Phase-sensitive reservoir modeled by beam splitters

Myung Shik Kim[1,2,*] and Nobuyuki Imoto[1,†]
[1] *NTT Basic Research Laboratories, 3-1 Morinosato-Wakamiya, Atsugi-shi, Kanagawa 243-01, Japan*
[2] *Physics Department, Sogang University, C.P.O. Box 1142, Seoul 100-611, Korea*
(Received 7 March 1995)

The superposition of input fields in a lossless beam splitter is studied in the Schrödinger picture by using the convolution of the positive P representations, and the convolution law for these representations is extended to other quasiprobability functions such as the Wigner and Q functions. We show that the reservoir can be modeled by an infinite array of beam splitters, and we use the convolution law and this model to derive the Fokker-Planck equation for a system coupled with a phase-sensitive reservoir. Solving this equation shows that a phase-sensitive attenuation and amplification can be described by the superposition of two independent quantum fields, one of which is the initial signal field and the other the squeezed thermal noise field representing the reservoir.

PACS number(s): 42.50.Dv, 42.50.Ar

I. INTRODUCTION

There has been considerable interest in the evolution of the quantum nature of an electromagnetic beam that passes through a beam splitter [1–6]. The phase properties of a signal beam can be measured by homodyne detection using a beam splitter [4,5], and an imperfect photodetector can be modeled by combining a beam splitter and a perfect detector [4]. The effect of a reservoir on a signal beam can be simulated by an infinite array of beam splitters and has been analyzed in the Heisenberg picture [7,8].

A beam splitter can superpose two fields [9], and Glauber has discussed the superposition of two fields from independent stationary radiation sources [10]. One source produces a field represented by density operator $\hat{\rho}_1$ whose fluctuation is described by the Glauber-Sudarshan P representation (GS P representation) [10,11] as

$$\hat{\rho}_1 = \int d^2\alpha \; P_1(\alpha)|\alpha\rangle\langle\alpha| \,, \tag{1.1}$$

where $P_1(\alpha)$ and $|\alpha\rangle$, respectively, denote the GS P representation and the coherent state with amplitude α. The other source produces the field whose state is described by

$$\hat{\rho}_2 = \int d^2\beta \; P_2(\beta)|\beta\rangle\langle\beta| \,. \tag{1.2}$$

The density operator for the superposition of the two fields is written as

$$\begin{aligned}\hat{\rho} &= \int\int d^2\alpha \; d^2\beta \; P_1(\alpha)P_2(\beta)|\alpha+\beta\rangle\langle\alpha+\beta| \\ &= \int d^2\gamma \; P(\gamma)|\gamma\rangle\langle\gamma| \,,\end{aligned} \tag{1.3}$$

where the P representation for the superposed field is given by

$$P(\gamma) = \int d^2\alpha \; P_1(\alpha)P_2(\gamma-\alpha) \,. \tag{1.4}$$

In this way Glauber has shown that the GS P representation for the superposed field is the simple convolution of the GS P representations for two component fields [10,12,13].

The GS P representation, however, is a diagonal expansion by coherent states so that it is not possible to describe a nonclassical field using a nonsingular GS P representation. There are other phase-space representations of the field, such as the Wigner, Q [14], and generalized P representations [15]. These are the so-called quasiprobabilities in phase space. In this paper we extend the simple convolution law (1.4) to describe the superposition of nonclassical fields. Once the convolution law for a quasiprobability is known, the other convolution relations for the other quasiprobabilities can be found straightforwardly by using the relations described by the characteristic functions of the quasiprobabilities. The convolution relation for the 50:50 beam splitter has been studied by Leonhardt within the framework of the simultaneous measurement of conjugate variables in phase space [16], but we will derive the convolution relation more rigorously here by using the quasiprobabilities.

