Two-ion superradiance theory

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A superradiance theory is developed for two identical hydrogenic ions (four states each) in a microtrap, as in recent experiments. The ions oscillate (micromotion) due to the trap's radio frequency (rf) electric quadrupole field. One signature of superradiance is a deviation of the two-ion average upper state decay rate $\bar{\gamma}$ from the one-ion value γ. A master equation is derived, giving a fractional change in the upper state lifetime $\gamma/\bar{\gamma}-1=\sin kR/2kR\,[J_0^2(z)-2J_1^2(z)+\ldots]\cos\Phi$, where $k=2\pi/\lambda$, λ is the emission wavelength, R is the ion-ion distance, $J_n(z)$ is a Bessel function of integer order n, and $z=ka$, a is the ion amplitude of motion, and Φ is the two-ion relative phase due to the preparation. In the Lamb-Dicke regime, $a<\lambda$, $J_0^2(z)\approx 1$ and thus superradiance is not influenced significantly by ion motion. This damped sinusoid is diluted by the factor of $\frac{1}{2}$ due to destructive interference. Superradiance vanishes in the absence of coherent preparation, e.g., with inversion, as indicated by the time evolution of the two-ion dipole correlation function. Fringes and a beat at the rf are predicted in forward scattering, the most elementary form of optical free induction decay.

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I. INTRODUCTION

The inspired article of Dicke entitled *Coherence in Spontaneous Radiation Processes* [1] contains in the Introduction the most elementary example of superradiance, the case of two neutrons. Assuming one neutron in an excited spin state, the other in the ground state, and a distance separating them far less than the radiation wavelength ($R\ll\lambda$), he argues that the rate of magnetic dipole radiation is either 2γ or 0 where γ is the rate of decay for one neutron. Notwithstanding the conceptual simplicity of two-particle superradiance and the desirability of observing this elementary process, these experiments have been elusive until recently [2]. The purpose of this article is to develop superradiance theory to interpret the current optical observations of two ions in a microtrap [2].

Previous experiments have been restricted to many-atom samples where the atom-atom distance is much greater than the radiation wavelength ($R\gg\lambda$), the regime known as superfluorescence. Typically, the initial condition is a population inversion, measurements having been performed in the far infrared [3], the near infrared [4], and the microwave region [5]. The literature of N-atom superradiance theory for $N>2$ is not reviewed here because it is extensive. Instead, we cite a few of the early references [6–10] and the excellent review by Gross and Haroche [11].

Following Dicke's article, numerous superradiance theories of two identical atoms appeared, most of them assuming two-level atoms [12]. Stephen [13] and Hutchinson and Hameka [14] utilized the radiation damping theory of Heitler [15] finding a two-atom damping rate and a Lamb-like shift as a function of the atom-atom distance R. Power then demonstrated that the two-atom damping rate can be obtained far more simply by first-order perturbation theory [16]. These calculations assume that the atoms are stationary, but later it was shown that the relative motion of the two atoms can reduce or even extinguish superradiance [17,18]. A third and more powerful approach is the master equation, the equation of motion of the reduced density matrix which contains the damping rates and frequency shifts at the outset [6,19].

Also, these theories implicitly assume that the two atoms interact instantaneously, ignoring that a photon emitted by one atom cannot propagate and excite a second atom faster than the speed of light c (Einstein causality). Indeed, Fermi [20] and subsequently others [21,22] showed that the second atom is excited by the first with a retardation time $t=R/c$. Recently, Fermi's result was disputed and a new analysis led to a causality paradox [23]. However, this statement prompted a response where the authors [24], using relativistic quantum field theory, find no paradox invalidating Fermi's retardation time. Milonni and Knight [25] generalized this problem to the case of multiple traversals where the retardation is $t=nR/c$, n being an integer; in the absence of retardation, their formalism reduces to the Stephen [13] result.

