Stability of systems of three arbitrary charges: General properties

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The stability of systems consisting of a negative charge $-q_1$ and two positive charges q_2 and q_3 is discussed as a function of the values of the charges q_i and of the constituent masses m_i. We give general results and list some open questions of interest, which hopefully will stimulate further investigations.

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I. INTRODUCTION

Recently, several authors have reviewed the problem of the stability of systems made of three unit charges, say $Q_i = \pm(-1, +1, +1)$ [1–4]. This includes some well-known cases such as the ionized hydrogen molecule H_2^+ ($e^- pp$), the hydrogen ion H^- (pe^-e^-), or the positronium ion ($e^+e^-e^-$), which are stable, and the positron-hydrogen system (e^-pe^+) which is unbound. The question is for which constituent masses m_i the system remains stable against dissociation into an atom and an isolated charge.

A natural extension consists of allowing the charges Q_i to vary, i.e., considering a system with

$$Q_i = \pm(-q_1, q_2, q_3), \quad q_i > 0, \tag{1}$$

and studying the stability domain as a function of the variables q_i. The practical applications are limited, at first sight. However, there are systems in nature like the proton, α particle, and electron for which one can ask the question of stability (in this particular case the answer is negative according to Chen and Spruch [5]). We hope, however, to understand more deeply the mechanisms by which collective binding sometimes becomes more favorable than splitting into small clusters.

In the course of our investigations, we have soon checked our expectation that this problem is extremely complicated. The system is governed by four parameters, two mass ratios and two charge ratios, and in most cases deciding the stability would require performing an accurate numerical calculation. We would like to stress that here we restrict ourselves to the problem of *strict* stability, i.e., stability of the ground state. For some systems, like proton−muon−α-particle, one can speak of "quasi-stability" of certain excited states according to the terminology of Gershtein and Ponomarev [6], i.e., stability in the limit where the masses of the nuclei become very large.

Our paper will therefore not display many concrete results. It will instead focus on selected open questions. Hopefully this will stimulate further developments on the rigorous properties of the stability domain, and perhaps some new numerical investigations.

The paper is organized as follows. In Sec. II, we briefly summarize the results obtained for unit charges $q_i = 1$. The simple properties of the stability domain for arbitrary charges are explained in Sec. III. In Sec. IV, we discuss the case where the positive charges are infinitely massive.

In Sec. V, we consider the case of nearly symmetric systems, with equal masses $m_2 = m_3$, and neighboring charges $q_2 \simeq q_3$. The possible shapes of the stability domain in the plane of inverse masses are presented in Sec. VI, while in Sec. VII we display the complementary slices of the domain: the stability as a function of the charges, for given masses. In the conclusions (Sec. VIII), we stress some of the most challenging open questions.

In a forthcoming paper [7], we shall study in more detail some critical cases which require specific developments, for instance, the case of small q_3.

II. RESULTS FOR UNIT CHARGES

In Ref. [1], we studied the properties of the stability domain for the case of three unit charges, i.e., $q_i = 1$ in
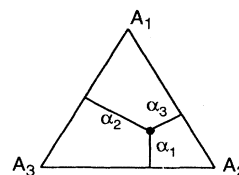

FIG. 1. Domain of possible inverse masses α_i submitted to $\sum \alpha_i = 1$.
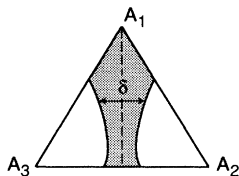
FIG. 2. Schematic shape of the stability domain in the case of equal charges $q_i = 1$.

our notation. A comprehensive review with many references is given in [4].

We first stressed that the inverse masses $\alpha_i \equiv 1/m_i$ are better variables than the masses m_i themselves to account for the regularities of the binding energy when one changes the constituents without modifying the potential. Using the scaling properties of the Coulomb interaction, we can impose, without loss of generality,

$$\sum_{i=1}^{3} \alpha_i = 1 , \tag{2}$$

so that all possible cases can be represented by the interior of an equilateral triangle, as shown in Fig. 1. The α_i are the barycentric coordinates in this plot.

