Algebraic approach to the Jaynes-Cummings models

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A so far most generalized Jaynes-Cummings (JC) model, which includes not only various ordinary JC models but also a special model describing spin-orbit interaction, is investigated from an algebraic point of view. We find that an alternative representation of su(2) algebra arises naturally from the system. This structure plays an essential role in the generalized system. As one result, the system behaves like a spin-$\frac{1}{2}$ system moving in a magnetic field. So it is rather easy to obtain the energy levels, eigenstates, and the time evolution operator, or any other quantities of physical interest. Two special cases are considered. Finally, a possible generalization to the three-level system is discussed.

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I. INTRODUCTION

Among the models describing the interaction between light and matter, the Jaynes-Cummings model (JCM) [1] seems to be ideal. It is simple enough to be exactly solved on the one hand and complicated enough to exhibit many fascinating quantum features on the other hand. These pure quantum effects include quantum collapse and revival of atomic inversion [2] and squeezing of the radiation field [3].

Recently superstructures, fraction revivals, and optical Schrödinger-cat states have been found in the JCM [4]. Also, this model can be tested experimentally, i.e., with the one-atom maser [5].

As a result many efforts have been made in order to generalize this model keeping these three advantages. In 1982 Singh discussed systematically some density-dependent and multiphoton interaction models [6]. The time evolution operators were given. In 1984 Sukumer and Buck studied the above models by using algebraic operator methods [7]. More recently, the JCM has been adopted with a Kerr nonlinearity [8], where the authors discovered classic beat phenomena under some suitable conditions. Later on, it was found by many authors that when the ordinary creation and annihilation operators in the JCM's were replaced by the q-deformed partners [9], namely, the q-deformed JCM's, these models can still be exactly solved ([10,11] and the references listed there). These systems also exhibit similar quantum features.

Naturally we tend to think that there must exist a fundamental structure that all these models have in common. In 1993 Bonatsos *et al.* after analyzing the methods involved in solving these systems, wrote in their paper [10]: "The similarities among the algebraic manipulations used in all the variations of the JC model suggest that there must be a unified formulation of the calculations and that all the forms of the model could be reformulated in a unified manner." They presented a unified formulation based on a generalized deformed oscillator algebra. All the models mentioned above can be included properly in their formulation. This is a great step, but we will see that this is not the last step, because a particular su(2) structure hidden in the system has not been revealed yet. This structure is a representation of su(2) algebra which can be constructed by combining the generalized deformed oscillator algebra with the Pauli matrices. We shall point out that it is this structure that makes all the JCM's akin. Surprisingly, a spin-orbit interaction can be included in our formulation too.

The purpose of this paper is to reveal this su(2) structure of JCM's. In Sec. II we generalize the quantized cavity field [12] based on slightly more generalized oscillator algebra than that of [10] and give the Hamiltonian of the generalized JCM. Additionally, an analysis of the energy levels of the system is given there. In Sec. III we construct a representation of su(2) algebra based on the generalized algebra and the Pauli matrices. The Hamiltonian is then simplified to an interaction between a spin-$\frac{1}{2}$ system and an external magnetic field. The evolution operator, the energy levels, and the eigenstates are given. In Sec. IV, we consider two special cases including a spin-orbit interaction. Finally there are some discussions and conclusions in Sec. V.

II. GENERALIZED JCM

Before presenting the Hamiltonian of the generalized JCM, we shall introduce the generalization of the quantized cavity field. To describe a quantized field, we need to know at least two things. One is the strength of the field or the number of its quanta. It can be described by a so-called number operator, say A_0, which is Hermitian. The other thing we must know is how to change this number. For a generalized

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cavity field, the processes of the changing are represented by creation and destruction operators $A_{\pm}$, which satisfy the following relations:

$$A_0 A_{\pm} = A_{\pm}(A_0 \pm m), \tag{1}$$

where m is a nonzero real number. We can see that A_+ increase the number of quanta by m and A_- decrease it by m. It can be expected that A_+A_-, representing a process in which we first destroy m quanta and then create m quanta, is a function of the number operator only: $A_+A_- = \chi(A_0)$. Since $A_-^{\dagger} = A_+$, the function χ is real, and we shall see later that the zero points of this function determine the spectra of the number operator. For the process where m quanta are created before they are destroyed, in order to be consistent with Eq. (1), we have $A_-A_+ = \chi(A_0+m)$. We recapitulate the requirements for these operators:

$$\begin{aligned} &[A_0, A_{\pm}] = \pm m A_{\pm}, \\ &A_+A_- = \chi(A_0), \\ &A_-A_+ = \chi(A_0+m), \\ &A_-^{\dagger} = A_+, \\ &A_0^{\dagger} = A_0, \end{aligned} \tag{2}$$

where m is a nonzero real constant and $\chi(A_0)$ is a well-defined real function of A_0. These three operators $\{A_0, A_+, A_-\}$ form a generalized deformed oscillator algebra [10].