When two atoms are sufficiently close, $R\ll\lambda$, and are driven continuously by an external field, superradiance and

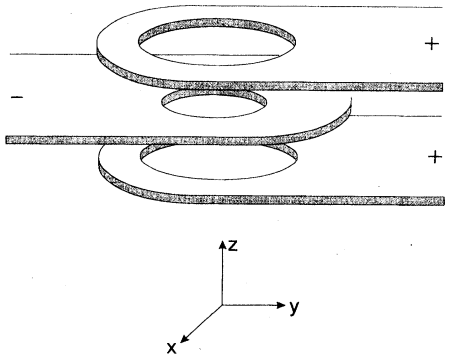

FIG. 1. Schematic of a planar Paul trap made from conducting sheets, after Brewer, DeVoe, and Kallenbach [28]. Typically, the radius of the central hole is 80 μm. The origin of the coordinate system has been translated from the central hole to avoid clutter.

macroscopic quantum jumps are predicted in spontaneous emission [12,19,26]. Randomly alternating long periods of brightness and darkness are expected as in earlier studies of a single three-level atom; e.g., see [27].

In this article, we derive a master equation for two Ba^+ ions in a microtrap, following recent experiments [2]. The ions are coherently prepared and radiate freely thereafter. The microtrap [28], shown in Fig. 1, reduces the ion-ion distance to $R \approx 2\lambda$, the region where superradiance begins. The ions can be made to lie either in the trap's $x-y$ plane of the central hole or along the z axis by applying a static and a radio frequency (rf) electric quadrupole field across these electrodes. Oscillation of the ions at the rf induces sidebands in their spontaneous emission with most of the energy being in the unshifted or central frequency due to laser cooling [29]. This is the Lamb-Dicke regime [30] where the ion motion does not significantly affect superradiance, unlike earlier calculations [17,18]. Another difference with earlier work is that the Ba^+ ion being hydrogenic has four relevant states instead of two. All these considerations are included in the master equation derivation.

II. ZERO-ORDER WAVE FUNCTIONS

Before developing solutions to a master equation, we generate zero-order wave functions for two Ba^+ ions, following recent superradiance experiments [2]. The two-ion states of interest in weak field are $^2P_{1/2}{}^2S_{1/2}$ and $^2S_{1/2}{}^2S_{1/2}$ where in the one-ion (jm) representation $\mathbf{j}=\mathbf{l}+\mathbf{s}$. The $^2P_{1/2}$ $^2P_{1/2}$ state may play a role also, but we neglect it here since its faster decay is not observed as yet [2]. At ion-ion distances of an optical wavelength, the ions weakly perturb each other, and thus in zero-order they act independently and their angular momenta j_1 and j_2 commute. The two-ion angular momentum $\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2$ gives rise in this (jj) coupling scheme to a two-ion (JM) representation. Since $j_1=j_2=\frac{1}{2}$, there are four ways of combining $(m_1 m_2)$, namely, $(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2})$, and $(-\frac{1}{2},\frac{1}{2})$, and these correspond to the two-ion triplet $(J=1,\ M=1,0,-1)$ and singlet $(J=0,\ M=0)$ states. Hence the two-ion electron ground and excited states each consist of a triplet and a singlet substate.

For the two-ion electronic ground state, the total wave function is a product of an angular momentum and an orbital wave function, Eqs. (2.5)–(2.8). The angular momentum eigenfunctions are well known [31] and are of the form of Eqs (2.5)–(2.8) while the orbital factor is a product of two spherical harmonics for the two S states. Here, it is unnecessary to antisymmetrize the total wave function because the electron clouds of the two ions do not overlap.

In discussing the two-ion excited state, imagine that ground-state ions are in the path of a light beam during the preparative stage so that either ion, but not both, can be excited. A wave function of the form $\phi(1,2) = 1/\sqrt{2}[^2P_{1/2}(1)^2S_{1/2}(2) - {}^2P_{1/2}(2)^2S_{1/2}(1)]$ expresses the potential for either ion (1,2) being excited, a superposition of the two possibilities, and our inability to know which one was excited. Similarly in emission, one cannot predict which ion will emit. Of course in a measurement, the apparatus introduces uncontrollable phase factors that destroy the superposition, and then it is possible to say which ion emitted. This superposition is responsible for the interference described in this article and can be viewed as an exchange of excitation. This exchange is a parity operation and for Fermi particles requires that the wave function be antisymmetric. The antisymmetric wave functions below, Eqs. (2.1)–(2.4), predict a superradiance signal with a phase that appears to agree with preliminary observations [2]. Reversing the sign produces symmetric wave functions and a phase error of π.