Although the quantum statistical theory of a beam splitter has been studied by a number of authors, the phase-space description has attracted attention just recently [17,18]. Agarwal used the evolution of the Wigner function in his study of quantum noise in interferometers [19], and Leonhardt has recently given considerable thought to using the beam splitter theory for making simultaneous measurements in phase space [16]. The most general formalism for the input and output fields of the beam splitter was published by Campos *et al.* [2], and their beam splitter unitary operator based on the SU(2) matrices is used in this paper.

An array consisting of an infinite number of beam splitters can model a reservoir coupled to a signal field

*Electronic address: mshkim@ccs.sogang.ac.kr
†Electronic address: nobu@will.ntt.jp

1050-2947/95/52(3)/2401(10)/$06.00

[7,8], and the Langevin equations have been derived for attenuation and amplification of a signal coupled with the vacuum. In the present paper we use the convolution relation to derive the Fokker-Planck equations both for attenuation and amplification of the signal coupled with the phase-sensitive reservoir [20]. We start from the beam splitter matrices for the amplification and dissipation cases and find the Fokker-Planck equations for each case.

The phase-sensitive reservoir [21,22], based on establishment of squeezed light [23] has been studied theoretically [24–26] and realized experimentally [27]. We find the general solution of the Fokker-Planck equation for the phase-sensitive reservoir, and this solution shows that the reservoir effect can be regarded as the superposition of two independent quantum fields: one is the signal field, which is either simply amplified (in the case of amplification) or attenuated (in the case of attenuation) without noise, and the other is either the thermal field (for phase-insensitive reservoir) or the squeezed thermal field (for phase-sensitive reservoir).

The convolution law is studied in Sec. II where we analyze the beam splitter output field when a squeezed thermal field is superposed on the input coherent state. In Sec. III the Fokker-Planck equations for attenuation and its solution are derived. Section IV shows that when the signal is amplified, the convolution relation is somewhat different from that for attenuation. Section V concludes this paper by discussing the physical interpretation and the potential applications of the present theory.

II. CONVOLUTION LAW

Consider that two fields in the two input ports of a beam splitter are superposed. For convenience we call one input field the signal and the other the noise (Fig. 1). The input signal mode b with its annihilation operator $\hat{b}$ is superposed on the noise mode a with its annihilation operator $\hat{a}$ by the beam splitter whose amplitude reflectivity $r=\sin\theta$ and transmittivity $t=\cos\theta$. Throughout the paper the beam splitter is assumed to be lossless. The two output field annihilation operators $\hat{c}$ and $\hat{d}$ are related to the beam splitter input fields by the transformation

$$\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}\hat{B}\begin{pmatrix}\hat{c}\\ \hat{d}\end{pmatrix}\hat{B}^\dagger=\begin{pmatrix}t\hat{c}+e^{i\varphi}r\hat{d}\\ t\hat{d}-e^{-i\varphi}r\hat{c}\end{pmatrix}, \tag{2.1}$$

for the unitary beam splitter operator [2]

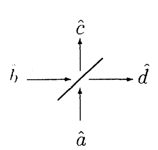

FIG. 1. Beam splitter with the signal input in mode b and the noise field input in mode a.

$$\hat{B}=\exp\left[\theta\left(e^{-i\varphi}\hat{c}\hat{d}^\dagger-e^{i\varphi}\hat{c}^\dagger\hat{d}\right)\right], \tag{2.2}$$

where r and t are real. Since the phase shift φ due to the beam splitter does not affect the theory, we assume $\varphi=0$ for simplicity. (Even when the phase shift is taken as $\varphi=\pi/2$ as in [23], for example, the phase can later be absorbed into the coherent amplitudes, and thus we can assume $\varphi=0$ without loss of generality.)