An atom with two energy levels can be most conveniently described by the well-known Pauli matrices $\vec{\sigma}$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They have the following properties: $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, where $i,j,k = x,y,z$. We denote the eigenstates of σ_z as $|\uparrow\rangle$ and $|\downarrow\rangle$, corresponding to eigenvalues 1 and -1, respectively. Finally we define raising and lowering operators as

$$\sigma_{\pm} = \tfrac{1}{2}(\sigma_x \pm i\sigma_y), \tag{3}$$

which change the atom from one energy level to the other.

Using these definitions, we can now construct a Hamiltonian for a two-level atom interacting with a generalized cavity field as

$$H = r(A_0) + s(A_0)\sigma_z + A_-\sigma_+ + A_+\sigma_-, \tag{4}$$

where $r(A_0)$ and $s(A_0)$ are well-defined real functions of A_0. The first term is the energy function of the cavity field, which is certainly a function of the number operator. The second term characterizes the energy of the two-level atom. Usually $s(A_0)$ is a constant function of the number A_0. The interactions are given by the last two terms, which are related to the following two processes. First, the atom absorbs m quanta of the field and jumps from the lower energy level to the upper one. Secondly, the atom jumps from the upper energy level to the lower one and emits m quanta of the field.

This Hamiltonian has almost the same form as that of [10], but here we have the coupling constant, which does not need to be real, included in the operators $A_{\pm}$. We shall see in Sec. IV that more models are included in this Hamiltonian.

As in the ordinary JC model, there exists a constant operator

$$\Delta = A_0 + m\frac{1+\sigma_z}{2}, \tag{5}$$

which commutes with H. In fact, it commutes with both $A_+\sigma_-$ and $A_-\sigma_+$:

$$[\Delta, A_{\mp}\sigma_{\pm}] = 0.$$

Because of the projecting property of the operator $\sigma_+\sigma_-$, for an arbitrary function of Δ, $f(\Delta)$, we have the following decomposition:

$$f(\Delta) = f(A_0)\frac{1-\sigma_2}{2} + f(A_0+m)\frac{1+\sigma_2}{2}. \tag{6}$$

For an arbitrary function of A_0, for example, $F(A_0)$, we have the following identity:

$$F(A_0) = F(\Delta)\frac{1-\sigma_z}{2} + F(\Delta-m)\frac{1+\sigma_z}{2}. \tag{7}$$

The Hamiltonian can consequently be divided into two parts $H = H_0 + H_i$ by using Eq. (7). One part of it is a function of Δ only, $H_0 = \omega(\Delta)$, where $\omega(\Delta)$ is the so-called "field energy function"

$$\omega(\Delta) = \frac{r(\Delta-m)+r(\Delta)}{2} + \frac{s(\Delta-m)-s(\Delta)}{2}. \tag{8}$$

The other part has the following form: $H_i = \delta(\Delta)\sigma_z + A_+\sigma_+ + A_-\sigma_-$, in which $\delta(\Delta)$ is the so-called "detuning function"

$$\delta(\Delta) = \frac{s(\Delta-m)+s(\Delta)}{2} + \frac{r(\Delta-m)-r(\Delta)}{2}. \tag{9}$$

Since the eigenvalues of H_0 are determined only by the constant operator Δ, whose eigenvalues depend on those of A_0 and σ_z, we shall give a brief discussion of the eigenstates and eigenvalues of Δ, assuming $m>0$. The situation $m<0$ can be treated in a similar manner.

The eigenstates of A_0 are denoted by $|n\rangle$ with eigenvalues n taking values from n_0 to n_1 with a step width m. In all the JCM's investigated so far the number operator has no upper limit. In Sec. IV we shall give an example with an upper limit. We know from the definition Eq. (5) that the eigenvalues of Δ run from n_0 to n_1+m, increasing also with step width m. The eigenstates corresponding to the eigenvalues n $(\neq n_0, n_1+m)$ can be both $|n-m,\uparrow\rangle$ and $|n,\downarrow\rangle$, which are degenerate. In fact, the operator Δ has an SU(2) symmetry group, which will be discussed in the next section.

