Geometric aspects of noncyclic quantum evolutions

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The geometric phase is defined for any arbitrary quantum evolution using a "reference section" of the bundle covering the curve in the projective Hilbert space. A canonical one-form is defined whose line integral gives the desired geometric phase. It is manifestly gauge, phase, and reparametrization invariant for all quantum evolutions. A simple proof of the vanishing nature of the geometric phase along the geodesic is given. Also, an elementary proof of the nonadditive nature of the geometric phase is given. In the limit of cyclic evolution of a pure quantum state, this phase reduces to the Aharonov and Anandan phase, precisely. It is observed that in addition to the geometric phase, other geometric structures exist, such as the "length" and "distance" during any arbitrary quantum evolution. The relations among all of these geometric quantities are pointed out. Finally, two simple examples are studied to illustrate the ideas introduced in this paper.

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I. INTRODUCTION

The search for admissible geometric structures during a cyclic evolution of a quantum system arose after Berry's [1] discovery of a nontrivial phase factor in adiabatic, cyclic, and parametric variations of a quantal system. Here, by geometric structures (in brief), we mean those that do not depend either on the details of the time dependence of the parameters by whose use the evolution curve is parametrized or on the phases of the wave function, and are independent of the detailed dynamics fixed by the particular Hamiltonian (but reflect the properties of the infinite number of possible Hamiltonians). Berry's phase is one of such nature. A nice interpretation was given by Simon [2] in terms of a natural Hermitian connection, as the parallel transport holonomy in a Hermitian line bundle. Aharonov and Anandan (AA) [3] showed the existence of a geometric phase irrespective of the adiabatic, cyclic variations of the parameters of the Hamiltonian. The AA phase was understood as the holonomy transformation for parallel transporting around a closed curve $\hat{C}$ in $\mathscr{P}$, with respect to the natural connection given by the inner product in $\mathscr{H}$ [4]. Anandan has interpreted this phase as the "area" of any surface spanned by $\hat{C}$ with respect to the natural symplectic structure in $\mathscr{P}$ determined by the inner product in $\mathscr{H}$. The present author [5] has given an interpretation of the AA phase as an integral of the contracted length of the curve traced by the single-valued vector during a cyclic evolution of the quantum system. Giving up the cyclicity condition, Samuel and Bhandari (SB) [6] were able to extract the geometric phase even for nonunitary evolutions of the Schrödinger type governed by a non-Hermitian Hamiltonian. However, the geometric phase of SB is an indirect definition since it rests on implicitly closing the initial and the final points of the open path by a geodesic. If the end points of the open path are not closed, the SB phase is not manifestly gauge invariant. Recently, Sudarshan, Anandan, and Govindarajan [7] obtained a geometric phase for an arbitrary infinitesimal triangle in the projective Hilbert space $\mathscr{P}$ using group-theoretic techniques. Aitchison and Wanelik [8] have redefined the SB phase in a nondynamical way. Recently, the noncyclic, nonunitary and non-Schrödinger geometric phase was derived by the present author by introducing the elementary ideas of paths and their lengths in the projective Hilbert space only [9]. Mukunda and Simon [10] have given a kinematic approach to the theory of geometric phases in general. For details on the subject of the geometric phase and its various applications, one is advised to see Ref. [11]. (However, the recent developments in the area of geometric phases are not discussed there.)

The present paper aims at studying some geometric aspects (such as the geometric phase, "distance," and "length") of the noncyclic evolutions of quantum systems. This paper is organized as follows. In Sec. II, we briefly spell out the meaning of the noncyclic evolution and review the existing definition of the geometric phase for Schrödinger evolutions. In Sec. III, we set up the geometrical skeleton to define the geometric phase for the noncyclic evolution of the Schrödinger type. We define a "reference section" of the bundle covering the curve in the projective Hilbert space $\mathscr{P}$ and obtain a connection one-form whose line integral gives the desired geometric phase. Various properties of the geometric phase are pointed out. It is shown that the geometric phase is not only phase invariant, but also gauge invariant. Its nonadditive nature is explicitly proven. Also, it is proven that the geometric phase vanishes along a geodesic. In Sec. IV, we introduce other geometric structures such as the length and distance for all quantum evolutions. We provide a topological reason for the appearance of the geometric phase based on an inequality between the length and the distance. In Sec. V, we calculate the geometric phase for a two-level atom and harmonic oscillator state, which is initially prepared in a coherent state. In the last section, we generalize the geometric phase for the nonunitary and non-Schrödinger evolutions, and the conclusion follows.

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II. NONCYCLIC GEOMETRIC PHASE DURING A SCHRÖDINGER EVOLUTION

Let $\{\Psi\}$ be a set of vectors in $\mathscr{H}=\mathbb{C}^{n+1}$ and $\{\Psi/||\Psi||\}$ be a set of vectors of norm one in $\mathscr{L}$. The state of a quantum system is determined by a ray of the Hilbert space $\mathscr{H}$. The set of rays of $\mathscr{H}$ is called the projective Hilbert space $\mathscr{P}$. The projection map $\Pi:\mathscr{L}\rightarrow\mathscr{P}$ is a principal fiber bundle $\mathscr{L}(\mathscr{P}, \mathrm{U}(1),\Pi)$, with structure group U(1). This can be seen by considering the action of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers on the space $\mathbb{C}^{n+1}-\{0\}$ given by the equivalence relation $(z_1,z_2,\ldots,z_{n+1})\lambda:=(z_1\lambda,z_2\lambda,\ldots,z_{n+1}\lambda)$, $\forall\lambda\in\mathbb{C}^*$. This is a free action and the orbit space in the space $\mathbb{C}\mathscr{P}^n$ of complex lines in the Hilbert space $\mathscr{H}=\mathbb{C}^{n+1}$. Thus, we get the principal bundle $\mathbb{C}^*\rightarrow\mathbb{C}^{n+1}-\{0\}\rightarrow\mathbb{C}\mathscr{P}^n=\mathscr{P}$, in which the projection map associates with each $(n+1)$-tuple $(z_1,z_2,\ldots,z_{n+1})$ the point in $\mathbb{C}\mathscr{P}^n$ with the homogeneous coordinates $(z_1,z_2,\ldots,z_{n+1})$. Thus, a quantum state at a given instant of time is represented by a point in $\mathscr{P}$ and the evolution of the system is given by a curve Γ in $\mathscr{H}$, which projects to a curve $\hat{\Gamma}=\Pi(\Gamma)$ in $\mathscr{P}$.