The stability region is displayed in Fig. 2. It consists of a rather narrow band around the symmetry axis $\alpha_2 = \alpha_3$ where like charges have the same mass. The shape of the frontier results from the three basic properties.

(i) Each instability region is star shaped with respect to the vertex of the triangle which it includes. This means that a straight line issuing from A_2 in Fig. 2 will cross at most once the stability frontier between A_2 and the vertical axis.

(ii) Each instability region is convex, i.e., if two points belong to the domain the whole segment connecting the two points belongs to the domain.

(iii) Every symmetric configuration ($m_2 = m_3$) is stable.

The proof of (iii) was given by Hill [8]. The proof of (i) and (ii) essentially relies on the binding energy being an increasing and concave function of each α_i, as explained in Ref. [1].

III. GENERAL RESULTS ON THREE-CHARGE SYSTEMS

Using the definition (1) of the charges, we write the three-body Hamiltonian as

$$H = \sum_i \frac{\alpha_i}{2} \vec{p}_i^2 + \frac{q_2 q_3}{r_{23}} - \frac{q_1 q_2}{r_{12}} - \frac{q_1 q_3}{r_{13}} , \tag{3}$$

with $r_{ij} - |\vec{r}_i - \vec{r}_j|$. It has the following properties.

A. Scaling

We already mentioned the elementary scaling properties of the Coulomb interaction. A simultaneous change $\alpha_i \rightarrow a\alpha_i$ and $q_i \rightarrow qq_i$ of all inverse masses and charges results in a change $E \rightarrow q^4 a^{-1} E$ and $r \rightarrow q^{-2} a r$ of typical energies and distances. We thus can fix $q_1 = 1$, or

$q_1 q_2 = 1$, etc., and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, or $\alpha_1 + \alpha_2 = 1$, etc., at our convenience.

B. Thresholds

To be stable, a three-body system should lie below its lowest dissociation threshold, which corresponds to the more negative of the (1,2) and (1,3) binding energies. The separation between the two possible thresholds is given by

$$(\alpha_1 + \alpha_2)^{-1}(q_1 q_2)^2 = (\alpha_1 + \alpha_3)^{-1}(q_1 q_3)^2 , \tag{4}$$

or, using (2),

$$q_2^2(1 - \alpha_2) = q_3^2(1 - \alpha_3) . \tag{5}$$

This is a straight line, hereafter referred to as (T), passing through A_1', the symmetric of the upper vertex A_1 with respect to the lower side of the triangle, as seen in Fig. 3. Without loss of generality, one can assume $q_2 \geq q_3$, so that $\alpha_2 \geq \alpha_3$ at the threshold separation, which is thus slanted toward the right, as shown in Fig. 3.

C. Star shape of the instability domains

The star-shape property mentioned for unit charges remains valid for $q_i \neq 1$. It can be phrased as follows: a straight line issuing from A_2 or A_3 enters at most once the stability region until it reaches either the border of the triangle or the threshold separation (T).

D. Convexity

The property of each instability region being convex also remains as the charges are no longer unity. The proofs given in Ref. [1] for the star shape and convexity properties are easily seen not to depend on $q_i = 1$.

E. Antisaturation

If $q_3 < q_1$, every point on the right of the threshold separation corresponds to a stable configuration. If, furthermore, $q_3 \leq q_2 < q_1$, every configuration is stable.

The proof easily results from the star-shape property. Let us, for instance, assume $q_3 < q_1$. For $\alpha_1 = \alpha_3 = 0$, i.e., for infinitely massive particles 1 and 3, we have a point-like (1,3) atom with a residual negative charge, which at-
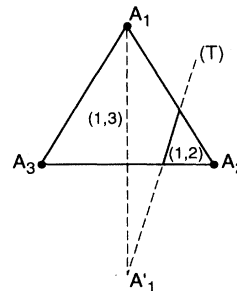

FIG. 3. Separation between the [(1,2)−3] and [(1,3)−2] thresholds.

tracts a particle 2 with mass $\alpha_2^{-1}=1$. Thus the point A_2 belongs to the stability domain. Hence every point on the right of (T) is stable. The same holds on the left side if $q_2 < q_1$.