When the field is fully described by the classical theory, the field can be represented in phase space by the positive nonsingular GS P representation as in Eqs. (1.1) and (1.2). For nonclassical fields, however, the GS P representation is either negative or not well behaved. This disadvantage is due to the GS P representation being an diagonal expansion by coherent states. The generalized P representation, especially the positive P representation, has been introduced to meet the growing awareness of the nonclassical fields [15]. The positive P function, even for a nonclassical state, is smooth and positive. Because we want to treat arbitrary signal and noise fields, we will use the positive P representation to describe the fields.

The noise field with density operator $\hat{\rho}_a$ can be written as the weighted integral of the coherent projection operators,

$$\hat{\rho}_a=\int d^2\alpha\, d^2\gamma\, P_a'(\alpha,\gamma)|\alpha\rangle\langle\gamma^*|, \tag{2.3}$$

where $P_a'(\alpha,\gamma)$ is the positive P representation $P(\alpha,\gamma)$ divided by the trace of the projection operator,

$$P'(\alpha,\gamma)=P(\alpha,\gamma)/\langle\gamma^*|\alpha\rangle. \tag{2.4}$$

Similarly the signal field with $\hat{\rho}_b$ is

$$\hat{\rho}_b=\int d^2\beta\, d^2\delta\, P_b'(\beta,\delta)|\beta\rangle\langle\delta^*|, \tag{2.5}$$

where

$$P'(\beta,\delta)\equiv P(\beta,\delta)/\langle\delta^*|\beta\rangle. \tag{2.6}$$

Coherent state $|\alpha\rangle$ is mathematically generated by the displacement of the vacuum with the displacement operator [10]

$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a}). \tag{2.7}$$

The density operator $\hat{\rho}_{\text{in}}$ for the total input field is then written as

$$\hat{\rho}_{\text{in}}=\int d^2\alpha\, d^2\beta\, d^2\delta\, d^2\gamma\, P_a'(\alpha,\gamma)P_b'(\beta,\delta)\times\hat{D}_a(\alpha)\hat{D}_b(\beta)|0\rangle\langle0|\hat{D}_b^\dagger(\gamma)\hat{D}_a^\dagger(\delta). \tag{2.8}$$

From the beam splitter transformation (2.1), we know that the density operator $\hat{\rho}_{\text{out}}$ for the output field from the beam splitter is

$$\hat{\rho}_{\text{out}}=\int d^2\alpha\, d^2\beta\, d^2\delta\, d^2\gamma\, P_a'(\alpha,\gamma)\times P_b'(\beta,\delta)|t\alpha-r\beta\rangle_{cc}\langle t\gamma^*-r\delta^*|\otimes|t\beta+r\alpha\rangle_{dd}\langle t\delta^*+r\gamma^*|. \tag{2.9}$$

The reduced density operator $\hat{\rho}_d$ for mode d is obtained by taking the trace over mode c,

$$\hat{\rho}_d = \int d^2\alpha\, d^2\beta\, d^2\delta\, d^2\gamma\; P_a(\alpha,\gamma)P_b(\beta,\delta) \times |t\beta + r\alpha\rangle_{dd}\langle t\delta^* + r\gamma^*| \frac{\langle t\gamma^* - r\delta^* | t\alpha - r\beta\rangle}{\langle\gamma^*|\alpha\rangle\langle\delta^*|\beta\rangle} , \tag{2.10}$$

where the relation (2.4) has been used. The density operator may also be written in the general form

$$\hat{\rho}_d = \int d^2\phi\, d^2\psi\; \frac{P_d(\phi,\psi)}{\langle\psi^*|\phi\rangle} |\phi\rangle_{dd}\langle\psi^*| . \tag{2.11}$$

Comparing Eqs. (2.10) and (2.11), we find the convolution law for the positive P representation,

$$P_d(\phi,\psi) = \frac{1}{t^4}\int d^2\alpha\, d^2\gamma\; P_a(\alpha,\gamma)P_b \times \left(\frac{\phi - r\alpha}{t}, \frac{\psi - r\gamma}{t}\right) . \tag{2.12}$$

The GS P representation $P(\alpha)$ is a special case of the positive P representation $P(\alpha,\alpha^*)$. The convolution relation (2.12) reduces to (1.4) when $\phi = \psi^*$ with appropriate weight rates to take the beam splitter t and r values into account.