Among all the eigenvalues of Δ, there are only the two zero points of the function $\chi(n)$. All the others satisfy $\chi(n)>0$. In fact, if n is an eigenvalue of A_0 then we have $\chi(n)=\langle n|A_+A_-|n\rangle\geq 0$. So n is a zero point of$\chi(n)$ if and only if $A_-|n\rangle=0$. When $m>0$ and A_- is similar to the ordinary annihilation operator, we can see that among all the eigenvalues of A_0 there is only one zero point of the function $\chi(n)$, namely, n_0. For a similar reason, we obtain $\chi(n_1+m)=0$ because $A_+|n_1\rangle=0$. Thus the two zero points of the function $\chi(n)$ which are also the eigenvalues of Δ are n_1+m. The eigenstates corresponding to them are $|n_0,\downarrow\rangle$ and $|n_1,\uparrow\rangle$, respectively.

These two states are also the eigenstates of the total Hamiltonian H, Eq. (4). Because $(A_-\sigma_+ + A_+\sigma_-)|n_1,\uparrow\rangle=0$, and $(A_-\sigma_+ + A_+\sigma_-)|n_0,\downarrow\rangle=0$, we have

$$H|n_0,\downarrow\rangle=[\omega(n_0)-\delta(n_0)]|n_0,\downarrow\rangle, \tag{10}$$

$$H|n_1,\uparrow\rangle=[\omega(n_1+m)+\delta(n_1+m)]|n_1,\uparrow\rangle. \tag{11}$$

As a result, we now have to concentrate only on the subspace in which $\chi(\Delta)\neq 0$.

When $m<0$, the eigenvalues of Δ take values from n_0+m to n_1, and the corresponding states are $|n_0,\uparrow\rangle$ and $|n_1,\downarrow\rangle$, which are still eigenstates of the total Hamiltonian.

III. SU(2) STRUCTURE

In the previous section, we have mentioned that in the subspace in which n, the eigenvalues of Δ, satisfy $\chi(n)\neq 0$, there is an SU(2) symmetry of Δ. Now we shall construct this symmetry group. Taking into account the commutativities of Δ and $A_\mp\sigma_\pm$, we can define three operators as below:

$$\Sigma_1=\frac{1}{2\sqrt{\chi(\Delta)}}(A_-\sigma_+ + A_+\sigma_-), \tag{12}$$

$$\Sigma_2=\frac{i}{2\sqrt{\chi(\Delta)}}(A_+\sigma_- - A_-\sigma_+), \tag{13}$$

$$\Sigma_3=\tfrac{1}{2}\sigma_z. \tag{14}$$

It is easy to see that all these operators commute with Δ. From the definition of the generalized algebra [Eq. (2)] and Eq. (6), we find that $A_\pm$ satisfy the following commutation and anticommutation rules:

$$[A_-\sigma_+, A_+\sigma_-]=\chi(\Delta)\sigma_z, \tag{15}$$

$$\{A_-\sigma_+, A_+\sigma_-\}=\chi(\Delta). \tag{16}$$

By using the above properties it is straightforward to obtain the following commutation and anticommutation rules:

$$[\Sigma_i,\Sigma_j]=i\epsilon_{ijk}\Sigma_k, \tag{17}$$

$$\{\Sigma_i,\Sigma_j\}=\tfrac{1}{2}\delta_{ij}, \tag{18}$$

where $i,j,k=1,2,3$. These two rules can be combined:

$$(2\Sigma_i)(2\Sigma_j)=\delta_{ij}+i\epsilon_{ijk}(2\Sigma_k), \tag{19}$$

which are exactly the properties that Pauli matrices have. So these three operators $\vec{\Sigma}$ form a representation of su(2) algebra and the corresponding group is the symmetry group of Δ. That is, if $|n,\downarrow\rangle$ is an eigenstate corres- ponding to n, then the state $\exp(i\vec{\theta}\cdot\vec{\Sigma})|n,\downarrow\rangle$, where $\vec{\theta}=(\theta_1,\theta_2,\theta_3)$, is also an eigenstate with the same eigenvalue.