F. Dependence on the charges

If q_1 changes, the threshold separation (T) does not move. One can rescale all charges so that q_1q_2 and q_1q_3, i.e., the threshold energies, remain constant, whereas q_2q_3 changes in the opposite direction. Thus increasing q_1 improves stability, and decreasing q_1 deteriorates it.

Consider now an increase of q_2, starting from a situation at the left of (T), where (1,2) is the lowest threshold. The line (T) becomes more slanted, so we stay in the same sector. A rescaling can make q_1q_2 constant, i.e., the threshold energy constant. In the three-body Hamiltonian, the repulsion q_2q_3 also remains constant, while the attraction q_1q_3 decreases. We thus have less binding, and the parts of the frontier which lie at the left of (T) move toward the right-hand side.

Similarly, a decrease of q_2 improves stability at the right of (T), with, however, the warning that (T) itself now moves to the left, so if one starts too close to (T) one might end in the sector where (1,3) is the lowest threshold.

Another case deals with a change $q_2 \rightarrow q_2 + \delta q_2$, when one starts from a point at the right of (T). If q_2 increases, (T) becomes more slanted toward the right, but we assume that δq_2 is small enough so that (1,3) remains the lowest threshold. The corresponding threshold energy is constant. If we write the three-body Hamiltonian as

$$H = h_{13} + \frac{\vec{p}_2^{\,2}}{2m_2} + q_2\left[\frac{q_3}{r_{23}} - \frac{q_1}{r_{12}}\right] , \tag{6}$$

necessarily

$$\left\langle \Psi \left| \frac{q_1}{r_{12}} \right| \Psi \right\rangle > \left\langle \Psi \left| \frac{q_3}{r_{23}} \right| \Psi \right\rangle \tag{7}$$

for a point inside the stability domain with wave function Ψ, otherwise H cannot become smaller than h_{13}. This means that the first-order correction

$$\delta E = \delta_{q2}\left\langle \Psi \left| \frac{q_3}{r_{23}} - \frac{q_1}{r_{12}} \right| \Psi \right\rangle \tag{8}$$

has a sign opposite to that of δq_2. If q_2 increases, stability is improved, and if q_2 decreases, it is deteriorated.

A similar study can be done with q_3 changing at fixed q_1 and q_2. An increase (decrease) of q_3 deteriorates (improves) binding at the right of (T), and the reverse is true at the left of (T).

To summarize, we have a simple monotonic behavior of the stability frontier in the triangular plot as the q_i vary. Changing q_1 makes the domain wider or narrower. A change in q_2 or in q_3 modifies the slope of the line (T) separating the thresholds, and the move of the frontier, if any, takes place in the same direction as (T). An example is schematically pictured in Fig. 4.
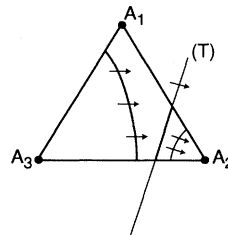

FIG. 4. Change of stability frontier as q_2 increases.

So far, we have used only qualitative arguments associated with the sign of the charges. One can also use the property of the lowest energy E being a concave function of each q_iq_j, since these strengths enter the Hamiltonian linearly. One can also combine scaling with convexity, to improve convexity constraints [9]: $-(-E)^{1/2}$ is a concave function of each q_iq_j.

IV. THE BORN-OPPENHEIMER LIMIT

Let us consider in this section the case where the masses of the positive charges are infinite, i.e., $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, corresponding to the vertex A_1 of the triangle. One can fix $q_1 = 1$ and scan the plane (q_2, q_3) to see where the system is stable. It is sufficient to calculate the ground-state energy $\epsilon_0(R)$ of the first particle as a function of the separation $R = r_{23}$ between the positive charges. Then one should look whether or not the effective potential

$$V_{\text{eff}}(R) = \epsilon_0(R) + \frac{q_2q_3}{R} \tag{9}$$

remains above the threshold energy $-q_2^2/2$. Solving the two-center Hamiltonian

$$h = \frac{\vec{p}_1^{\,2}}{2} - \frac{q_2}{r_2} - \frac{q_3}{r_3} \tag{10}$$

to compute $\epsilon_0(R)$ is a standard problem [10] (up to a scale factor, it depends only on the ratio q_2/q_3).

