Compensation of losses in photodetection and in quantum-state measurements

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We show how losses in photodetection and in quantum-state measurements can be numerically compensated after the measurements have been performed. When the overall efficiency exceeds $\frac{1}{2}$, our recipe works for all quantum states. For smaller efficiencies, however, the convergence of the compensation procedure depends on the quantum state under investigation.

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Detector inefficiencies and losses are present in every real experiment in quantum optics. Apart from attenuating the signal they create extra noise as a consequence of the fluctuation-dissipation theorem. This noise causes quantum decoherence and diminishes our ability to observe subtle quantum phenomena such as interference in phase space [1,2]. The effect of losses is especially important in the recent measurements of the quantum state of light [3,4]. How can we compensate detection losses? It can be done physically by preamplification [5] or numerically by deconvolution of the recorded data. In a recent proposal [6] for the tomographic reconstruction of the density matrix, the deconvolution is woven into the reconstruction algorithm. There the compensation of losses is possible, but only when the detection efficiency η exceeds the critical value $\frac{1}{2}$. Is this an artifact of the particular algorithm or is $\eta=\frac{1}{2}$ the general bound?

In this Brief Report we separate the detection from the compensation procedure. We assume a photon-number distribution or, more generally, a density matrix as given. We show how the compensation of losses can be achieved. Again, only when the efficiency is larger than the critical value $\frac{1}{2}$ is this possible for every density matrix. In this respect, $\eta=\frac{1}{2}$ is a bound also for our method. On the other hand, we show that in certain cases the critical η can be less than $\frac{1}{2}$.

The detection efficiency and other losses (e.g., those due to mode mismatch) can be effectively taken into account with a simple beam-splitter model [7], where a fictitious semitransparent mirror is placed in front of the ideal photodetector. The same simple picture can be applied to a homodyne detection scheme [8]. Here the measuring apparatus can be considered as an ideal homodyne detector with a single beam splitter placed in front of it that accounts for all the losses (Fig. 1). Mode 1 is the signal being in the state $\hat{\rho}_{\rm sig}$. It is attenuated by the beam splitter, while mode 2 is the channel of the losses where a vacuum input $\hat{\rho}_{\rm vac}=|0\rangle\langle 0|$ is formally introduced. This vacuum mode models the extra quantum noise involved in inefficient detection.

An elegant way [9,10] of treating the beam splitter is to apply the Jordan-Schwinger formalism, originally developed in the theory of angular momenta. Setting the phase parameters to zero for simplicity, we find for the unitary transformation of the beam splitter

$$\hat{B}(\eta)=e^{-i2\arccos\sqrt{\eta}\hat{L}_2}\ , \tag{1}$$

where

$$\hat{L}_2=\frac{1}{2i}(\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)\ . \tag{2}$$

Here $\hat{a}_1$ and $\hat{a}_2$ denote the annihilation operators for the signal and the noise mode, respectively. The transmittance η of the beam splitter is identified with the overall detection efficiency. With this notation the signal density matrix is transformed according to

$$\hat{\rho}_{\rm meas}={\rm Tr}_2\{\hat{B}^\dagger(\eta)\hat{\rho}_{\rm sig}\hat{\rho}_{\rm vac}\hat{B}(\eta)\}\ , \tag{3}$$

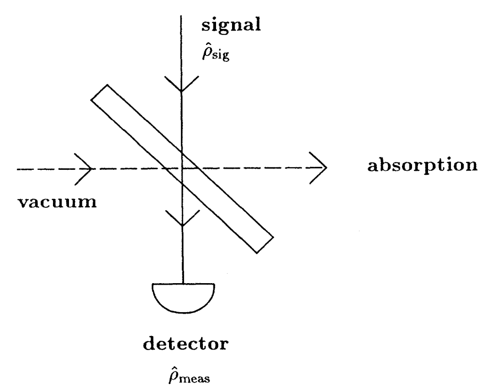

FIG. 1. Model for inefficient detection. The signal is attenuated by a fictitious beam splitter placed in front of an ideal detector. The vacuum mode entering the second port of the beam splitter models the extra noise involved in inefficient detection. The detector stands for a simple photodetector or a homodyne apparatus, respectively.