First, we consider the unitary Schrödinger evolution of a quantum system and exhibit how to separate out the geometric phase for an open path in $\mathscr{P}$, without explicitly closing the initial and the final points by a geodesic. In the subsequent section, we will sketch how to generalize our expression for nonunitary and non-Schrödinger evolutions of a quantal system. Let $|\Psi(t)\rangle\in\mathscr{H}$ be the state vector of a quantum system and it evolves according to the Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle=H(t)|\Psi(t)\rangle, \tag{1}$$

where $H(t)$ is the Hamiltonian of the system. Contrary to the case of a cyclic evolution, in the case of noncyclic evolution we can not, in general, factor out the total phase from the initial-state vector, i.e., $|\Psi(t)\rangle\neq\exp(i\Phi)|\Psi(0)\rangle$, during an evolution from $t=0$ to t. The projection of the open curve $\Gamma:t\rightarrow|\Psi(t)\rangle$ is $\Pi(\Gamma)=\hat{\Gamma}$ and it lies in $\mathscr{P}$. The curve $\hat{\Gamma}$, in general, is an open path and a nonzero distance is maintained between the points $\Pi(|\Psi(0)\rangle)$ and $\Pi(|\Psi(t)\rangle)$. In terms of the minimum-normed distance function [5], the cyclic evolutions are those for which $D^2(\Psi(0),\Psi(t))=[2-2|\langle\Psi(0)|\Psi(t)\rangle|]=0$ and (strictly) noncyclic evolutions are those for which $D(\Psi(0),\Psi(t))>0$. In the case of noncyclic evolution, the open path $\hat{\Gamma}$ is lifted to $\mathscr{H}$, then the initial and the final points correspond to two different rays. (Whereas for cyclic evolution, the initial and final points belong to the same ray.) To compare the phases of the state vectors belonging to two different rays, we use the Pancharatnam [12] connection. While dealing with the interference of light, Pancharatnam came up with a brilliant idea that is simple and physically motivated. If the system evolves from an initial state $|\Psi_0\rangle=|\Psi(0)\rangle$ to a final state $|\Psi\rangle=|\Psi(t)\rangle$, the relative phase difference between them is given by

$$e^{i\Phi_P}=e^{i[\Phi]_0^t}=\frac{\langle\Psi(0)|\Psi(t)\rangle}{|\langle\Psi(0)|\Psi(t)\rangle|}. \tag{2}$$

Here all we require is that the initial state and the final state should not be orthogonal. If $\langle\Psi(0)|\Psi(t)\rangle$ is real and positive, then the quantum system does not acquire any phase during an evolution from time $t=0$ to t (say). This is the well known Pancharatnam connection. It should be stressed that the phase difference given by (2) is true, in general, irrespective of closing the initial and the final points; it is the total phase [9,12] acquired by a quantum system during an arbitrary quantum evolution between $[0,t]$. (Here after we will not mention that the evolution is from time $t=0$ to t.) Thus, the total phase is given by

$$\Phi_{\mathrm{total}}=\Phi_P=[\Phi]_0^t=\arg\langle\Psi(0)|\Psi(t)\rangle. \tag{3}$$

Equivalently, we can express the total phase as

$$[\Phi]_0^t=\arctan\left(\frac{\mathrm{Im}\langle\Psi_0|\Psi\rangle}{\mathrm{Re}\langle\Psi_0|\Psi\rangle}\right), \tag{3a}$$

where $\langle\Psi(0)|\Psi(t)\rangle=\mathrm{Re}\langle\Psi_0|\Psi\rangle+i\ \mathrm{Im}\langle\Psi_0|\Psi\rangle$. The total phase is insensitive to any changes in the modulus of the inner product $\langle\Psi(0)|\Psi(t)\rangle$, namely, $|\langle\Psi(0)|\Psi(t)\rangle|=R$ (say). Thus, we can say that the total phase is R invariant.

For any arbitrary quantum evolution, the geometric phase is the difference between the total phase and the dynamical phase (where the dynamical phase is given by the time integral of the expectation value of the Hamiltonian). Mukunda and Simon have expressed the geometric phase as [10]

$$[\Phi_g]_0^t=[\Phi]_0^t+(1/\hbar)\int_0^t\langle\Psi(t)|H(t)|\Psi(t)\rangle dt. \tag{4}$$

But then, the calculation of the geometric phase from the above expression (4) requires the knowledge of the Hamiltonian of the quantum system. We intend to given an expression that will be independent of the detail dynamics of the system and bring out its full geometric nature by showing its dependence uniquely only on the image of the curve Γ in the projective Hilbert space $\mathscr{P}$. To provide such an expression, we consider the following geometrical setups.

III. REFERENCE SECTION AND THE NONCYCLIC GEOMETRIC PHASE

Let $\Pi:\mathscr{L}\rightarrow\mathscr{P}$ be a principal bundle with structure group U(1) and $T_{|\Psi\rangle}\mathscr{L}$ be the tangent space to $\mathscr{L}$ at point $|\Psi\rangle$, and $T\mathscr{L}=U_{|\Psi\rangle}T_{|\Psi\rangle}\mathscr{L}$ is the tangent bundle. Let $\Gamma:t\rightarrow|\Psi(t)\rangle$ be a smooth curve mapping the closed interval $[0,t]\subset\mathbb{R}$ into $\mathscr{L}$ and $(d/dt)|\Psi(t)\rangle$ is the tangent vector to the curve Γ at point $|\Psi(t)\rangle$. Now any tangent vector $(d/dt)|\Psi(t)\rangle$ can be decomposed [13–15] uniquely into a sum of vertical and horizontal components lying in $V_{|\Psi\rangle}\mathscr{L}$ and $H_{|\Psi\rangle}\mathscr{L}$, where

$$V_{|\Psi\rangle}\mathscr{L}:=\left\{\frac{D}{dt}\bigg|\Psi(t)\right\rangle\in T_{|\Psi\rangle}\mathscr{L}\left|\Pi_*\frac{D}{dt}\bigg|\Psi(t)\right\rangle=0\right\} \tag{5}$$

and

$$H_{|\Psi\rangle}\mathscr{L}:=\left\{\frac{\delta}{dt}\bigg|\Psi(t)\right\rangle\in T_{|\Psi\rangle}\mathscr{L}\ |\langle\Psi(t)|\delta\Psi(t)/dt\rangle=0\bigg\}. \tag{6}$$

Thus, a curve is called vertical if the projection of the point $|\Psi(t)\rangle$ in $\mathscr{P}$ is constant, and the tangent to a vertical curve is called a vertical vector. A horizontal vector is one that is perpendicular to the fiber at that point. The set of horizontal vectors at a given point in $\mathscr{L}$ is called the horizontal space at that point. To obtain the geometric phase, it is necessary to define a connection. A connection [13] in a principal bundle $U(1)\rightarrow\mathscr{L}\rightarrow\mathscr{P}$ is a smooth assignment to each point $|\Psi\rangle \in\mathscr{L}$ of a subspace $H_{|\Psi\rangle}\mathscr{L}$ of $T_{|\Psi\rangle}\mathscr{L}$ such that (i) $T_{|\Psi\rangle}\mathscr{L} \approx V_{|\Psi\rangle}\mathscr{L}\oplus H_{|\Psi\rangle}\mathscr{L}$ for all $|\Psi\rangle\in\mathscr{L}$, and (ii) $\delta_c^*(H_{|\Psi\rangle}\mathscr{L}) = H_{c|\Psi\rangle}\mathscr{L}$ for all $c\in U(1)$, $|\Psi\rangle\in\mathscr{L}$ with $c^*c=1$, where $\delta_c(|\Psi\rangle)=c|\Psi\rangle$ denotes the induced action of the phase c on $\mathscr{L}$.