A detailed study has been performed recently by Hogreve [11]. Our investigations confirm his results, and supplement them, as we shall see, by an analysis of the behavior near $q_3 \rightarrow 0$. The stability domain is shown in Fig. 5. Let us repeat that our definition of "stability" is kept as the one used in the finite-mass case, namely, that
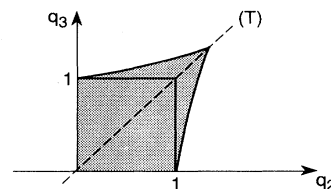

FIG. 5. Domain of stability in the Born-Oppenheimer limit where the like-sign charges q_2 and q_3 have infinite masses.

the ground-state energy of the three-body system is lower than that of either of the two possible two-body systems.

The stability domain includes the square $(q_2 \leq 1, q_3 \leq 1)$, due to the antisaturation property mentioned in Sec. III. It extends mostly along the symmetry axis $q_2 = q_3$, with an extreme peak near $q^2 = q_3 \simeq 1.24$ (1.236 67, according to Hogreve [11]).

A higher value $q_2 = q_3 \simeq 1.44$ is sometimes quoted. It corresponds to the limit for metastability. For $q_2 = q_3$ and $1.24 < q_2 < 1.44$, the effective potential has a local minimum in which the heavy particles are trapped, but this minimum lies above the dissociation threshold.

When q_3 is small, one can use perturbation theory to show that the frontier behaves as

$$q_2 - 1 \simeq 18 \frac{q_3}{(-\ln q_3)^3} , \tag{11}$$

and thus starts vertically, as pictured in Fig. 5.

The proof is the following. At zeroth order in q_3, we have a (1,2) atom, with energy $-q_2^2/2$ and wave function

$$\pi^{-1/2} q_2^{3/2} \exp(-q_2 r) = \frac{1}{\sqrt{4\pi}} \frac{u(r)}{r} , \tag{12}$$

with $\vec{r} = \vec{r}_1 - \vec{r}_2$, and $r = |\vec{r}|$. At first order, the charge q_3, located at $\vec{r}_3 = \vec{r}_2 + \vec{R}$, has an energy

$$\delta E_1 = q_3 \left[\frac{q_2}{R} - \left\langle \frac{1}{|\vec{r} - \vec{R}|} \right\rangle \right] . \tag{13}$$

The expectation value is computed using the above wave function, with the well-known result

$$\langle \ \rangle = \frac{1}{R} \int_0^R u^2(r) dr + \int_R^\infty \frac{u^2(r)}{r} dr . \tag{14}$$

Thus

$$\langle \ \rangle \leq \frac{1}{R} \int_0^\infty u^2(r) dr = \frac{1}{R} , \tag{15}$$

and $\delta E_1 \geq q_3(q_2 - 1)/R$ is positive if $q_2 > 1$. Thus the stability frontier ends at $(q_2 = 1, q_3 = 0)$.

To make more precise the behavior of the frontier near the end point, we push the perturbative expansion up to second order, and write the effective potential between the heavy charges as

$$V_{\text{eff}}(q_2, q_3, R) = -\frac{q_2^2}{2} + \delta E_1 + \delta E_2 , \tag{16}$$

with an explicit computation of δE_1 of Eq. (13) as

$$\delta E_1 = q_3 \left[\frac{q_2}{R} - \frac{f(q_2 R)}{R} \right] , \tag{17}$$

where the form factor is

$$f(x) = 1 - (1 + x)\exp(-x) . \tag{18}$$

We now assume, and will shortly check, that if there is any binding for small q_3 it occurs at large separation R. Then the second-order effect is calculated in the approximation of a constant electric field q_3/R^2 acting on the atom. The Stark effect is well known and reads