By using this representation of su(2) algebra, we cannot rewrite the Hamiltonian H_i as follows:

$$H_i=\vec{B}\cdot\vec{\Sigma}, \tag{20}$$

where $\vec{B}=2(\sqrt{\chi(\Delta)},0,\delta(\Delta))$ can be regarded as an effective magnetic field. We rewrite it as $2\Omega_\Delta(\sin\theta_\Delta,0,\cos\theta_\Delta)$, where the so-called Rabbi frequency is $\Omega_\Delta=\sqrt{\delta^2(\Delta)+\chi(\Delta)}$ and

$$\cos\theta_\Delta=\frac{\delta(\Delta)}{\Omega_\Delta},\quad \sin\theta_\Delta=\frac{\sqrt{\chi(\Delta)}}{\Omega_\Delta}. \tag{21}$$

This form of the Hamiltonian H_i expresses exactly an interaction between a spin-$\frac{1}{2}$ system represented by $\vec{\Sigma}$ and an effective magnetic field $\vec{B}$. As a result, all the methods and results for that interaction can be used directly here. This equivalence is supported experimentally by Ref. [13], in which the authors use an unpolarized neutron moving in a classical magnetic field to simulate the results of the JCM.

The evolution operator of the system has the following form:

$$U(t)=e^{-itH}=e^{-it\omega(\Delta)}U_i, \tag{22}$$

where

$$U_i=e^{-itH_i}=\cos(\Omega_\Delta t)+i\,\sin(\Omega_\Delta t)\frac{\vec{B}\cdot\vec{\Sigma}}{\Omega_\Delta}. \tag{23}$$

In the investigation of the properties of the JCM, the most important quantity is the time dependence of the quantum inversion or the expectation value of σ_z. So in the following we specify the rule that the operators $\vec{\Sigma}$ obey. We suppose that the operator at $t=0$ is $\vec{\Sigma}$; in Heisenberg's picture, at time t, the operator becomes

$$\vec{\Sigma}(t)=U(t)\vec{\Sigma}[U(t)]^\dagger=S(t)\vec{\Sigma}, \tag{24}$$

where the evolution $S(t)$ is

$$S(t)=\begin{pmatrix} 1-\cos^2\theta_\Delta(1-\cos2\Omega_\Delta t) & \cos\theta_\Delta\sin2\Omega_\Delta t & \cos\theta_\Delta\sin\theta_\Delta(1-\cos2\Omega_\Delta t) \\ -\cos\theta_\Delta\sin2\Omega_\Delta t & \cos2\Omega_\Delta t & \sin\theta_\Delta\sin2\Omega_\Delta T \\ \sin\theta_\Delta\cos\theta_\Delta(1-\cos2\Omega_\Delta t) & -\sin\theta_\Delta\sin2\Omega_\Delta T & 1-\sin^2\theta_\Delta(1-\cos2\Omega_\Delta t) \end{pmatrix}. \tag{25}$$

As a result we obtain the time dependence of Σ_3:

$$\Sigma_3(t)=\cos\theta_\Delta\sin\theta_\Delta(1-\cos2\Omega_\Delta t)\Sigma_1-\sin\theta_\Delta\sin(2\Omega_\Delta t)\Sigma_2 +[1-\sin^2\theta_\Delta(1-\cos2\Omega_\Delta t)]\Sigma_3 . \quad (26)$$

If we assume that the system is in the following state initially:

$$|\pi\rangle=\sum_{n_0\leq n\leq n_1}(w_n|n,\uparrow\rangle+v_n|n,\downarrow\rangle),$$

the quantum inversion is then

$$\langle\pi|\sigma_z(t)|\pi\rangle=\langle\pi|\sigma_z|\pi\rangle-2\sum_{n_0\leq n\leq n_1}\sin\theta_{n+m}\sin\Omega_{n+m}t \times(P_n\sin\Omega_{n+m}t+Q_n\cos\Omega_{n+m}t), \quad (27)$$

where

$$P_n=(|w_n|^2-|v_{n+m}|^2)\sin\theta_{n+m} -(v^*_{n+m}w_n+v_{n+m}w^*_n)\cos\theta_{n+m},$$

$$Q_n=i(v^*_{n+m}w_n-v_{n+m}w^*_n). \quad (28)$$

To obtain the energy levels and the eigenstates, we diagonalize the Hamiltonian H_i through the following transformation:

$$U=\exp(i\theta_\Delta\Sigma_2). \quad (29)$$

Then the total Hamiltonian is transformed to

$$H'=UHU^\dagger=\omega(\Delta)+2\Omega_\Delta\Sigma_3 . \quad (30)$$

The eigenstates of H' are obviously $|n-m,\uparrow\rangle$ corresponding to the energy level $E_{n,+}$ and $|n,\downarrow\rangle$ corresponding to the energy $E_{n,-}$ where

$$E_{n,\pm}=\omega(n)\pm\Omega_n . \quad (31)$$

These are of course the energy levels of H too.