With the help of the above-mentioned connection, we can now describe the parallel transport of a vector in a principal bundle. Consider the horizontal lift of the open curve $\hat{\Gamma}$ in $\mathscr{P}$ as a curve $\bar{\Gamma}:[0,t]\rightarrow\mathscr{L}$, which is horizontal (i.e., vert$[\Gamma]=0$), and such that $\Pi(\bar{\Gamma})=\hat{\Gamma}(t)$ for all time and is, in general, open in $\mathscr{L}$. Let $\bar{\Gamma}:t\rightarrow|\bar{\Psi}(t)\rangle$ be the horizontal lift of the curve $\hat{\Gamma}$, such that

$$\langle\bar{\Psi}(t)|d\bar{\Psi}(t)/dt\rangle=0. \tag{7}$$

Then $|\bar{\Psi}(t)\rangle$ is said to undergo parallel transport with respect to this connection [3,4,14]. It can be seen that if $|\Psi(t)\rangle$ undergoes the time evolution according to (1), with a Hamiltonian $H(t)$, along a curve Γ then

$$|\bar{\Psi}(t)\rangle=\exp\left[(i/\hbar)\int_0^t\langle\Psi(t')|H(t')|\Psi(t')\rangle dt'\right]|\Psi(t)\rangle \tag{8}$$

obeys (7). In terms of the horizontal vector $|\bar{\Psi}(t)\rangle$, the geometric phase factor would be given by

$$e^{i[\Phi g]_0^t}=\frac{\langle\bar{\Psi}(0)|\bar{\Psi}(t)\rangle}{|\langle\bar{\Psi}(0)|\bar{\Psi}(t)\rangle|}. \tag{9}$$

Note that in the case of a cyclic evolution, (9) reduces to $\exp(i[\Phi g]_0^T)=\exp[i\beta(C)]=\langle\bar{\Psi}(0)|\bar{\Psi}(T)\rangle$, where $\beta(C)$ is the AA phase, which is nothing but the holonomy transformation for parallel transporting the vector around a closed loop. (This is because $|\bar{\Psi}(T)\rangle=\exp[i\beta(C)]|\bar{\Psi}(0)\rangle$ for cyclic evolutions of quantum systems.) Therefore, Eq. (9) can be regarded as the generalized holonomy transformation law for parallel transporting the vector along an open path in $\mathscr{P}$.

If we lift an open path in $\mathscr{P}$ to $\mathscr{L}$, there may be many open curves in $\mathscr{L}$. But there exists one special curve, which is traced out by a "reference state." This reference state is defined with respect to the initial state vector $|\Psi(0)\rangle$ and with the help of this we will soon define the geometric phase for noncyclic evolutions of quantum systems. Furthermore, in Sec. IV, we will see that the geometric phase arises because of a fundamental inequality between the length of the horizontal curve and the length of the above-mentioned special, but nonhorizontal curve.

To define this curve, we consider a "reference section" $|\chi_0(t)\rangle$ of the bundle covering $\rho(t)=\Pi(|\Psi(t)\rangle)$. This is a map $s:\mathscr{P}\rightarrow\mathscr{L}$ such that the image of each point $\rho(t)\in\mathscr{P}$ lies in the fiber $\Pi(\rho)$ over ρ, i.e., $\Pi\circ s=id_p$. We define the new "reference section" (with respect to the initial point) as a mapping of the state curve $\hat{\Gamma}$ through the section s and is given by

$$|\chi_0(t)\rangle=\frac{\langle\Psi(t)|\Psi(0)\rangle}{|\langle\Psi(t)|\Psi(0)\rangle|}|\Psi(t)\rangle \tag{10}$$

or

$$|\chi_0(t)\rangle=\xi(t,0)|\Psi(t)\rangle.$$

Here the subscript 0 in $|\chi(t)\rangle$ means that it is always defined with respect to the initial state $|\Psi(0)\rangle$. [If we denote the initial state as $|\Psi(t_1)\rangle$, then there will be a subscript t_1 in $|\chi(t)\rangle$, i.e., the "reference section" will be denoted as $|\chi_{t_1}(t)\rangle$.] The "reference section" $|\chi_0(t)\rangle$ defined above is a "local section" in the fiber bundle $\mathscr{L}$ (i.e., $s:U_\alpha\subset\mathscr{P}\rightarrow\mathscr{L}$, where U_α is an open neighborhood of $\mathscr{P}$ and has the following properties.

(i) $s\Pi(|\Psi(0)\rangle)=|\Psi(0)\rangle=|\chi_0(0)\rangle$. This means that the mapping s sends $\Pi(|\Psi(0)\rangle)\in\mathscr{P}$ to $|\Psi(0)\rangle$. The initial vector $|\Psi(0)\rangle$ and the initial reference section $|\chi_0(0)\rangle$ begin at the same point in the same ray.

(ii) $\Pi(|\chi_0(t)\rangle)=\Pi(|\Psi(t)\rangle)$ for all time during the evolution. This means that the curves $\Gamma_0(t)$ and $\Gamma(t)$ project to the same open curve.

(iii) $\langle\chi_0(0)|\chi_0(t)\rangle$ is real and positive for all time during the evolution of the quantum system. This means that at any later time t, the evolving "reference section" $|\chi_0(t)\rangle$ remains always in phase with the initial "reference section" $|\chi_0(0)\rangle$, in accordance with the Pancharatnam condition.

Among all the properties mentioned above, only (ii) will be satisfied by all local "sections." From the mathematical definition of the local section, there is no requirement for a local section to satisfy either condition (i) or (iii). However, what we have defined is not just a local section, but rather a local reference section. Thus, our careful insight into the problem has led us to define the above reference section with extra conditions (i) and (iii). Here, we emphasize that those reference sections, that satisfy all three conditions can only be used to define the geometric phase for noncyclic evolutions; subsequently, we will see that the condition (iii) ensures the gauge invariance of the geometric phase.

This "reference section" of the bundle covering the curve $\hat{\Gamma}$ in $\mathscr{P}$ will be used to define the geometric phase for any arbitrary quantum evolution. In this sense, Eq. (10) is an important step in this paper. Now it is easy to see that the following equation is obtained on differentiating (10) and upon sandwiching with $|\chi_0(t)\rangle$:

$$i\langle\chi_0(t)|\dot{\chi}_0(t)\rangle dt=i\xi^*(t,0)\frac{d\xi}{dt}(t,0)dt+(1/\hbar)\times\langle\Psi(t)|H(t)|\Psi(t)\rangle dt. \tag{11}$$

On integrating both sides and realizing that the first term on the right-hand side is the total phase, i.e.,

$$\int_0^t i\xi^*(t',0)\frac{d\xi}{dt'}(t',0)dt'=[\Phi]_0^t=\arg\langle\Psi(0)|\Psi(t)\rangle \tag{12}$$

(in fact, this is another important step in our derivation), we have the desired expression for the geometric phase during an arbitrary quantum evolution as

$$[\Phi_g]_0^t = i\int_{\hat{\Gamma}} \langle \chi_0(t)|\dot{\chi}_0(t)\rangle dt. \tag{13}$$

[Hereafter we do not put explicitly the limits of the integrations. Unless otherwise stated, it is understood that the integrations are from an initial point $\Pi(|\Psi_0\rangle)$ to a final point $\Pi(|\Psi\rangle)$.]