$$\delta E_2 = -\frac{9 q_3^2}{4 R^4 q_2^4} . \tag{19}$$

If one rewrites (16) as

$$V_{\text{eff}} = -\frac{q_2^2}{2} + \frac{F(R)}{R} , \tag{20}$$

the stability frontier corresponds to $F(R) = 0$ and $F'(R) = 0$, which in the limit $q_2 \to 1$ and $q_3 \to 0$ reads

$$(q_2 - 1) + (1 + R)\exp(-2R) - \frac{9 q_3}{4 R^3} = 0 ,$$

$$-(2R + 1)\exp(-2R) + \frac{27 q_3}{4 R^4} = 0 . \tag{21}$$

Disregarding the spurious small-R solution due to approximation (19), we see that the second equation of (21) shows that R is, indeed, very large, $R \sim |\ln q_3|/2$. A simple elimination leads to the desired result (11).

V. STABILITY OF NEARLY SYMMETRIC SYSTEMS

In Ref. [1], we considered the case of unit charges. Many accurate calculations exist for symmetric systems, with inverse masses proportional to α_1, $\alpha_2 = (1 - \alpha_1)/2$, and $\alpha_3 = \alpha_2$. If their binding energy is written as

$$E(\alpha_1) = [1 + g(\alpha_1)] E_{\text{th}} , \tag{22}$$

where $g(\alpha_1)$ denotes the fraction of binding below the threshold, one can set a lower limit to the extension of the stability domain for a given α_1 [1],

$$\delta \geq \frac{2}{\sqrt{3}} \frac{g(\alpha_1)}{1 + g(\alpha_1)} (1 + \alpha_1) . \tag{23}$$

The width δ, defined in Fig. 2, is the maximal allowed value for $2|\alpha_2 - \alpha_3|/\sqrt{3}$.

The proof is very simple [1]. The threshold of the system $(\alpha_1, \alpha_2, \alpha_3)$ is exactly known, and its energy is bounded by rewriting

$$H(\alpha_1, \alpha_2, \alpha_3) = H\left[\alpha_1, \frac{\alpha_2 + \alpha_3}{2}, \frac{\alpha_2 + \alpha_3}{2} \right]$$
$$+ \frac{\alpha_2 - \alpha_3}{4} (\vec{p}_3^2 - \vec{p}_2^2) . \tag{24}$$

This ensures that

$$E(\alpha_1, \alpha_2, \alpha_3) \leq E\left[\alpha_1, \frac{\alpha_2 + \alpha_3}{2}, \frac{\alpha_2 + \alpha_3}{2} \right] , \tag{25}$$

since the antisymmetric term in (24) necessarily lowers the ground state, a simple consequence of the variational principle.

The same reasoning can be applied to the strengths $s_1 = q_2 q_3$, $s_2 = q_3 q_1$, and $s_3 = q_1 q_2$. For a given symmetric mass distribution $(\alpha_2 = \alpha_3)$, one has

$$E(s_1, s_2, s_3) \leq E\left[s_1, \frac{s_2 + s_3}{2}, \frac{s_2 + s_3}{2} \right] . \tag{26}$$
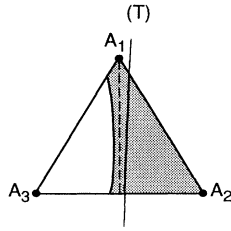
FIG. 6. For charges $q_1 = 1$, $q_2 = 1.02$, and $q_3 = 0.98$, the stability domain extends at least from A_2 to the median axis of the triangle.

If, for instance,

$$s_1 = 1, \quad s_2 = 1 - x, \quad s_3 = 1 + x , \tag{27}$$

stability is guaranteed at least for

$$(1 + x)^2 \leq 1 + g . \tag{28}$$

One can solve (27) for the charges, rescale to $q_1 = 1$, and then say that (28) means stability for configurations corresponding to parameters

$$\alpha_i = \left[\alpha_1, \frac{1 - \alpha_1}{2}, \frac{1 - \alpha_1}{2} \right], \quad q_i = \left[1, \frac{1}{1-x}, \frac{1}{1+x} \right] . \tag{29}$$

When α_1 varies, the minimum of g is between 4% and 5%. Then, if $x \leq 0.02$, we are sure that the symmetry axis of the triangle entirely belongs to the stability domain, as shown in Fig. 6.