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where Tr_2 stands for tracing in the noise mode 2. One theoretically remarkable feature of beam splitters is that they are not only nice models for losses in photodetection or homodyne measurements, but they describe rather general damping processes as well [10,11]. In particular, the density matrix $\hat{\rho}_{\mathrm{meas}}$ given by Eq. (3) may be also interpreted as the result of a formal dissipation process described by the master equation [10]

$$\frac{d\hat{\rho}}{dt}=\frac{1}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a}) \; . \tag{4}$$

Here t means a formal time that is related to the transmittance η of the beam splitter or, in our terminology, to the efficiency by

$$\eta=e^{-t} \; . \tag{5}$$

How can we compensate the effect of dissipation? Let us run the process (4) backwards in time. Then we arrive at the original density matrix $\hat{\rho}_{\mathrm{sig}}$. Replacing t by $-t$ means replacing η by η^{-1}. In this way we obtain a simple recipe for getting the original density matrix; calculate the density matrix in a given basis according to Eq. (3) and then replace η by η^{-1}. That this is correct is immediately verified by considering the density matrix in the Fock basis. Using the general formula (3) the calculation of the matrix elements is straightforward. In the Fock basis we obtain

$$\langle n_1|\hat{\rho}_{\mathrm{meas}}|n_1'\rangle = \eta^{1/2(n_1+n_1')}\sum_{j=0}^{\infty}\langle n_1+j|\hat{\rho}_{\mathrm{sig}}|n_1'+j\rangle \times\left[\binom{n_1+j}{n_1}\binom{n_1'+j}{n_1'}\right]^{1/2}(1-\eta)^j \; . \tag{6}$$

This *generalized Bernouli transformation* can be directly inverted,

$$\langle n_1|\hat{\rho}_{\mathrm{sig}}|n_1'\rangle = \eta^{-1/2(n_1+n_1')}\sum_{k=0}^{\infty}\langle n_1+k|\hat{\rho}_{\mathrm{meas}}|n_1'+k\rangle \times\left[\binom{n_1+j}{n_1}\binom{n_1'+j}{n_1'}\right]^{1/2} \times\left[1-\frac{1}{\eta}\right]^k , \tag{7}$$

as is easily seen. Note that the special case of the main diagonals was already obtained by Lee [12]. The existence of such an inversion formula may be surprising, in particular if we take into account that dissipation is regarded as an irreversible process. The losses, however, have a statistically well-defined character and therefore they can be compensated. So, if the corresponding expressions converge, the original density matrix can be reconstructed, provided that the overall detection efficiency η is exactly given and the complete $\hat{\rho}_{\mathrm{meas}}$ is known with high accuracy. These conditions, however, can evidently never be fulfilled perfectly in a real experimental situation. Only a finite part of the density matrix can be measured with some accuracy, determined by the statistical and other errors. We will study the role of the statistics in some detail for the simplest photodetection measuring scheme.

Before doing that, let us examine the form of the expressions (6) and (7). Introducing the diagonal vectors of the density matrix

$$(\mathbf{d}^p)_n=\begin{cases}\langle n|\hat{\rho}|n+p\rangle, & p\geq 0\\ \langle n-p|\hat{\rho}|n\rangle, & p<0 \; ,\end{cases} \tag{8}$$

where p denotes the number of the diagonal and $p=0$ corresponds to the main diagonal we arrive at a simple matrix form for both the transformation (6) and its inverse (7)

$$\mathbf{d}^p_{\mathrm{meas}}=\mathbf{T}^p(\eta)\mathbf{d}^p_{\mathrm{sig}}, \quad \mathbf{d}^p_{\mathrm{sig}}=\mathbf{R}^p(\eta)\mathbf{d}^p_{\mathrm{meas}} \; , \tag{9}$$

where we use the matrix notation

$$[\mathbf{T}^p(\eta)]_{nm}=\begin{cases}\eta^{n+|p|/2}\left[\binom{m}{p}\binom{m+|p|}{n+|p|}\right]^{1/2} \\ \times(1-\eta)^{m-n}, \quad m\geq n\\ 0, \quad m<n\end{cases} \tag{10}$$

and

$$\mathbf{R}^p(\eta)=\mathbf{T}^p(\eta^{-1}) \; . \tag{11}$$

We illustrate the effect of the finiteness of the measured density matrix and the statistical error considering the simple photocounting scheme. The photon-number distribution is given by the main diagonals of the corresponding density matrix

$$p_n=(\mathbf{d}^0_{\mathrm{sig}})_n, \quad q_n=(\mathbf{d}^0_{\mathrm{meas}})_n \; . \tag{12}$$