Wave functions in the weak field regime can be generated [32] by using the angular momentum lowering operator $j^- = j_x - ij_y$ in

$$j^-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle,$$

where $j=j_{1,2}$ or J. We now label the state of each ion $(\epsilon=1,2)$ by $|a_\epsilon\rangle$ or $|b_\epsilon\rangle$ in the S state, corresponding to $m=1/2$ or $-1/2$, and similarly by $|c_\epsilon\rangle$ or $|d_\epsilon\rangle$ in the P state. The following two-ion wave functions result assuming an initial wave function and then applying the lowering operator:

$$|1\rangle \equiv |1,1\rangle_{ps} = \frac{1}{\sqrt{2}}(|c_1 a_2\rangle - |a_1 c_2\rangle), \tag{2.1}$$

$$|2\rangle \equiv |1,0\rangle_{ps} = \tfrac{1}{2}(|d_1 a_2\rangle - |a_1 d_2\rangle + |c_1 b_2\rangle - |b_1 c_2\rangle), \tag{2.2}$$

$$|3\rangle \equiv |1,-1\rangle_{ps} = \frac{1}{\sqrt{2}}(|d_1 b_2\rangle - |b_1 d_2\rangle), \tag{2.3}$$

$$|4\rangle \equiv |0,0\rangle_{ps} = \tfrac{1}{2}(-|d_1 a_2\rangle + |a_1 d_2\rangle + |c_1 b_2\rangle - |b_1 c_2\rangle), \tag{2.4}$$

$$|5\rangle \equiv |1,1\rangle_{ss} = |a_1 a_2\rangle, \tag{2.5}$$

$$|6\rangle \equiv |1,0\rangle_{ss} = \frac{1}{\sqrt{2}}(|a_1 b_2\rangle + |b_1 a_2\rangle), \tag{2.6}$$

$$|7\rangle \equiv |1,-1\rangle_{ss} = |b_1 b_2\rangle, \tag{2.7}$$

$$|8\rangle \equiv |0,0\rangle_{ss} = \frac{1}{\sqrt{2}}(|a_1 b_2\rangle - |b_1 a_2\rangle). \tag{2.8}$$

For example, applying the above lowering operator, $J^-|1,1\rangle_{ps} = \hbar\sqrt{2}|1,0\rangle_{ps}$, and since $J^- = j_1^- + j_2^-$, we have the relation $J^-|1,1\rangle_{ps} = (\hbar/\sqrt{2})(|d_1 a_2\rangle - |a_1 d_2\rangle + |c_1 b_2\rangle - |b_1 c_2\rangle)$, which gives Eq. (2.2). Repeating the operation gives Eq. (2.3), and a change of signs in Eq. (2.2) yields Eq. (2.4), the wave function being antisymmetric and $J^2 = J_z = 0$. The above wave functions are orthogonal, linearly independent, and normalized, and lead to matrices of dimensionality 8×8.

III. THEORETICAL FRAMEWORK

A. Hamiltonian

To derive a master equation, we adopt the Hamiltonian

$$\mathscr{H}=\tfrac{1}{2}\hbar\omega_0\sum_{\epsilon=1,2}S^z_\epsilon+\hbar\sum_\lambda\omega_\lambda b^\dagger_\lambda b_\lambda+\hbar\sum_{j\lambda\epsilon}g_{j\lambda\epsilon}\mathbf{S}^+_{j\epsilon}\cdot\mathbf{b}_\lambda e^{i\mathbf{k}\cdot\mathbf{r}_\epsilon(t)}+\text{H.c.} \tag{3.1}$$

The first term is the two-ion energy, the second the electromagnetic energy of the field at wavelength λ, and the third the two-ion-field energy of interaction. The operators for a single ion undergoing a transition $|k\rangle\rightarrow|l\rangle$ are

$$\mathbf{s}^+_{kl\epsilon}=(|l\rangle\langle k|)_\epsilon\hat{\mu}_{kl},\quad \mathbf{s}^-_{lk\epsilon}=(\mathbf{s}^+_{kl\epsilon})^*,$$

$$s^z_{kl\epsilon}=\tfrac{1}{2}(|k\rangle\langle k|-|l\rangle\langle l|)_\epsilon,$$

which obey the commutator $[s^+,s^-]=2s^z$ where $\hbar=1$, $s^\pm=\mathbf{s}^\pm\cdot\hat{\mu}^*$, and $\hat{\mu}$ is the unit dipole vector. Taking the sums $\mathbf{S}^\pm=\Sigma_{kl}\mathbf{s}^\pm_{kl}$ and $S^z=\Sigma_{kl}s^z_{kl}$ over all transitions of a given ion yields the equations