The geometric phase defined above has the following properties. It is independent of the choice of the phase of the state $|\Psi(t)\rangle$ of the quantum system. Because under a phase transformation $|\Psi(t)\rangle \to e^{i\alpha(t)}|\Psi(t)\rangle$, however, the reference section $|\chi_0(t)\rangle$ undergoes a global phase transformation by some fixed amount, i.e., $|\chi_0(t)\rangle \to e^{i\alpha(0)}|\chi_0(t)\rangle$ and hence the geometric phase remains the same. It is also reparametrization invariant, i.e., by changing the parameter from t to t' with $dt/dt'>0$, the geometric phase remains unaltered and hence it is a property of only the unparametrized path $\hat{\Gamma}$ in $\mathscr{P}$. In addition to this, the real geometric phase is independent of the particular Hamiltonian used to evolve the quantum system along a given curve Γ in $\mathscr{L}$; rather, it depends uniquely only on the curve $\hat{\Gamma}$ in $\mathscr{P}$. In Sec. IV, we will discuss some other geometric structures during a noncyclic evolution, which will respect the above-mentioned properties also. Therefore, we can say that these properties constitute the set of properties that characterize the geometric nature of some structures associated with any arbitrary quantum evolution.

We can also express (13) as an integral over a connection one-form as

$$[\Phi_g]_0^t = i\int_{\hat{\Gamma}} \langle \chi_0(\lambda)|\partial_\mu|\chi_0(\lambda)d\lambda^\mu = \int_{\hat{\Gamma}} \omega_\mu(\lambda)d\lambda^\mu$$
$$= \int_{\hat{\Gamma}} \omega(\lambda), \tag{14}$$

where $\partial_\mu = \partial/\partial\lambda^\mu$ and λ's are coordinates in the projective Hilbert space $\mathscr{P}$. $\hat{\Gamma}$ is a curve in the projective Hilbert space connecting the initial point $\Pi(|\Psi_0\rangle)$ and the final point $\Pi(|\Psi\rangle)$. Here, $\omega_\mu(\lambda) = i\langle \chi_0(\lambda)|\partial_\mu \chi_0(\lambda)\rangle$ is the connection form whose line integral gives the geometric phase. It is to be noted that this line integral cannot be converted to a surface integral using a Stoke's theorem (since, in general, we do not have a closed path in the projective Hilbert space). However, if we join the end points of the evolution curve by the shortest geodesic, then we can convert the line integral to a surface integral.

Next, we show that the geometric phase for noncyclic evolution is manifestly gauge invariant. In fiber bundle language [13], a gauge transformation is a mapping from one local section $(s: U_\alpha \subset \mathscr{P} \to \mathscr{L})$ to another $(s': U'_\alpha \subset \mathscr{P} \to \mathscr{L})$, which results from the intersection of two different coordinate neighborhoods in the projective space $\mathscr{P}$. Another choice of the local section means we change $|\chi_0(t)\rangle \to |\chi_0(t)'\rangle$, where $|\chi_0(t)'\rangle = e^{i\eta(t)}|\chi_0(t)\rangle$ and the change in the section is given by the structure group U(1), which acts on the fibers [14,15]. Under this gauge transformation, the connection one-form $\omega_\mu(\lambda)$ transforms as

$$\omega_\mu(\lambda) \to \omega'_\mu(\lambda) = \omega_\mu(\lambda) - \partial_\mu \eta(\lambda). \tag{15}$$

The local section $|\chi_0(t)\rangle$ maps an open path $\hat{\Gamma}$ in $\mathscr{P}$ into an open path Γ_0 in $\mathscr{L}$ in such a way that the initial and final points of the path Γ_0 are in phase. [Recall the Pancharatnam condition, i.e., $\langle \chi_0(0)|\chi_0(t)$ is real and positive.] The above-mentioned gauge transformation gives a different open path $\Gamma'_0: t \to |\chi_0(t)'\rangle$ in $\mathscr{L}$. In order for $|\chi_0(t)'\rangle$ to be a local reference section, we require the Pancharatnam condition [namely, $\langle \chi'_0(0)|\chi_0(t)'\rangle$ is real and positive] to be satisfied on the path Γ'_0. This gives the relation on the gauge function $\eta(t)$ as $\eta(t) = \eta(0) + 2\pi n$, since the phase angle is only invariant mod 2π. [For cyclic evolution, we would require $\eta(T) = \eta(0) + 2\pi n$, where n is an integer.] Under this gauge transformation, the geometric phase transforms as

$$[\Phi_g]_0^t = i\int_{\hat{\Gamma}} \langle \chi_0(t)|\dot{\chi}_0(t)\rangle dt \to i\int_{\hat{\Gamma}} \langle \chi'_0(t)|\dot{\chi}_0(t)'\rangle dt$$
$$= i\int_{\hat{\Gamma}} \langle \chi_0(t)|\dot{\chi}_0(t)\rangle dt - [\eta(t) - \eta(0)]. \tag{16}$$

Using the condition on the gauge function, we have

$$i\int_{\hat{\Gamma}} \langle \chi'_0(t)|\dot{\chi}_0(t)'\rangle dt = i\int_{\hat{\Gamma}} \langle \chi_0(t)|\dot{\chi}_0(t)\rangle dt, \quad \text{mod } 2\pi. \tag{16a}$$

This shows that it is manifestly gauge invariant. Here, we need a little elaboration on the gauge-transformation properties of the geometric phase. The proof of invariance of the geometric phase under gauge transformations is based on choosing special local reference sections, which satisfy the "in-phase" condition, namely, the inner product $\langle \chi_0(0)|\chi_0(t)\rangle$ is real and positive. One may argue that there may be many local reference sections that do not satisfy the above-mentioned condition. Hence, the gauge-transformation properties we present in this paper are for a restricted set of gauge transformations in the sense that the transformed reference section should also satisfy the reality and positivity condition. In my view, this is not a restriction but rather a criterion for chosing a reference section so as to be able to define the geometric phase for an open path during an arbitrary quantum evolution. If this condition is not satisfied by a reference section, then we cannot define a connection form whose line integral will give the correct geometric phase. Those reference sections that do not satisfy the condition (iii) will not be compatible with the Pancharatnam connection and the geometric phase defined using them will not be the gauge-invariant one.