The Wigner and Q functions are other quasiprobability functions and are well defined even for the nonclassical state, although the Wigner function can be negative for certain nonclassical states. The positive P representation is also well behaved for a nonclassical state, but it is defined in four-dimensional space. The Q or Wigner function, both of which are defined in two-dimensional space, is, therefore, sometimes easier to treat, so we will extend the convolution law for the uses of the Q and Wigner functions.

Drummond and Gardiner have proven that the positive P representation may also be defined as the Fourier transform of the characteristic function,

$$C^{(p)}(\xi) = \mathrm{Tr}[\hat{\rho}\exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})] , \tag{2.13}$$

for the field represented by density operator $\hat{\rho}$ [15]. This function for the positive P representation is the same as that for the GS P representation and is related to characteristic functions $C^{(q)}(\xi)$ and $C^{(w)}(\xi)$ for the Q and Wigner functions as shown in the following equation [28]:

$$C^{(p)}(\xi)\exp(-|\xi|^2/2) = C^{(w)}(\xi) = C^{(q)}(\xi)\exp(|\xi|^2/2) . \tag{2.14}$$

By using the convolution theorem, we can factorize the inverse Fourier transform of the convolution law (2.12) as

$$C_d^{(p)}(\xi) = C_a^{(p)}(r\xi)C_b^{(p)}(t\xi) , \tag{2.15}$$

where $C_a^{(p)}$ and $C_b^{(p)}$ are the characteristic functions for $P_a(\alpha,\gamma)$ and $P_b(\beta,\delta)$. Using Eq. (2.14), we can write Eq. (2.15) as

$$\begin{aligned} C_d^{(q)}(\xi) &= C^{(p)}(\xi)e^{-|\xi|^2} \\ &= C_a^{(p)}(r\xi)e^{-|r\xi|^2}C_b^{(p)}(t\xi)e^{-|t\xi|^2} \\ &= C_a^{(q)}(r\xi)C_b^{(q)}(t\xi) , \end{aligned} \tag{2.16}$$

where $C_a^{(q)}$ and $C_b^{(q)}$ are the characteristic functions for $Q_a(\alpha,\gamma)$ and $Q_b(\beta,\delta)$ and where the relation $r^2 + t^2 = 1$ for the lossless beam splitter has been used. The above equation is Fourier transformed to give the convolution law for the Q function,

$$Q_d(\phi) = \frac{1}{t^2}\int d^2\alpha\; Q_a(\alpha)Q_b\left(\frac{\phi - r\alpha}{t}\right) . \tag{2.17}$$

Similarly, the Wigner function convolution law for superposition of the two fields with the Wigner functions $W_a(\alpha)$ and $W_b(\beta)$ in the beam splitter is

$$W_d(\phi) = \frac{1}{t^2}\int d^2\alpha\; W_a(\alpha)W_b\left(\frac{\phi - r\alpha}{t}\right) . \tag{2.18}$$

Leonhardt has used the wave-function method to study the convolution of the superposition field in the 50:50 beam splitter [16], and if we take $r = t = 1/\sqrt{2}$ the convolution law (2.18) agrees with his result. The convolution law is also related to the simultaneous measurement of noncommuting observables [16,29–31]. When the signal field is split into two beams by a beam splitter and one quadrature component is measured in one beam and the other quadrature component is measured in the other beam, the joint probability of obtaining $\mathrm{Re}[\phi]$ and $\mathrm{Im}[\phi]$ is given by (2.18) [16].

Example: squeezed thermal noise field

There have been a number of works on phase-insensitive amplification and attenuation using superposition of the signal field and added noise fields. The amplification and attenuation are assumed to be linear, and in this case, the quantum state of the amplified or dissipated signal is calculated as the superposition of the original signal and the thermal noise field [32–35].