$$\mathbf{S}^+_\epsilon=(|c\rangle\langle a|\hat{\mathbf{k}}+|d\rangle\langle a|\frac{\hat{\mathbf{i}}+i\hat{\mathbf{j}}}{\sqrt{2}}+|c\rangle\langle b|\frac{\hat{\mathbf{i}}-i\hat{\mathbf{j}}}{\sqrt{2}}-|d\rangle\langle b|\hat{\mathbf{k}})_\epsilon, \tag{3.2a}$$

$$S^z_\epsilon=(|c\rangle\langle c|+|d\rangle\langle d|-|a\rangle\langle a|-|b\rangle\langle b|)_\epsilon, \tag{3.2b}$$

which obey the commutator $[S^+,S^-]=2S^z$ with $S^\pm=\mathbf{S}^\pm\cdot\Sigma_{kl}\hat{\mu}^*_{kl}$. Since each state of an ion can emit on two transitions, each state is counted twice in S^z, which explains the factor of 1/2 in the first term of Eq. (3.1).

The field raising ($b^\dagger$) and lowering (b) operators can be written as a vector, $\mathbf{b}=b\mathbf{e}$, for example, where the polarization $\mathbf{e}=\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2$ consists of two orthogonal unit vectors

$$\hat{\mathbf{e}}_1=(-\cos\theta\cos\phi,-\cos\theta\sin\phi,\sin\theta),$$

$$\hat{\mathbf{e}}_2=(\sin\phi,-\cos\phi,0),$$

the propagation vector $\mathbf{k}$ making angles θ and ϕ with the z and x axes [33]. Thus the dot product, the third term of Eq. (3.1), when squared gives $|\hat{\mu}\cdot\mathbf{e}|^2$ or

$$\left|(\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2).\frac{1}{\sqrt{2}}(\hat{\mathbf{i}}\pm i\hat{\mathbf{j}})\right|^2=\tfrac{1}{2}(1+\cos^2\theta), \tag{3.3a}$$

$$|(\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{k}}|^2=\sin^2\theta \tag{3.3b}$$

for circularly and linearly polarized light.

We assume in the third term of Eq. (3.1) that the two ions oscillate harmonically about their common center at a frequency Ω, the rf of the trap. This classical motion [34] (micromotion) occurs in the radial plane of the trap with a vanishing axial motion, due to laser cooling, and is given by

$$r_1(t)=r_1^0+a\cos\Omega t+\Phi/k, \tag{3.4a}$$

$$r_2(t)=r_2^0-a\cos\Omega t, \tag{3.4b}$$

where the time-average separation of ions 1 and 2 is $R=r_1^0-r_2^0$ and Φ is the two-ion relative phase acquired in the preparative period at $t=0$. Equations (3.4) also apply when the ion-ion axis is collinear with the trap z axis. Thus the micromotion itself can determine Φ. The secular motion can be included as well, but it is neglected here because of its lower frequency. A quantum mechanical treatment of the ions' motion [35], without using a pseudopotential, is complicated and unnecessary in the present theory. This motion introduces sidebands in the spontaneous emission with a Bessel function distribution characteristic of the Lamb-Dicke regime [30] where the amplitude of motion is less than the wavelength of light, $a<\lambda$. This follows in the derivation by expanding the third term of Eq. (3.1) as

$$e^{iz\cos\Omega t}=\sum_{n=-\infty}^{\infty}i^nJ_n(z)e^{in\Omega t}, \tag{3.5}$$

where $z=\mathbf{k}\cdot\mathbf{a}$. The condition $a<\lambda$ ensures that most of the energy radiated resides in the central frequency corresponding to the $J_0(z)$ term.

