression of the unpolarized spectrum for an arbitrary input polarization $\hat{\mathbf{e}}_i$:

$$\mathscr{S}(\omega;r,\hat{\mathbf{e}}_i,\hat{\mathbf{n}})=W(\hat{\mathbf{n}},\hat{\mathbf{e}}_i)I(r)\frac{\mathscr{R}^2}{\delta^2}\sum_{\mathbf{k}}\delta\left(\omega+\omega_R-\frac{\hbar\mathbf{k}\cdot\Delta\boldsymbol{\kappa}}{m}\right)\times\bar{n}_{\mathbf{k}}(1\pm\bar{n}_{\mathbf{k}-\Delta\boldsymbol{\kappa}}).\tag{77}$$

Here the + and − signs apply to bosons and fermions, respectively, and the angular distribution function W is given by

$$W(\hat{\mathbf{n}},\hat{\mathbf{e}}_i)=\sum_{\hat{\mathbf{e}},m_1,m_2}\hat{\mathbf{e}}\cdot\mathsf{M}^{m_1m_2}_{m_2m_1}(\hat{\mathbf{e}}_i,\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}^*.\tag{78}$$

The sum over $\hat{\mathbf{e}}$ stands for a sum of arbitrary two orthonormal polarization vectors also orthogonal to $\hat{\mathbf{n}}$.

The result (78) tells us that our simplest boson and fermion examples have already captured the essential physics for any level structure. Besides atom statistics, the only difference between different level schemes is the overall angular distribution of the scattered light.

E. Ideal gas

The examples of the present section were all based on the Hamiltonian (52) of an ideal gas. We now inquire about the validity of the ideal-gas assumption. A quick criterion may be obtained by comparing the collision frequency of the ground state atoms

$$\omega_C=n\langle\sigma v\rangle,\tag{79}$$

thermal expectation value involving the relative velocity v and the cross section σ, with the scale of the frequency spectrum. The most relevant such scale is the effective recoil frequency ω_R, so that the condition for the ideal-gas approximation to be passable is $\omega_C\lesssim\omega_R$.

The quantity ω_R is expected to be in the neighborhood of atomic recoil frequencies, $\omega_R\sim\epsilon_R\sim 100$ kHz as a rule of thumb. In the absence of experiments with low-density degenerate samples, the validity of the ideal-gas model is an unanswerable question, but we suspect that the ideal-gas assumption will be one of the first casualties of real experiments. In fact, the numerical parameters used in Ref. [3] and a scattering length of a few nanometers would give $\omega_C\sim\epsilon_R$.

The calculations of Sec. IV may turn out to be examples of a method rather than predictions. Nonetheless, we point out that our key scattering results such as Eq. (47) in no way depend on the ideal-gas assumption. The question simply is to obtain the relevant correlation functions for an interacting gas, a staple problem in condensed matter physics. Our em-

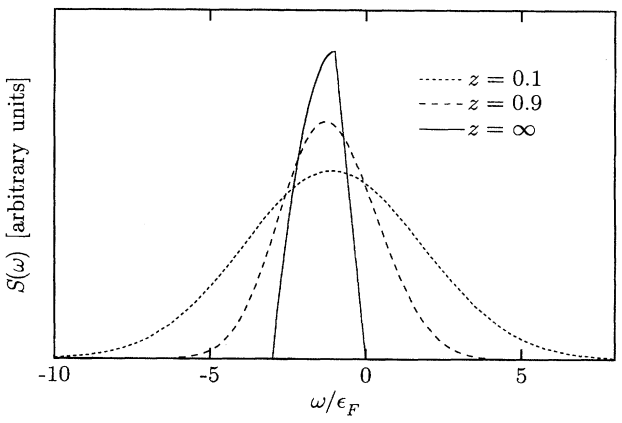

FIG. 2. Spectra of light scattered from a Fermi gas with various degrees of degeneracy. The effective recoil frequency is chosen equal to the Fermi energy, $\omega_R=\epsilon_F$. The curves are parametrized by the fugacity z, which for a fixed Fermi energy is a measure of temperature.

phasis here is on photons and so we rule such correlation functions outside the scope of the present paper, but we plan to return to probing of a true interacting gas in a future communication.