The importance of our definition is that the geometric phase (13) has its own existence and physical meaning even if we do not close the end points explicitly by a geodesic. In fact, in our formulation we do not have to close the end points by a geodesic because along the geodesic the geometric phase defined in (13) is identically zero. To prove this in the simplest way, we use the geodesic equation and its solution. Geodesics in $\mathscr{P}$ can be defined as those curves for which the "energy" of the horizontal curve is stationary. The energy of a curve is an important concept in studying the geometry of quantum evolutions, which we have introduce in a recent paper [17]. (For further details on this concept we advise the readers to see Ref. [17].) We define the energy associated with the horizontal curve as the number

$$[E(\bar{\Psi})]_0^t=\hbar\int_0^t\langle\dot{\bar{\Psi}}(t')|\dot{\bar{\Psi}}(t')\rangle dt'. \qquad (17)$$

If we carry out a variational calculation, we obtain the geodesic equation [10,17] that is satisfied by the horizontal vector $|\bar{\Psi}(t)\rangle$

$$\frac{d^2}{dt^2}|\bar{\Psi}(t)\rangle+[V_p(t)]^2|\bar{\Psi}(t)\rangle=0, \qquad (18)$$

where $V_p(t)$ is the speed of transportation of the state vector in $\mathscr{P}$ and is given by $V_p(t)=\Delta E(t)/\hbar$. Here, $\Delta E(t)$ is the uncertainty in the energy of the system. For simplicity, we assume that $\Delta E(t)$ is independent of time. Then, the solution to the geodesic equation (18) is

$$|\bar{\Psi}(t)\rangle=\cos(V_p t)|\bar{\Psi}(0)\rangle+\frac{\sin(V_p t)}{V_p}|\dot{\bar{\Psi}}(0)\rangle. \qquad (19)$$

Suppose we have an evolution of the quantum system from time $t=0$ to t (say) along the geodesic. Then along this special path the reference section $|\chi_0(t)\rangle$ is given by

$$|\chi_0(t)\rangle=\frac{\langle\bar{\Psi}(t)|\bar{\Psi}(t)\rangle}{|\langle\bar{\Psi}(t)|\bar{\Psi}(0)\rangle|}|\bar{\Psi}(t)\rangle. \qquad (20)$$

Since $\langle\bar{\Psi}(t)|\bar{\Psi}(0)\rangle$ is real [from (19)], we have $|\chi_0(t)\rangle=|\bar{\Psi}(t)\rangle$. Therefore, the geometric phase along the geodesic vanishes identically. (Here, we use the fact that $\langle\bar{\Psi}(t)|\dot{\bar{\Psi}}(t)\rangle=0$.) This is the simplest and most direct proof of the vanishing nature of the geometric phase along a geodesic.

The expression (13) is an integral of a nonlocal integrand and provides a closed-from expression for the geometric phase during an arbitrary quantum evolution. The dynamical phase is a locally additive functional of Γ whereas the geometric phase is a nonlocal and nonadditive functional of $\hat{\Gamma}$. To see the nonadditive nature of the geometric phase (13) clearly, let us consider the evolution of a quantum system from point $\Pi(|\Psi(t_1)\rangle)$ to a point $\Pi(|\Psi(t_2)\rangle)$, and then from $\Pi(|\Psi(t_2)\rangle)$ to $\Pi(|\Psi(t_3)\rangle)$. We will prove that the geometric phase acquired by the system during an evolution from a point $\Pi(|\Psi(t_1)\rangle)$ to a point $\Pi(|\Psi(t_3)\rangle)$ is not equal to the sum of the geometric phases acquired by the system during the evolution from point $\Pi(|\Psi(t_1)\rangle)$ to $\Pi(|\Psi(t_2)\rangle)$, and $\Pi(|\Psi(t_2)\rangle)$ to $\Pi(|\Psi(t_3)\rangle)$. The geometric phase acquired by the system during the evolution from time t_1 to t_2 is given by $[\Phi_g]_1^2=i\int_1^2\chi_1(t)|\dot{\chi}_1(t)\rangle dt$, from t_2 to t_3 is given by $[\Phi_g]_2^3=i\int_2^3\langle\chi_2(t)|\dot{\chi}_2(t)\rangle dt$, and from time t_1 to t_3 is given by $[\Phi_g]_1^3=i\int_1^3\langle\chi_1(t)|\dot{\chi}_1(t)\rangle dt$. The reference sections $|\chi_1(t)\rangle$ and $|\chi_2(t)\rangle$ are defined through

$$|\chi_1(t)\rangle=e^{-i\Phi_P(t,t_1)}|\Psi(t)\rangle$$

and

$$|\chi_2(t)\rangle=e^{-i\Phi_P(t,t_2)}|\Psi(t)\rangle, \qquad (21)$$

where $\Phi_P(t,t_1)=\arg\langle\Psi(t_1)|\Psi(t)\rangle$ and $\Phi_P(t,t_2)=\arg\langle\Psi(t_2)|\Psi(t)\rangle$. Now, the geometric phase $[\Phi_g]_1^3$ can be expressed as

$$[\Phi_g]_1^3=[\Phi_g]_1^2+i\int_1^2\langle\chi_1(t)|\dot{\chi}_1(t)\rangle dt. \qquad (22)$$

On using (21), (22) can be expressed as

$$[\Phi_g]_1^3=[\Phi_g]_1^2+[\Phi_g]_2^3-\int_2^3[d\Phi_p(t,t_2)-d\Phi_p(t,t_1)].$$

Therefore, the excess geometric phase is

$$\begin{aligned}[\Phi_g]_{\text{excess}}&=[\Phi_g]_1^3-([\Phi_g]_1^2+[\Phi_g]_2^3)\\&=\Phi_P(t_3,t_1)-[\Phi_P(t_3,t_2)-\Phi(t_2,t_1)]\\&=\arg\langle\Psi(t_1)|\Psi(t_3)\rangle-[\arg\langle\Psi(t_2)|\Psi(t_3)\rangle\\&\quad+\arg\langle\Psi(t_3)|\Psi(t_1)\rangle].\end{aligned}$$

This also can be related to the three-point Bargmann invariant [10] as follows:

$$\begin{aligned}[\Phi_g]_{\text{excess}}&=-\arg\Delta^{(3)}\\&=-\arg[\langle\Psi(t_1)|\Psi(t_2)\rangle\langle\Psi(t_2)|\Psi(t_3)\rangle\\&\quad\times\langle\Psi(t_2)|\Psi(t_1)\rangle].\end{aligned} \qquad (23)$$

In the above expression, the left-hand side contains only the excess geometric phase, whereas the right-hand side contains only the excess total phase, implying that the excess dynamical phase is identically zero. This clearly shows the additive nature of the dynamical phase and brings the full nonadditive nature of the geometric phase.