VI. STABILITY DOMAIN FOR GIVEN CHARGES

We have a variety of scenarios for the stability region in the triangle of normalized inverse masses. We first fix $q_1 = 1$, and consider the case of equal charges $q_2 = q_3$. For $q_2 < 1$, every configuration is stable.

For $q_2 = 1$, we have the central band of Fig. 2. There is a sudden change as q_2 becomes 1, when one compares in the stability plot the three-body energy $E^{(3)}$ with the two-body energy $E^{(2)}$, but this does not contradict the property of $E^{(3)}$ and $E^{(2)}$ being both continuous.

As q_2 further increases, the band becomes narrower, and then presumably breaks slightly below the center of the triangle. Indeed, for $q_2 = 1$, the relative excess energy $g(\alpha_1)$ of Eq. (22) is maximal for $\alpha_1 = 1(H_2^+$ with $m_p = \infty)$, but is not minimal for $\alpha_1 = 0$. Near $\alpha_1 = 0$, H^- becomes less stable as the proton mass becomes finite, as seen from existing accurate calculations [12]. The minimum of $g(\alpha_1)$ occurs around $\alpha_1 \simeq 0.2$ (see, for instance, the plot in Fig. 2 of Ref. [1]), intermediate between Ps^- and H^-. We thus expect the breaking as q_2 increases to take place near this $\alpha_1 \simeq 0.2$.

For values of q_2 just above this breaking, two islands of stability remain, one in the Born-Oppenheimer region A_1 (H_2^+), and another at the middle of $A_2 A_3$ (H^-). This is illustrated in Fig. 7.
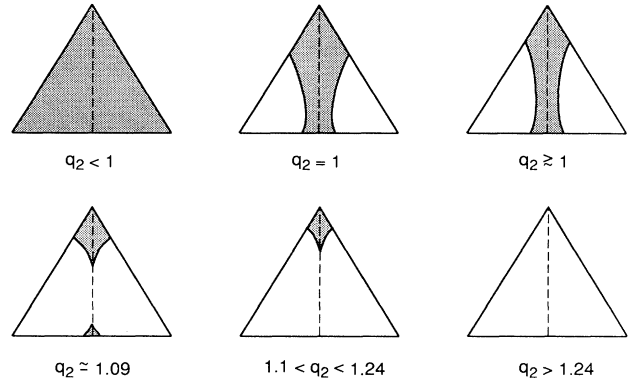

FIG. 7. Shape of the stability region for $q_1 = 1$ and increasing values of the equal charges $q_2 = q_3$.

As shown by Baker et al. [13], stability is lost for H^- near $q_2 = q_3 = 1.0975$, while the study of Hogreve [11] and the results of Sec. IV show that H_2^+ survives up to $q_2 = q_3 = 1.24$. From these results it is easy to see by concavity considerations that stability is completely lost for $q_2 = q_3 > 1.24$ for all masses. The fact that the stability region splits into two separate domains for $q_2 \simeq 1.09$ follows from the work of Pekeris [14] on recoil corrections in Ze^-e^-.

Let us now turn to the case where $q_3 < q_2$. The ratio q_2/q_3 uniquely determines the slope of (T), the line separating the thresholds, according to Eq. (5). Some examples are given in Fig. 8, and commented upon below.

For $q_3 < q_2 < 1$, every configuration is stable. For $q_3 < q_2 = 1$, one should observe a sudden change, as some configurations near A_3 become unstable.