B. Master equation

Using the Hamiltonian Eq. (3.1) and the expansion Eq. (3.5), we derive a master equation [6,36] which can be written compactly as

$$\frac{d\rho}{dt}=-\tfrac{1}{2}i\omega_0\sum_\epsilon[S^z_\epsilon,\rho]+\frac{1}{2}\sum_{ij\epsilon\epsilon'}\gamma^{\epsilon\epsilon'}_{ij}([S^-_{i\epsilon}\rho,S^+_{j\epsilon'}]+[S^-_{i\epsilon},\rho S^+_{j\epsilon'}]) \tag{3.6}$$

and contains one-ion ($\epsilon=\epsilon'$) and two-ion ($\epsilon\neq\epsilon'$) terms where the labels i,j are a shorthand for the indices $i=kl$ and $j=lk'$ of the one-ion matrix elements of $S^\pm$ in a two-ion representation. When $k\neq k'$, the transition matrix product g_ig_j results in a frequency shift through a virtual transition, but when $k=k'$, damping occurs, allowing us to use i in place of ij. In this article, frequency shift terms are neglected. The one-ion decay rate is then of the form

$$\gamma^{\epsilon\epsilon}_{kl}=\frac{2\pi}{\hbar}\sum_\lambda|\hat{\mu}_{kl}\cdot\mathbf{e}|^2|g_{kl\lambda\epsilon}|^2\sum_{n=-\infty}^{\infty}J_n^2(z)\,\delta_n, \tag{3.7}$$

where $|\hat{\mu}_{kl}\cdot\mathbf{e}|^2$ is given by Eq. (3.3), the sum Σ_λ is to be replaced by the integrals $\int\int\rho_\omega d\omega d\Omega_s$ with ρ_ω the density of final states and the differential solid angle $d\Omega_s=\sin\theta d\theta d\phi$ [15], $J_n(z)$ is a Bessel function of integer order n and argument $z=\mathbf{k}\cdot\mathbf{a}$, and the delta function $\delta_{\pm n}=\delta(\omega_0-\omega_\lambda\pm n\Omega)$ defines the resonance condition of a moving ion with an optical sideband of order n. Here, the Bessel function sum is factored out of the angular integral because it varies slowly with z in the Lamb-Dicke regime, and we assume that $z=\mathbf{k}\cdot\mathbf{a}\approx ka$. The total one-ion decay rate of an upper state $|k\rangle$ to all lower states $|l\rangle$ in sidebands of order n and frequency ω is

$$\sum_{l,\epsilon}\gamma^{\epsilon\epsilon}_{kl}=\gamma\sum_{n=-\infty}^{\infty}J_n^2(z)|_{\omega=\omega_0+n\Omega}=\gamma, \tag{3.8}$$

where γ is the Einstein decay rate of a single ion and the sum of the Bessel function squared is unity [37]. As an example, for the upper state $|1\rangle$, $\Sigma_{l,\epsilon}\gamma_{kl}^{\epsilon\epsilon}=\frac{1}{2}[\gamma_{15}^{11}+\gamma_{15}^{22}+\frac{1}{2}(\gamma_{16}^{11}+\gamma_{16}^{22}+\gamma_{18}^{11}+\gamma_{18}^{22})]$, the first two terms corresponding to a linearly polarized transition. The last four terms correspond to two circularly polarized transitions in the two-ion representation with a total transition probability equal to that of the single circularly polarized transition in the one-ion representation. For either representation, the total one-ion decay rate is γ. Also, the one-ion decay rate is the same whether the ion is stationary or moving. Each one-ion emission event involves one sideband and one lower state from the distribution, but in a measurement of the upper state lifetime, the sum of the individual decay rates is the relevant quantity.

The effect of two-ion interference on the decay rate is

$$\gamma_{kl}^{\epsilon\epsilon'}=2\pi/\hbar\sum_{\lambda}K(\theta,\omega_\lambda)|\hat{\mu}_{kl}\cdot\mathbf{e}|^2 g_{kl\lambda\epsilon}g_{lk\lambda\epsilon'}, \tag{3.9}$$

$$\begin{aligned}K(\theta,\omega_\lambda)=&\cos(\mathbf{k}\cdot\mathbf{R}+\Phi)\sum_{n=-\infty}^{\infty}(-1)^n J_n^2(z)\delta_n\\&-\sin(\mathbf{k}\cdot\mathbf{R}+\Phi)\\&\times J_0(z)J_1(z)\cos(\Omega t)(2\delta_0+\delta_1+\delta_{-1})+\ldots,\end{aligned} \tag{3.10}$$

$$\gamma'=\gamma\sum_{n=-\infty}^{\infty}(-1)^n J_n^2(z)|_{\omega=\omega_0+n\Omega}, \tag{3.11}$$