We can see that the geometric phase (13) will reduce to the adiabatic Berry phase and the nonadiabatic AA phase in the appropriate limit. For example, in the case of cyclic evolution of quantum systems during an interval $[0,T]$, the state vector $|\Psi(t)\rangle$ satisfies $|\Psi(T)\rangle=\exp(i\Phi)|\Psi(0)\rangle$, $\Phi\in\mathbb{R}$ being the total phase. The reference section $|\chi_0(t)\rangle$ during cyclic evolution satisfies $|\chi_0(T)\rangle=|\chi_0(0)\rangle$, i.e., it is single valued. (Compare this with the state $|\tilde{\Psi}(t)\rangle$, which was used to define the AA phase.) Therefore, the geometric phase during a cyclic evaluation of a quantum system is given by

$$[\Phi_g]_0^T=i\int_0^T\langle\chi_0(t)|\dot{\chi}_0(t)\rangle dt=i\oint\langle\chi_0|d\chi_0\rangle. \qquad (24)$$

We will show that this phase will be same as that of the AA phase, $\beta(C)$. Recall that the section $|\tilde{\Psi}(t)\rangle$ defined by AA [3] is given by $|\tilde{\Psi}(t)\rangle=\exp[-if(t)]|\Psi(t)\rangle$, where $f(t)$ is any smooth function satisfying $f(T)-f(0)=\Phi$, and Φ is the total phase. The AA section $|\tilde{\Psi}(t)\rangle$ is single valued, i.e., $|\tilde{\Psi}(T)\rangle=|\tilde{\Psi}(0)\rangle$. Taking out the dynamical phase from Φ, the geometric phase for cyclic evolution is given by

$$\beta(C)=i\int_0^T\langle\tilde{\Psi}(t)|\dot{\tilde{\Psi}}(t)\rangle dt=i\oint_{\hat{C}}\langle\tilde{\Psi}(t)|d\tilde{\Psi}(t)\rangle. \qquad (25)$$

Now our reference section $|\chi_0(t)\rangle$ and the AA section $|\tilde{\Psi}(t)\rangle$ are related by

$$|\chi_0(t)\rangle = e^{if(0)} \frac{\langle\tilde{\Psi}(t)|\tilde{\Psi}(0)\rangle}{|\langle\tilde{\Psi}(t)|\tilde{\Psi}(0)\rangle|}|\tilde{\Psi}(t)\rangle. \tag{26}$$

Using our definition (24), the cyclic geometric phase is given by

$$\begin{aligned}[\Phi_g]_0^T &= i\int_0^T \langle\chi_0(t)|\dot{\chi}_0(t)\rangle dt \\ &= i\int_0^T \left(\frac{\langle\tilde{\Psi}(t)|\tilde{\Psi}(0)\rangle}{|\langle\tilde{\Psi}(t)|\tilde{\Psi}(0)\rangle|}\right)^* \frac{d}{dt} \\ &\quad \times \left(\frac{\langle\tilde{\Psi}(t)|\tilde{\Psi}(0)\rangle}{|\langle\tilde{\Psi}(t)|\tilde{\Psi}(0)\rangle|}\right) + i\int_0^T \langle\tilde{\Psi}(t)|\dot{\tilde{\Psi}}(t)\rangle dt \\ &= \arg\langle\tilde{\Psi}(0)|\tilde{\Psi}(T)\rangle + i\int_0^T \langle\tilde{\Psi}(t)|\dot{\tilde{\Psi}}(t)\rangle dt. \end{aligned} \tag{27}$$

Therefore, $[\Phi_g]_o^T = i\int_0^T\langle\tilde{\Psi}(t)|\dot{\tilde{\Psi}}(t)\rangle dt = \beta(C)$, which is nothing but the AA phase. In providing this we have used the single valuedness of the AA section $|\tilde{\Psi}(t)\rangle$. The geometric phase $[\Phi_g]_0^T$ is obviously invariant under phase and gauge transformations. For the detail transformation properties of the AA phase, readers are advised to see the papers of Bohm, Boya, and Kendrick [14,15] and of Kendrick [16].

IV. OTHER GEOMETRIC STRUCTURES IN NONCYCLIC EVOLUTIONS

In this section we look for other admissible geometric structures such as the length and distance during any arbitrary evolution of quantum systems. Also, we sought for a topogolical reason for the origin of the geometric phase. The basic geometric structures in dictating the topology of a manifold are the "length element" and the "distance element." We will bring out an inequality between the length and distance and argue that the geometric phase arises because of a fundamental inequality between them. Consider the curves $\Gamma_0:[0,t]\rightarrow\mathscr{L}$ and $\bar{\Gamma}:[0,t]\rightarrow\mathscr{L}$ traced out by the reference section $|\chi_0(t)\rangle$ and the horizontal curve $|\bar{\Gamma}(t)\rangle$, respectively. It is well known that the inner product in $\mathscr{H}$ induces a metric in $\mathscr{P}$ [5] and the presence of a metric allows the definition of the length of a differentiable curve in $\mathscr{L}$.

Let $t\rightarrow|\chi_0(t)\rangle$ be a curve $\Gamma_0(t)$ during an arbitrary evolution of a quantum system. Then the total length of the differentiable curve Γ_0 from a point $|\chi_0(0)\rangle$ to a point $|\chi_0(t)\rangle$ is a number defined as

$$l(\dot{\chi}_0(t))|_0^t = \int \langle\dot{\chi}_0(t)|\dot{\chi}_0(t)\rangle^{1/2} dt, \tag{28}$$

where $|\dot{\chi}_0(t)\rangle$ is the velocity vector in $\mathscr{L}$ of the curve $t\rightarrow|\chi_0(t)\rangle$ at time t along the path of evolution of $|\chi_0(t)\rangle$ (relative to the initial point).

Next, let $t\rightarrow|\bar{\Psi}(t)\rangle$ be a curve $\bar{\Gamma}(t)$ during an arbitrary evolution of a quantum system. Then the total length of the differentiable curve $\bar{\Gamma}$ from a point $|\bar{\Psi}(0)\rangle$ to a point $|\bar{\Psi}(t)\rangle$ is a number defined as

$$l(\bar{\Psi}(t))|_0^t = \int \langle\dot{\bar{\Psi}}(t)|\dot{\bar{\Psi}}(t)^{1/2} dt, \tag{29}$$

where $|\dot{\bar{\Psi}}(t)\rangle$ is the velocity vector in $\mathscr{L}$ of the curve $t\rightarrow|\bar{\Psi}(t)\rangle$ at time t along the path of evolution of $|\bar{\Psi}(t)\rangle$.

We can verify that the above two lengths defined through (28) and (29) are also geometric structures associated with any quantum evolution. The integrals (28) and (29) exist in the interval $[0,t]$, since the integrand is continuous and the numbers calculated on using them are real. The above two lengths have an important property of reparametrization invariance, i.e., all curves deduced from $\Gamma_0/\bar{\Gamma}$ by a change of parameter t to t' with $dt/dt'>0$, the length of the curves remain unaltered. Hence, the length of the curves are independent of the parametrization of their image set, are properties of the geometrical curves, and are t invariant. Also, they are invariant under the phase transformation, i.e., when $|\Psi(t)\rangle\rightarrow e^{i\alpha(t)}|\Psi(t)\rangle$, $l(\chi_0(t))$, and $l(\bar{\Psi}(t))$ remain the same. In addition to this, the lengths are independent of the particular Hamiltonian used to evolve the quantum system along a given path $\hat{\Gamma}$ in $\mathscr{P}$.