The eigenstates of H can be obtained by applying the transformation U on the eigenstates of H'. For energy $E_{n,+}$ and $E_{n,-}$ the eigenstates are

$$|n,+\rangle=U^\dagger|n-m,\uparrow\rangle=\cos\tfrac{1}{2}\theta_n|n-m,\uparrow\rangle-\sin\tfrac{1}{2}\theta_n|n,\downarrow\rangle, \quad (32)$$

$$|n,-\rangle=U^\dagger|n,\downarrow\rangle=\sin\tfrac{1}{2}\theta_n|n-m,\uparrow\rangle+\cos\tfrac{1}{2}\theta_n|n,\downarrow\rangle, \quad (33)$$

where $n_0<n<n_1+m$.

IV. SPECIAL CASES

In this section we turn to two special models. For the first examples we consider a generalized JCM which is called the density-dependent multiphoton JCM ([6,10] and the references listed there). The Hamiltonian reads

$$H=\omega a^\dagger a+\omega_1\sigma_z+g\rho(N)a^{\dagger k}\sigma_-+g^*\sigma_+a^k\rho(N), \quad (34)$$

where g is a coupling constant, which may be a complex number. When $k=1$ and $\rho(N)=1$, it becomes the original Jaynes-Cummings model.

Obviously this model is based on the generalized oscillator algebra generated by $\{N,\rho(N)a^{\dagger k},a^k\rho(N)\}$, where $N=a^\dagger a$ and $[a,a^\dagger]=1$ are the ordinary number operator and the commutation rule of the normal creation and annihilation operators. $\rho(N)$ is an arbitrary function of N. Comparing with the Hamiltonian Eq. (4), we obtain the following correspondences:

$$A_0\mapsto N,$$

$$A_+\mapsto g\rho(N)a^{\dagger k}, \quad (35)$$

$$A_-\mapsto g^*a^k\rho(N).$$

In this case we have $m=k>0$, and

$$\chi(N)=|g|^2\rho^2(N)\frac{N!}{(N-k)!} .$$

The constant operator becomes

$$\Delta=N+k\frac{1+\sigma_z}{2} .$$

Its eigenvalue has the form n_0+jk, where n_0 is a nonnegative integer smaller than k and $j=0,1,\ldots$. Since $r(N)=\omega N$ and $s(N)=\omega_1$, the energy function is

$$\omega(\Delta)=\omega(\Delta-\tfrac{1}{2}k),$$

and the detuning function is

$$\delta(\Delta)=\omega_1-\tfrac{1}{2}k.$$

In this model there is no upper limit of the number operator, so $|n_0,\downarrow\rangle$ is the only nongenerate eigenstate of Δ. This state is also the eigenstate of the Hamiltonian Eq. (34) which is related to the energy level ω_1. The other eigenvalues and eigenstates can be obtained by using the formulas of the previous section.

The second example comes from the interaction between a hydrogenlike atom and an external magnetic field [14]. The total Hamiltonian is

$$H=\frac{P^2}{2m}+V_c(r)+H_1,\quad H_1=\alpha\vec{L}\cdot\vec{S}+\beta(L_z+2S_z), \quad (36)$$

where the spin operator $\vec{S}$ differs from the Pauli matrix by a factor $\frac{1}{2}$ and the operators $\{L_x,L_y,L_z\}$ generate algebra so (3):

$$[L_i,L_j]=i\epsilon_{ijk}L_k . \quad (37)$$

The coupling constant α depends on the central potential $V_c(r)$ only and β depends only on the strength of the external magnetic field. Here we concentrate on the interaction H_1 only.

The first term of H_1 is a spin-orbit interaction and the second term is an interaction of a magnetic field with the angular momentum. To obtain the eigenvalue of the system, there are commonly two approximations. One is the Zeeman approximation, when the magnetic field is so weak that the second term can be regarded as a perturbation. The other one is the Paschen-Back limit: the magnetic field is so intense that the first term can be treated as a perturbation. In the following we shall see that this model is just a special case of our generalized JCM. As a result, we can give the quantities of physical interest exactly.