V. CONCLUDING REMARKS

For an ordinary nondegenerate thermal gas, the spectrum of scattered light is Gaussian, basically an image of the velocity distribution. For a degenerate gas, the effects of statistics fall into two categories. First, the velocity distribution is modified, with attendant modifications in the spectrum. Second, statistics may enhance or inhibit the recoil of an atom during a photon scattering event, which leads to a direct qualitative influence on the spectrum. The direct effects of the statistics depend on the relative magnitude of the effective recoil frequency and the effective Doppler width. In an experiment, this comparison may be controlled to a certain degree by varying the scattering angle of light.

There are limits to the sensitivity of the spectrum to the statistics of the atoms. For instance, while the spectrum of a gas containing a Bose condensate may display two peaks, these peaks would, in practice, evolve continuously as the gas moves across the Bose condensation point. The spectrum cannot pinpoint the precise onset of Bose condensation. In addition to the macroscopic population of the ground state, at the phase transition point the Bose gas should abruptly acquire nontrivial phase properties in a process known as spontaneous symmetry breaking [19]. Unfortunately, as far as we can tell, the spectrum of scattered light is also totally insensitive to such phases.

Our example theory was derived for an ideal, homogeneous Bose or Fermi gas, whereas a trapped, weakly interacting gas is a more likely result of an experiment. The momentum distribution of a real (presumably trapped) gas is broadened [20], both because of the uncertainty principle and because of the interparticle interactions embodied in $\mathscr{H}_{gg}$, see Eq. (18b). Condensate features analogous to the δ functions in Eq. (68) should thus have a nonzero width. Nevertheless, as long as the sample is much larger than the wavelength of the driving light, at least the width of the spectral features imposed by the uncertainty principle remains much smaller than ϵ_R.

Doppler velocimetry experiments, as in Refs. [8], serve as a blueprint for the measurements of the spectrum of scattered light. There apparently is no reason why the existing experimental methods would not work on the scale of a photon recoil ϵ_R. The problem really is to produce the degenerate gas. At this writing there is no predicting which experimental setups will lead to observable statistics effects. Perforce, one cannot predict the actual parameters of the gas. While we have offered some numerical estimates in [3], these are just straightforward extrapolations of the present experiments. Our theory needs to be refined as the experiments proceed. For instance, the ideal-gas treatment may prove to be an oversimplification.

Neutron scattering [21] has been used to measure the condensate fraction in the quantum fluid ^{4}He for quite a while, see [22] and references therein. Now, with the assumption of large detuning, we have eliminated adiabatically the excited state atoms. This amounts to introducing an atom-field interaction equivalent to the neutron-nucleus contact interaction responsible for neutron scattering. Conversely, it is instructive to delineate the differences between neutron scattering and light scattering. Neutron scattering in ^{4}He is carried out at large momentum and energy transfers and probes short spatial and temporal scales, over which the atoms are effectively free. Neutron scattering therefore misses any and all correlations in the atomic motion, and just yields the velocity distribution. The direct qualitative effects of the statistics are not accessible. On the other hand, in light scattering the spatial resolution is of the order of the effective wavelength λ'. If the typical interatomic distance is less than λ', direct statistics effects make their appearance.

In Ref. [23] a loose principle was introduced, according to which the absence or presence of phenomena associated with particle statistics depends on whether the particles could or could not be distinguished in the given situation. Remarkably enough, here we have another example of the principle: The onset of direct statistics effects in light scattering occurs at densities such that the resolution of a probe based on light would no longer suffice to tell an atom from its nearest neighbors.

Two types of optical degeneracy effects are known at present: the enormous linewidth of a Bose condensate [3,4], and the spectral features discussed in [7] and in the present paper. We regard the search for the optical manifestations of spontaneous symmetry breaking as a high-priority item. It is not even known if any such phenomenon can exist, but we hope to have laid the foundation for the eventual settlement of the question.

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