where only the leading terms are shown and $\mathbf{k}\cdot\mathbf{R}=kR\cos\theta$. The $\sin\mathbf{k}\cdot\mathbf{R}$ term contains a beat at the rf in forward scattering and is discussed below. Changing the sum to two integrals as before and integrating $K(\theta,\omega)$, the $\sin\mathbf{k}\cdot\mathbf{R}$ term vanishes, but the $\cos\mathbf{k}\cdot\mathbf{R}$ term yields

$$\gamma_{kl\sigma}^{12}=\gamma_{kl\sigma}^{21}=q\gamma', \tag{3.12a}$$

$$q=\left(\frac{\sin kR}{kR}+\frac{\cos kR}{(kR)^2}-\frac{\sin kR}{(kR)^3}\right)\cos\Phi \tag{3.12b}$$

for circularly polarized emission, and

$$\gamma_{kl\pi}^{12}=\gamma_{kl\pi}^{21}=p\gamma', \tag{3.13a}$$

$$p=\left(-\frac{\cos kR}{(kR)^2}+\frac{\sin kR}{(kR)^3}\right)\cos\Phi \tag{3.13b}$$

for linearly polarized emission, the indices σ and π labeling the two cases. In the limit of motionless ions, $\lim_{z\to 0}\gamma'=\gamma$ and with Eq. (3.8), we recover one- and two-ion decay rates corresponding to calculations of two two-level atoms [19]. The total decay rate, being a sum of one- and two-ion terms, exhibits constructive or destructive interference depending on the signs of the two-ion terms. The signs are determined by the two-ion operator products of the master equation, which we now consider.

The one-ion raising and lowering operator matrix elements for the states of Eq. (2.1)–(2.8), written as 4×4 submatrices, are

$$S_{kl1}^{+}=(S_{lk1}^{-})^*=\frac{1}{2}\begin{pmatrix}\sqrt{2}&1&0&-1\\1&0&1&\sqrt{2}\\0&1&-\sqrt{2}&1\\-1&\sqrt{2}&1&0\end{pmatrix},$$

$$S_{kl2}^{+}=(S_{lk2}^{-})^*=\frac{1}{2}\begin{pmatrix}-\sqrt{2}&-1&0&-1\\-1&0&-1&\sqrt{2}\\0&-1&\sqrt{2}&1\\1&-\sqrt{2}&-1&0\end{pmatrix}, \tag{3.14}$$

where the columns span the states $|5\rangle-|8\rangle$ and the rows $|1\rangle-|4\rangle$, and the indices 1,2 label the two ions.

Applying these matrix elements in the master equation gives diagonal and off-diagonal equations of motion

$$\frac{d\rho_{kk}}{dt}=-\gamma_k(R)\rho_{kk}, \tag{3.15}$$

$$\frac{d\rho_{kl}}{dt}=-[\tfrac{1}{2}\gamma_k(R)+i\omega_0]\rho_{kl}, \tag{3.16}$$

where $\gamma_k(R)$ is the total decay rate of the upper state $|k\rangle$ to all lower states $|l\rangle$, the sum of one- and two-ion rates, Eqs. (3.8), (3.12), and (3.13). These equations apply, as we shall see, following a coherent preparative period that establishes an initial condition with induced dipoles $\rho_{kl}(0)\neq 0$ that radiate coherently. We then find for the four upper states that

$$\gamma_1(R)=\gamma-p\gamma', \tag{3.17a}$$

$$\gamma_2(R)=\gamma+(p-q)\gamma', \tag{3.17b}$$

$$\gamma_3(R)=\gamma-p\gamma', \tag{3.17c}$$

$$\gamma_4(R)=\gamma-(p+q)\gamma'. \tag{3.17d}$$

Each upper state decays to three lower states via one linearly and two circularly polarized transitions, the latter exhibiting an important two-ion interference. Thus in Eqs. (3.17a) and (3.17c) the interference is destructive with the q terms vanishing whereas in Eqs. (3.17b) and (3.17d) the interference is constructive. It is the q terms that will give an observable superradiance signal due to the leading term $\sin kR/kR$ in Eq. (3.12b).