Apart from sharing the geometric properties, the two lengths are basically different. The length $l(\chi_0(t))$ is nonadditive in nature, because if we have an evolution (say) from time t_1 to t_2 and then from t_2 to t_3, the length of the curve between t_1 to t_3 is not equal to the sum of the lengths between t_1 to t_2 and t_2 to t_3. This length has an analogous property to that of the geometric phase, namely, it is nonintegrable in nature. The other length $l(\bar{\Psi}(t))$ is, in fact, equal to the total distance traveled by the state vector $|\bar{\Psi}(t)\rangle$ along a given curve $\hat{\Gamma}$ in the projective Hilbert space $\mathscr{P}$ as measured by the Fubini-Study metric [18–20,5]. The projective Hilbert space $\mathscr{P}$ admits a natural metric structure (namely, the Fubini-Study metric) and the total distance is equal to the time integral of the uncertainty in the energy of the system during an arbitrary evolution, i.e.,

$$\begin{aligned} l(\bar{\Psi}(t))|_0^t &= \int [\langle\dot{\Psi}(t)|\dot{\Psi}(t)\rangle - (i\langle\Psi(t)|\dot{\Psi}(t)\rangle)^2]^{1/2} dt \\ &= \int \Delta E(t) dt/\hbar. \end{aligned} \tag{30}$$

Moreover, they are distinct geometric objects. It can be seen from the fact that the length of the curve $l(\chi_0(t))$ is greater than the length of the curve $l(\bar{\Psi}(t))$. To see this explicitly, we evaluate the square of the infinitesimal length of the curve $l(\chi_0(t))$ as follows:

$$\begin{aligned} dl^2(\chi_0(t)) &= \langle\dot{\chi}_0(t)|\dot{\chi}_0(t)\rangle dt^2 \\ &= \left[\frac{d\xi^*}{dt}(t,0)\frac{d\xi}{dt}(t,0) + 2\frac{d\xi^*}{dt}(t,0)\xi(t,0)\right. \end{aligned}$$

$$\times\langle\Psi(t)|\dot{\Psi}(t)\rangle+\langle\dot{\Psi}(t)|\dot{\Psi}(t)\rangle\Bigg]. \quad (31)$$

On using the following expressions,

$$\frac{d\xi^*}{dt}(t,0)\frac{d\xi}{dt}(t,0)=[i\langle\chi_0(t)|\dot{\chi}_0(t)\rangle-i\langle\Psi(t)|\dot{\Psi}(t)\rangle]^2,$$

$$\frac{d\xi^*}{dt}(t,0)\xi(t,0)=\langle\Psi(t)|\dot{\Psi}(t)\rangle-\langle\chi_0(t)|\dot{\chi}_0(t)\rangle,$$

we have

$$dl^2(\chi_0(t))-dl^2(\bar{\Psi}(t))=[i\langle\chi_0(t)|\dot{\chi}_0(t)\rangle dt]^2. \quad (32)$$

Since $i\langle\chi_0(t)|\dot{\chi}_0(t)\rangle$ is real, we have

$$dl^2(\chi_0(t))>dl^2(\psi(t)). \quad (33)$$

This is what we have mentioned above. Equation (33) says that (infinitesimally) the length of the nonhorizontal curve Γ_0 is greater than the length of the horizontal curve $\bar{\Gamma}$—a fundamental feature of the quantum evolution and because of this fundamental inequality between the length and the distance the geometric phase arises. Equation (32) contains much more information about the geometric phase than that of the expression (13). For the special case of a cyclic quantum evolution, an expression similar to that of (29) was derived and explored in greater detail by the present author [5]. For quantum evolutions during an infinitesimal time interval Δt, Eq. (32) can be put in a form

$$\delta l^2(\chi_0)-\delta l^2(\bar{\Psi})=\delta\Phi_g^2, \quad (34)$$

where $\delta l=(dl/dt)\Delta t$, etc. Equation (34) has a beautiful geometric meaning. Locally it says that when we move from one fiber to another, which are infinitesimal nearby, then the infinitesimal changes in reference sectional curve, horizontal curve, and the geometric phase can be represented by the sides of a right triangle. So (34) states the Pythagorean theorem for the length, distance, and phase in an infinitesimal neighborhood of a fiber bundle. Similar reasoning was also pointed out in the case of a cyclic quantum evolution [23].

V. TWO EXAMPLES

In this section we illustrate the ideas introduced in this paper in some simple examples. Consider the following simple, yet nontrivial example that is of a two-level atom (which is isomorphic to a spin-$\frac{1}{2}$ particle interacting with magnetic field). Here, the Hilbert space $\mathscr{H}=\mathbb{C}^2$ and $\mathscr{P}=\mathscr{P}_1(\mathbb{C})$ is the real two-dimensional sphere S^2. The state vector at any time t is given by

$$|\Psi(t)\rangle=\sin(\theta/2)\exp(i\phi)\exp(-i\omega t)|+\rangle + \cos(\theta/2)\exp(i\omega t)|-\rangle, \quad (35)$$

where θ and ϕ are related to the physical parameters of the two-level system. The basis vectors $|+\rangle$ and $|-\rangle$ are orthogonal vectors that span the Hilbert space $\mathbb{C}^2$. During a noncyclic evolution, the state vector traces an arc over a cone with polar angle θ and azimuthal angle $\varphi=2\omega t$. We assume that θ is constant during this spin evolution and φ only changes with time. The reference section $|\chi_0(t)\rangle$ is given by

$$|\chi(t)\rangle=e^{-i\tan^{-1}(\tan\omega t\cos\theta)}[\sin(\theta/2)e^{i\phi}e^{-i\omega t}|+\rangle + \cos(\theta/2)e^{i\omega t}|-\rangle]. \quad (36)$$

It is easy to see that this reference section satisfies all the properties mentioned in Sec. III. Using this we can calculate the noncyclic geometric phase $[\Phi_g]_0^t$ for a two-level atom as

$$[\Phi_g]=\tan^{-1}(\tan\omega t\cos\theta)-\omega t\cos\theta. \quad (37)$$

This is an evolving geometric phase and it varies nonlinearly with time in contrast to the dynamical phase, which varies linearly with time. The correct measurement of such noncyclic geometric phases for spin-$\frac{1}{2}$ particles (neutrons) has been proposed by Wagh and Rakhecha [21]. We can also see that for a cyclic quantum evolution, the above geometric phase reduces to $[\Phi_g]_0^T=\beta(C)=\pi(1-\cos\theta)$, which is nothing but the AA phase for a fictitious spin-$\frac{1}{2}$ particle and gives the usual half the solid angle law for the geometric phase.