Let the total number of measurements be N and the number of those where we found n photons X_n. The set of variables X_n $(n=0,1,\ldots,M)$ obeys the multinomial distribution

$$P(X_0=k_0,\ldots,X_M=k_M) = \frac{N!}{k_0!\cdots k_M!K!}q_0^{k_0}\cdots q_M^{k_M}\left[1-\sum_{n=0}^{M}q_n\right]^K , \quad K=N-\sum_{n=0}^{M}k_n \; , \tag{13}$$

where M is a cutoff parameter for large photon numbers. According to the multidimensional central limit theorem [13] the relative frequency $(h_n\equiv X_n/N)$ is distributed normally in the weak limit, with vector notation

$$\lim{}^{*}\mathbf{h}^M\sim\mathcal{N}(\mathbf{q}^M\Sigma^{h,M}) \; , \tag{14}$$

where $\mathcal{N}$ stands for the multidimensional normal distribution with $\mathbf{q}^M$ mean (the index M refers to the finite dimension of the vectors due to the cutoff). The error matrix is given by

$$\Sigma_{nn}^{h,M}=\frac{q_n(1-q_n)}{N},\quad \Sigma_{nm}^{h,M}=\frac{-q_nq_m}{N}\ . \tag{15}$$

The transformed distribution

$$\mathbf{g}^M=\mathbf{R}^{0,M}\mathbf{h}^M \tag{16}$$

follows in the weak limit a transformed normal distribution

$$\lim{}^*\mathbf{g}^M\sim\mathcal{N}(\mathbf{p}^M,\Sigma^{g,M})\ . \tag{17}$$

The corresponding error matrix is

$$\Sigma^{g,M}=\mathbf{R}^{0,M}\Sigma^{h,M}(\mathbf{R}^{0,M})^T\ . \tag{18}$$

We should emphasize at this point that p_n^M is not equal to the original p_n probability because of the cutoff. Only in the limit

$$p_n=\lim_{M\to\infty}p_n^M \tag{19}$$

does it hold. The statistical uncertainty is characterized by the main diagonals of the error matrix

$$\Sigma_{nn}^{g,M}=\frac{1}{N}\left[\sum_{l=n}^{M}\binom{l}{n}^2\left[\frac{1-\eta}{\eta}\right]^{2l}(1-\eta)^{-2n}q_l(1-q_l)\right.$$
$$\left.-\sum_{l,k=n}^{M}\binom{k}{n}\binom{l}{n}(-1)^{k+l}\left[\frac{1-\eta}{\eta}\right]^{k+l}\times(1-\eta)^{-2n}q_kq_l\right]\ . \tag{20}$$

For a given finite cutoff M there exists a minimal N_ϵ^M for every small ϵ that if $N>N_\epsilon^M$, then $\Sigma_{nn}^{g,M}<\epsilon$. Thus, in the presence of a cutoff, the statistical uncertainty can be reduced by increasing the number of measurements. Tending with the parameter M to infinity we have two different cases depending on the behavior of N_ϵ^M.

(a) $\lim_{M\to\infty}N_\epsilon^M<\infty$. In this case there is an upper limit N_ϵ above which the statistical error, according to the normal distribution, is smaller than ϵ. Hence, in the weak limit, the original distribution can be recovered.

(b) $\lim_{M\to\infty}N_\epsilon^M=\infty$. Now p_n cannot be recovered with a precision ϵ. The statistical fluctuations in the high photon numbers have a dramatic influence, making an accurate reconstruction impossible.

We sum up the consequences in some simple statements, which can easily be proved using the properties of (20). First, all finite distributions can always be reconstructed, with arbitrary η. Second, if $\eta>0.5$, then every p_n distribution can be recovered, with arbitrary statistical precision, from a large enough number of data. The elements of the reconstruction matrix tend to zero for large photon numbers, thus the effect of their uncertainty is negligible. Third, for an efficiency $\eta<0.5$ there are counterexamples, where the reconstruction is impossible. For a thermal distribution

$$p_n=\frac{(\bar{n})^n}{(1+\bar{n})^{n+1}}\ , \tag{21}$$

the critical efficiency needed takes the form

$$\eta_{\rm cr}=\left[\frac{1}{\bar{n}}+2\right]^{-1}\ . \tag{22}$$

Here $\bar{n}$ denotes the mean photon number of the original distribution. Above this critical value the statistical errors have a finite effect in the weak limit. Note that in this case ($\eta<0.5$) other errors (e.g., the experimental uncertainty of η itself) are also amplified during the reconstruction process, adding extra noise beyond statistics.

To summarize, we have shown that losses in photodetection and in quantum-state measurements can be numerically compensated. The inefficiencies are modeled by a fictitious beam splitter placed in front of an ideal measuring apparatus. Our recipe for compensation is the following: derive the formula for the attenuated density matrix in a given basis and then replace the efficiency η by η^{-1}. In this way we obtained a compensation formula for the density matrix in the Fock basis. For detector efficiencies smaller than or equal to $\frac{1}{2}$, however, our procedure does not work for all quantum states.

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