IV. PREDICTIONS

A. Second-order dipole correlation

Superradiance requires that the transition dipoles be correlated with a fixed relative phase. We calculate the second-order correlation defined by the quantum mechanical average $\langle S_{kl\epsilon}^{+}S_{lk\epsilon'}^{-}\rangle$ with $\epsilon\neq\epsilon'$ [36]. Multiplying both sides of the master equation by these matrix elements and taking the trace yields the equation of motion

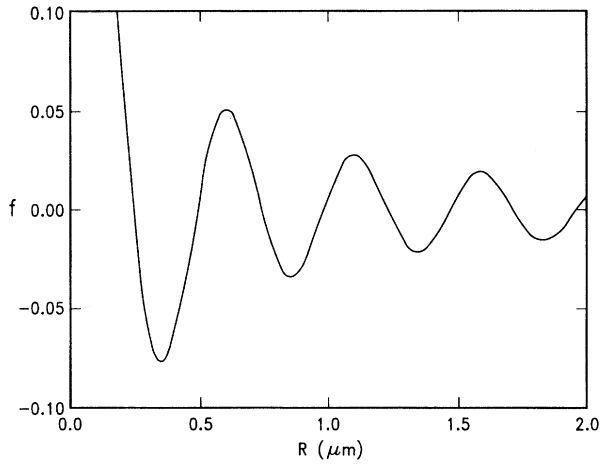

FIG. 2. Fractional change in the two-ion upper state lifetime vs the ion-ion distance, f vs R of Eq. (4.4). Here, for Ba^+, $\lambda=0.49$ μm, $z=0.5$, and $\Phi=0.0$.

$$\frac{d\langle S^+_{kl\epsilon}S^-_{lk\epsilon'}\rangle}{dt}=-\gamma_k\langle S^+_{kl\epsilon}S^-_{lk\epsilon'}\rangle, \tag{4.1}$$

with the obvious solution

$$\langle S^+_{kl\epsilon}(t)S^-_{lk\epsilon'}(t)\rangle=\langle S^+_{kl\epsilon}(0)S^-_{lk\epsilon'}(0)\rangle e^{-\gamma_k t}. \tag{4.2}$$

This says that the two dipoles are correlated only through the preparation which provides an initial condition at $t=0$. In the absence of this preparation, the correlation will vanish in the transition $|k\rangle\rightarrow|l\rangle$ and the two-ion contribution to the decay rate γ_k will vanish also. When all possible transitions are coherently prepared, Eqs. (3.15) and (3.16) apply with γ_k given by Eq. (3.17). In the other extreme when no dipoles are correlated, $\gamma_k\rightarrow\gamma$, the one-ion decay rate. We also predict that superradiance will not be observed when the population is inverted, since the dipoles vanish.

B. Superradiance lifetime

One signature of superradiance is the deviation of the two-ion upper state spontaneous emission lifetime from the one-ion value. We now consider examples corresponding to different initial conditions. For the case of all transitions being coherently prepared, we find from Eqs. (3.12), (3.13), and (3.17) the average upper state decay rate

$$\bar{\gamma}=\frac{1}{4}\sum_{k=1}^{4}\gamma_k=\gamma-\frac{\sin kR}{2kR}[J_0^2(z)-2J_1^2(z)+\ \ldots\ ]\cos\Phi. \tag{4.3}$$

The $\cos\Phi$ term and the negative sign preceding $\sin kR$ set the phase. The latter is a result of the excited-state wave functions Eqs. (2.1)–(2.4) being antisymmetric, in agreement with experiment [2] and the two ions being Fermi particles. The $\frac{1}{2}$ factor dilutes the superradiance because one-half of the transitions interfere constructively and the other half destructively. In the large R limit, $\lim_{R=\infty}\bar{\gamma}=\gamma$, the one-ion decay rate is as expected for ions that do not interact. In the small R limit, $\lim_{R=0}\bar{\gamma}\approx\frac{1}{2}\gamma$, which is a subradiance peak rather than the superradiance peak of earlier theories [1,13,14,16,19]. We can write this decay rate as the fractional change in the lifetime $f=\gamma/\bar{\gamma}-1$,

$$f=\frac{\sin kR}{2kR}[J_0^2(z)-2J_1^2(z)+\ \ldots\ ]\cos\Phi, \tag{4.4}$$

which is plotted in Fig. 2.