The picture is qualitatively similar for $q_3 < 1 < q_2$, with stability at the right-hand side of (T), and by continuity somewhat at the left of (T). If, for instance, $q_3 = 0.95$
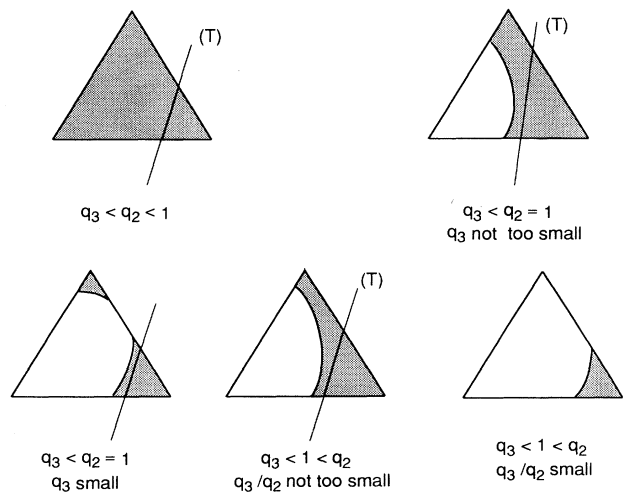

FIG. 8. Shape of the stability region for $q_1 = 1$ and various values of the positive charges $q_3 < q_2$.

and $q_2 = 1.05$, corresponding to the lower middle case, the stability domain includes the Born-Oppenheimer point A_1, while for, say, $q_3 = 0.5$ and $q_2 = 1.2$, roughly the situation of the lower right case, we are restricted to a limited area near Λ_2. This will be studied into more detail in Ref. [14].

For $q_3 = 1 < q_2$, we observe another sudden change: some configurations near A_2 become unstable. For $1 < q_3 < q_2$, the stability region includes at most a band around (T), which can break into two or three islands, and eventually disappear. For a moderate ratio q_2/q_3, we have a deformed version of the situation depicted for $q_2 = q_3$ in Fig. 7. The situation becomes more delicate for large values of q_2/q_3: here, one does not expect any stability, except perhaps for q_3 very close to 1. A study of the large-q_2/q_3 limit will be given in [7].

VII. STABILITY DOMAIN FOR GIVEN MASSES

We now consider the stability domain for given masses in the (q_2, q_3) plane. In this section the concavity of $-\sqrt{-E}$ with respect to q_2 for fixed q_3 and to q_3 for fixed q_2 is essential. From it we can prove that the two arcs forming the border are intersected only once by horizontal or vertical lines. The normalization is again $q_1 = 1$ in this section. We note that the square $(q_2 < 1, q_3 < 1)$ always corresponds to stability. The nontrivial question is thus to find some extension outside this square. In the case where $m_2 = m_3 = \infty$, shown in Fig. 4, we have seen that this extension essentially consists of a spike along the symmetry axis.

Consider first equal but finite masses $m_2 = m_3$. For very small q_3, and $q_2 \geq 1$, the system is unstable. The frontier actually departs from the square $(q_2 < 1, q_3 < 1)$ at some finite value of q_3, as shown in Fig. 9(a). There is an extremum on the symmetry axis $q_2 = q_3$, below the value $q_2 = 1.24$ obtained for infinite masses. If, indeed, we look at the limit $m_2 = m_3 \rightarrow \infty$, we can approximate the Born-Oppenheimer potential by a harmonic-oscillator

well near its minimum, resulting in a binding energy of the type

$$E^{(3)} \simeq A + \frac{B}{\sqrt{m_2}}, \quad B > 0 . \tag{30}$$

Thus $E^{(3)}$ varies quite sharply, while the threshold energy $E^{(2)}$ is a smooth function of m_2. Stability becomes harder when m_2 is finite.

Let us now assume an asymmetric $m_2 < m_3$. In the (q_2, q_3) plot, the separation (T) between the thresholds is also a straight line, given by Eq. (5).

Consider first the case where the masses are such that we have stability for unit charges. The picture in the (q_2, q_3) plot is then qualitatively similar to the previous ones. An illustration is given in Fig. 9(b). Stability is ensured in the square $(q_2 < 1, q_3 < 1)$ and in the triangle limited by $q_3 = 1$ and (T), and extends somehow outside.

We can finally imagine a larger asymmetry between m_2 and m_3, so that the point $q_2 = q_3 = 1$ does not correspond to stability. (T) can now be extremely slanted. This means stability for $q_1 = 1$, $q_3 \simeq 1$, and very large q_2. This is illustrated in Fig. 9(c).