Denoting $L_{\pm}=L_x \pm iL_y$, we rewrite the interaction as follows:

$$H_1=\beta L_z+(\tfrac{1}{2}\alpha L_z+\beta)\sigma_z+\tfrac{1}{2}\alpha(L_-\sigma_+ + L_+\sigma_-). \quad (38)$$

Comparing with the Hamiltonian Eq. (4), we find the following correspondence:

$$\begin{aligned} A_0 &\mapsto L_z, \\ A_- &\mapsto \tfrac{1}{2}\alpha L_+, \\ A_- &\mapsto \tfrac{1}{2}\ \alpha L_-. \end{aligned} \quad (39)$$

In this case we have $m=1$ and

$$\chi(L_z)=\frac{\alpha^2}{4}[L^2-L_z(L_z-1)],$$

where $L^2=L_x^2+L_y^2+L_z^2=l(l+1)$, and here

$$r(L_z)=\beta L_z, \quad (40)$$

$$s(L_z)=\tfrac{1}{2}\alpha L_z+\beta. \quad (41)$$

The energy function and detuning function are

$$\omega(\Delta)=\beta(\Delta-\tfrac{1}{2})-\frac{\alpha}{4}, \quad \delta(D)=\tfrac{1}{2}\alpha(\Delta-\tfrac{1}{2})+\tfrac{1}{2}\beta. \quad (42)$$

The Rabi frequency is

$$\Omega_\Delta=\sqrt{(\beta^2)/4+\alpha^2/4(l+\tfrac{1}{2})^2+\tfrac{1}{2}\alpha\beta(\Delta-\tfrac{1}{2})}. \quad (43)$$

So the energy levels are

$$E_{\Delta,\pm}=\beta(\Delta-\tfrac{1}{2})-\frac{\alpha}{4}\pm\Omega_\Delta. \quad (44)$$

Here we have come to a situation different from the first one: the eigenvalues of operator $A_0=L_z$ are bounded between $-l$ and l, because of the conditions $\chi(L_z)\geq 0$ and $\chi(L_z+1)\geq 0$. Since the constant operator is now the total momentum of the system plus a constant $\frac{1}{2}$,

$$\Delta=L_z+\frac{1+\sigma_z}{2}=L_z+S_z+\tfrac{1}{2}, \quad (45)$$

its eigenvalue n consequently runs from $-l$ to $l+1$. When $n=-l$ and $n=l+1$, the eigenstates of the Hamiltonian H_1 are $|-l,\downarrow\rangle$ and $|l,\uparrow\rangle$, respectively. For the other values of n, the eigenstates are expressed in Eqs. (32) and (33).

We now suppose that the orbital angular momentum of the system is $\frac{1}{2}$ and the external magnetic field is zero, that is, $\beta=0$. Here it is a case that two spin-$\frac{1}{2}$ systems interact with one another by the Hamiltonian Eq. (36). Initially, we assume that the system is in the state

$$|\pi\rangle=\tfrac{1}{2}(|-\rangle+e^{i\gamma}|+\rangle)(|\uparrow\rangle+|\downarrow\rangle) \ . \quad (46)$$

We can determine the phase difference of the orthogonal states $|-\rangle$ and $|+\rangle$, by simply measuring the quantum inversion. In fact, by using Eq. (28) we have

$$\langle\pi|\sigma_z(t)|\pi\rangle=\tfrac{1}{2}\sin\gamma\ \sin\alpha t, \quad (47)$$

in which the amplitude of the oscillation with time is just the sine of the phase difference γ.

V. CONCLUSIONS AND DISCUSSIONS

We have unified all the JCM's based on a fundamental SU(2) structure. After the revealing of this structure, the Hamiltonian of the system is simplified to an interaction of a spin-$\frac{1}{2}$ system and an effective static magnetic field. This is the reason why various kinds of JCM's can be exactly solved. Quite unexpectedly, a model of spin and orbit interaction is also included in our formulation.

In addition, it has the following two consequences. First, our generalized Hamiltonian can be generalized to a time-dependent interaction. For example, since the spin-$\frac{1}{2}$ system in rotational magnetic fields has been thoroughly studied, when the coupling constant depends on time according to a proper function making the effective magnetic fields rotate with time, the system remains to be solved. Second, since when the two-level atom interacts with a generalized quantized cavity field an SU(2) structure can be extracted, we expect to find an SU(3) structure behind the interaction between a three-level system and a cavity field. It will be discussed in a forthcoming presentation.

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