Consider now the two cases of either linearly or circularly polarized transitions being prepared. Imagine the two ions confined to the microtrap's radial plane, and aligned along the x axis, taken to be the axis of quantization. A pulse of laser light, a $\pi/2$ pulse, propagating along the symmetry axis of the trap, the z axis, prepares linearly polarized transitions with its electric vector parallel to the x axis and circularly polarized transitions with the electric vector parallel to the y axis. The corresponding decay rates are

$$\bar{\gamma}_\pi=\gamma-\frac{1}{2}p\gamma', \tag{4.5}$$

$$\bar{\gamma}_\sigma=\gamma-\frac{1}{2}q\gamma', \tag{4.6}$$

where γ', q, and p are given by Eqs. (3.11), (3.12), and (3.13). The p term, being of higher order in $1/R$ than the q term, is typically one order of magnitude smaller. Thus we predict that a 90° rotation of the pulse's plane of polarization changes the superradiance contribution to the decay rate from a value that is nearly maximum, Eq. (4.6) versus Eq. (4.3), to one that nearly vanishes.

In a real measurement, the count rate a photomultiplier detects is of the form of Eq. (4.7a) below but with $d\gamma_k$ essentially given by Eqs. (3.7) and (3.9) integrated over the viewing angle. The time dependence of the decay is given by $\rho_{kk}(t)$, Eq. (4.7c) below, which reflects the reduction in population due to all sources.

C. Free induction decay

Inspection of Eq. (3.10) shows the possibility of a beat at the rf Ω due to two-ion coherent emission, one ion emitting at the carrier frequency δ_0 and the other at one sideband $\delta_\pm 1$. This assumes the two ions are coherently prepared and properly aligned. The beat term is a maximum in forward scattering when the ions are aligned along the trap z axis with $\theta=0$ and the phase angle $\Phi=0$. The beat vanishes when the ions are in the radial plane of the trap, as in the preceding section, with $\theta=\pi/2$ and $\Phi=0$. In contrast, one ion cannot produce a beat because it cannot emit on two sidebands simultaneously, but radiates only on one sideband at a time, as a similar calculation shows.

The population decay rate of state $|k\rangle$ due to two ions on the z axis scattering light in the forward direction about $\theta=0$ in a differential cone angle $d\theta$ is

$$\left(\frac{d\rho_{kk}}{dt}\right)_f=-d\gamma_k\rho_{kk}(t), \tag{4.7a}$$

$$d\gamma_k=\tfrac{1}{2}d\theta^2\gamma[\cos(kR+\Phi)(\gamma'/\gamma)-2\sin(kR+\Phi)$$
$$\times J_0(z)J_1(z)\cos\Omega t+\cdots], \tag{4.7b}$$
$$\rho_{kk}(t)=\rho_{kk}(0)e^{-\gamma_k t}. \tag{4.7c}$$

The rate $d\gamma_k$ is the differential form of Eq. (3.9) summed over both ions and all lower states $|l\rangle$, assuming cohererent preparation. To order $d\theta^2$, only the circularly polarized transitions contribute, the linearly polarized being of higher order does not. The term $\rho_{kk}(t)$ is the solution to Eq. (3.15) with γ_k given by Eq. (3.17). Equations (4.7) show a damped modulation, the depth of modulation being typically 15%, in agreement with recent observations [2]. We note that this forward scattering is the most elementary form of optical free induction decay [38]. Previous studies involved a many-atom sample where the off-axis scattering is completely destructive while for two ions the off-axis interference is only partially destructive. In both cases the full coherence is preserved in forward scattering. Another difference is that the master equation is more fundamental than the phenomenological Bloch equations used in earlier theories.

D. Fringes

Spatial interference in forward scattering is predicted also. This is the transient analog of recent steady-state measurements [39] which we will use here as a model. Assume the ions are aligned along the y axis and are prepared by a pulse of linearly polarized light with $kR\gg 1$. The normalized scattering rate into a differential solid angle at a scattering angle θ is

$$S(\theta,R)=\tfrac{1}{2}\{\sin(\theta)(1+\cos^2\theta)+\sin^3(\theta)$$
$$\times[1-\tfrac{1}{2}\cos(kR\cos\theta)]\}e^{-\gamma t}. \tag{4.8}$$

The derivation follows from the differential form of the one- and two-ion decay rates and resembles that of Eq. (4.7). The correspondence between transient and steady-state solutions is discussed elsewhere [40] and results in an independent validation of this theory.

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