We illustrate the preceding ideas in another simple example, namely, a one-dimensional harmonic oscillator (HO). Here, the state vector of the HO belongs to an infinite-dimensional Hilbert space. The Hamiltonian of the HO in one dimension is

$$H=P^2/2m+\tfrac{1}{2}kX^2 \quad (38)$$

and the state vector at any time $t>0$, is given by

$$|\Psi(t)\rangle=\sum_{n=0}^{n=\infty}e^{-i\omega t(n+1/2)}C_n|n\rangle, \quad (39)$$

where the expansion coefficients C_n's depend on the choice of the initial state $|\Psi(0)\rangle$. We consider the initial state $|\Psi(0)\rangle$ to be in a coherent state, then $|\Psi(t)\rangle$ is given by

$$|\Psi(t)\rangle=e^{-|z|^2/2}\sum_{n=0}^{n=\infty}\frac{z^n}{\sqrt{n!}}e^{-i\omega t(n+1/2)}|n\rangle, \quad (40)$$

where z is a complex number. The reference section $|\chi_0(t)\rangle$, which is normalized, is in phase with the initial state and projects to the same open curve $\hat{\Gamma}$ in $\mathscr{P}$, is given by

$$|\chi_0(t)\rangle=\exp\left[-|z|^2/2+i\tan^{-1}\left(\frac{\sum_{n=0}^{\infty}[(z\bar{z})^n/n!]\sin(n+\frac{1}{2})\omega t}{\sum_{n=0}^{\infty}[(z\bar{z})^n/n!]\cos(n+\frac{1}{2})\omega t}\right)\right]\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}e^{-i\omega t(n+1/2)}|n\rangle\ . \tag{41}$$

Once $|\chi_0(t)\rangle$ is known, we can calculate the evolving geometric phase as

$$[\Phi_g]_0^t=\tan^{-1}\left[-\frac{\sum_{n=0}^{\infty}\frac{(z\bar{z})^n}{n!}\sin(n+\frac{1}{2})\omega t}{\sum_{n=0}^{\infty}\frac{(z\bar{z})^n}{n!}\cos(n+\frac{1}{2})\omega t}\right]+\omega t(z\bar{z}+\tfrac{1}{2}). \tag{42}$$

For cyclic evolutions, we have the well known result for the geometric phase, which is given by

$$[\Phi_g]_0^T=\beta(C)=2\pi z\bar{z}. \tag{43}$$

This is nothing but the area in-phase space enclosed by the phase space trajectory.

VI. GENERALIZATION AND CONCLUSION

The purpose of the last section is to discuss briefly how to generalize the formula (13) to the case of nonunitary and non-Schrödinger (in fact any type of quantum evolution, meaning, without presupposing anything concerning the form of the equation of motion [8]) quantum evolutions.

Consider a smooth curve $t\rightarrow|\Psi(t)\rangle$, i.e., $\Gamma:[0,t]\rightarrow\mathcal{H}$ along which $||\Psi(t)||$ varies with time (here, t could be any other continuous parameter and need not be necessarily identified with time) and, consequently, $\langle\Psi(t)|\dot{\Psi}(t)\rangle$ is no more purely imaginary. Facilitating the idea of the inner product in $\mathcal{H}$ and using the notion of smoothness of the curve, we can define the total phase and the dynamical phase, without any recourse to the evolution equation. Thus, the total phase factor is given by

$$e^{i[\Phi]_0^t}=\frac{\left\langle\frac{\Psi(0)}{||\Psi(0)||}\middle|\frac{\Psi(t)}{||\Psi(t)||}\right\rangle}{\cos(\theta/2)}, \tag{44}$$

where $\theta/2$ is the Bargmann angle between the initial and final points in $\mathcal{P}$ on to which $|\Psi(0)\rangle$ and $|\Psi(t)\rangle$ project via the projection map $\Pi:\mathcal{H}\rightarrow\mathcal{P}$. The dynamical phase could be defined from the elementary idea of the smoothness of the curve Γ and is given by

$$[\Phi_d]_0^t=-i\int\left\langle\frac{\Psi(t)}{||\Psi(t)||}\middle|\frac{d}{dt}\middle|\frac{\Psi(t)}{||\Psi(t)||}\right\rangle dt. \tag{45}$$

We can easily be convinced that for the case of the unitary and norm-one Schrödinger evolution of a quantum system (44) and (45) reduces to (2) and the second term in (4), respectively. Now we would obviously write the expression for the geometric phase as

$$[\Phi_g]_0^t=\arg\left\langle\frac{\Psi(0)}{||\Psi(0)||}\middle|\frac{\Psi(t)}{||\Psi(t)||}\right\rangle+i\int\left\langle\frac{\Psi(t)}{||\Psi(t)||}\middle|\frac{d}{dt}\middle|\frac{\Psi(t)}{||\Psi(t)||}\right\rangle dt. \tag{46}$$

But this is not the end. We would like to present a compact formula for the geometric phase as that of Eq. (13). In order to do this, we follow the prescription given in the earlier pages of our paper by defining a reference section of the bundle covering $\Pi(|\Psi(t)\rangle)$ and the final expression for the geometric phase is given by

$$[\Phi_g]_0^t=i\int\left\langle\frac{\chi_0(t)}{||\chi_0(t)||}\middle|\frac{d}{dt}\middle|\frac{\chi_0(t)}{||\chi_0(t)||}\right\rangle dt. \tag{47}$$

To the best of our knowledge, this is the most general formula for the geometric phase that has been derived for the first time [22]. We can easily check that (47) reduces to (13) for unitary and norm-one Schrödinger evolutions. Our geometric phase enjoys the following properties. It is manifestly gauge invariant not only under U(1) action but also under a general transformation of the type $|\Psi(t)\rangle\rightarrow|\Psi(t')\rangle=Z(t)|\Psi(t)\rangle$, where $Z(t)$ is an arbitrary smooth complex function and $Z(t)\in\mathbb{C}^*$ with $|Z(t)|\neq1$. Also under phase transformation, $|\chi_0(t)\rangle/||\chi_0(t)||$ transforms by a global phase factor and hence the geometric phase remains invariant. This property of the reference section entails that the geometric phase is a property of the projection of $|\Psi(t)\rangle$ in $\mathcal{P}$ rather than of $|\Psi(t)\rangle$ itself. It is also reparametrization invariant and is a property of the geometrical curve, defined by the equivalence classes of parametrized paths. Last but not least, it is too independent of the detail dynamics of the evolution of the state vector.

Here, also, we can argue as how the geometric phase arises in the context of nonunitary non-Schrödinger evolutions based on the inequality between the length of the curve and the distance. But we skip the proof because similar steps as that of the earlier one would convince the readers that this is so.

To conclude this paper, we present a closed-form expression for the geometric phase in the case of noncyclic but Schrödinger quantum evolution. A canonical one-form is defined whose line integral gives the geometric phase. The geometric phase is shown to be invariant under phase and gauge transformations. The nonadditive and vanishing nature (along a geodesic) of the geometric phase is explicitly shown. For cyclic evolutions, we prove that it reduces to the

AA phase. Then we generalize it to the case of noncyclic and non-Schrödinger evolutions (in fact, for all arbitrary quantum evolutions). Other geometric structures such as the length and distance are defined for all evolutions of the quantum systems. The geometric phase, length, and distance are found to be related. Infinitesimally they satisfy a Pythagorean theorem. This nonlocal phase arises because of a fundamental inequality between the length and the distance function. The geometric phase is calculated in two simple examples and the issue is clarified. Thus, this paper provides a complete treatment of the noncyclic geometric phase, distance, and length for all evolutions of quantum systems.

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