VIII. OUTLOOK

In this paper, we have sketched the possible shapes of the stability domain for various values of the masses and charges. The main lesson is the following. We always obtain stability when the like-sign charges q_2 and q_3 are smaller than q_1, whatever masses are involved. If, say, only $q_3 < q_1$, we have a rather obvious stability when both masses m_1 and m_3 are relatively heavy, so that $(1,3)$ acts as a localized attractive center for particle 2; more precisely, $q_3 < q_1$ implies stability at least as long as $(1,3)$ remains the lowest threshold. It is much less obvious that one can obtain stability with both $q_2 \geq q_1$ and $q_3 \geq q_1$, and indeed this usually does not lead to a stable compound. The remarkable exceptions correspond to situations where the thresholds energies are nearly equal. This is seen as a band around the threshold separation (T) in the triangular plot of inverse masses at fixed charges, or as a spike along (T) in the (q_2, q_3) plot at fixed masses and fixed $q_1 = 1$.

In other words, there are only two ways for a three-charge system to be stable: the trivial way $(q_2 < 1$ and/or $q_3 < 1)$, corresponding to an obvious excess of attraction; otherwise the two threshold energies being nearly equal. In this case the wave function contains a coherent superposition of [1-2]-3 and [1-3]-2 cluster structures. This mechanism is well known in few-body physics. In particular, experts in variational calculations are aware that the nearest thresholds should be introduced in the trial wave function, and that competing thresholds tend to generate attraction.

The present work could be extended in several directions. Let us list a few of these.

(i) The behavior of the frontier near end points could be further elaborated.

(ii) Many variational methods have been developed for such Coulomb systems. They could be applied to esti-
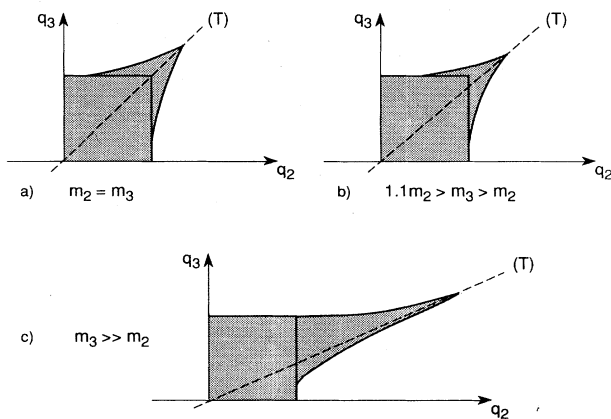

FIG. 9. Shape of the stability region in the (q_2, q_3) plane for $q_1 = 1$ and various values of the masses.

mate quantitatively the location of the frontier, with a high accuracy.

(iii) One would like to know the envelope of the stability curves in the (q_2, q_3) plane when one varies the masses. There is, in the (q_2, q_3) plane, a domain where the three-body system is unstable for all masses. At present, very little is known about this domain.

(iv) A similar study could be done with other potentials. For instance, with power-law potentials $V \propto r_{ij}^\beta$, we have similar scaling laws and virial relations as for the $\beta = -1$ case considered here, and thus the same number of variables in the stability plots. The convexity arguments developed in [1] and in the present paper hold in fact for any potential that does not depend on the masses.

(v) Generalization to four particles seems technically difficult, but rather tempting. One could guess one would again obtain stability in two cases: an obvious excess of attraction, i.e., $\sum q_i q_j < 0$ ($i \neq j$), and also a conspiration

of thresholds, i.e., with an appropriate numbering

$$\frac{(q_1 q_3)^2}{\alpha_1 + \alpha_3} + \frac{(q_2 q_4)^2}{\alpha_2 + \alpha_4} \simeq \frac{(q_1 q_4)^2}{\alpha_1 + \alpha_4} + \frac{(q_2 q_3)^2}{\alpha_2 + \alpha_3} . \tag{31}$$

(vi) One could then look whether the stability conditions are likely to be fulfilled in other systems. For instance, it is an open problem in hadron physics whether or not stable multiquark systems exist [15].

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