On the other hand, phase-sensitive linear processes are also important. The signal can be amplified or attenuated by a phase-sensitive reservoir [20–22]. There also is a phase-sensitive measurement process, in which the signal field is coupled to a reservoir and is measured by heterodyne detection with phase-dependent accuracy and phase-dependently added noise [36].

In this example, we analyze the superposition of the phase-sensitive (i.e., squeezed) thermal noise field on the signal prepared in an arbitrary field. A squeezed state has reduced fluctuations in one quadrature at the expense of the enhanced noise in the other quadrature. The density operator $\hat{\rho}_{\mathrm{sq}}$ for the squeezed thermal state is defined [37] as

$$\hat{\rho}_{\mathrm{sq}} = \hat{S}(s)\hat{\rho}_{\mathrm{th}}\hat{S}^\dagger(s) , \tag{2.19}$$

where $\hat{\rho}_{\text{th}}$ is the density operator for the thermal field [38] and the squeeze operator $\hat{S}(s)$ for the real squeeze parameter s is

$$\hat{S}(s) = \exp\left[\frac{s}{2}(\hat{a}^\dagger)^2 - \frac{s}{2}\hat{a}^2\right] . \tag{2.20}$$

The state of the quantum-mechanical system is characterized by the set of expectation values of the system operators. The normally ordered moments of the bosonic operators are evaluated, with the help of the characteristic function $C^{(p)}(\xi)$ defined in Eq. (2.13), as follows:

$$\langle (a^\dagger)^m a^n \rangle = D_\xi^m D_{\xi^*}^n C^{(p)}(\xi)|_{\xi=0} , \tag{2.21}$$

where the partial differentials are

$$D_\xi = \frac{\partial}{\partial \xi} \quad \text{and} \quad D_{\xi^*} = \frac{\partial}{\partial(-\xi^*)} . \tag{2.22}$$

The characteristic function for the superposition of the squeezed thermal field on the signal field is factorized as shown in Eq. (2.15). Accordingly, the normally ordered moments are found to be

$$\begin{aligned}\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle &= D_\xi^m D_{\xi^*}^n C_{\text{sq}}^{(p)}(\xi) C_{\text{s}}^{(p)}(\xi)|_{\xi=0} \\ &= \sum_{\ell=0}^{m}\sum_{k=0}^{n} \binom{m}{\ell}\binom{n}{k} r^{k+\ell} t^{m+n-k-\ell} \\ &\quad \times \langle (\hat{a}^\dagger)^\ell \hat{a}^k \rangle_{\text{sq}} \langle (\hat{a}^\dagger)^{m-\ell} \hat{a}^{n-k} \rangle_{\text{s}} ,\end{aligned} \tag{2.23}$$

where the subscripts sq and s, respectively, denote the squeezed thermal and signal fields.

According to the definition (2.19) the expectation value of the electric field for the squeezed thermal field is zero. From Eq. (2.23) the mean photon number $\langle \hat{n} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle$ of the superposition field is found to be the sum of the mean photon numbers of the component fields,

$$\langle \hat{n} \rangle = R\langle \hat{n} \rangle_{\text{sq}} + T\langle \hat{n} \rangle_{\text{s}} , \tag{2.24}$$

where $R = r^2$ and $T = t^2$ are the intensity reflectivity and transmittivity of the beam splitter. This is obvious because the superposition does not violate the conservation of energy.

We are particularly interested in the quadrature noise added by the superposition because the squeezed thermal state has the phase-dependent noise. The quadrature operators are defined by

$$\hat{\chi}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2} \quad \text{and} \quad \hat{\chi}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i} . \tag{2.25}$$

The quadrature variances, $\langle (\Delta\hat{\chi}_i)^2 \rangle = \langle \hat{\chi}_i^2 \rangle - \langle \hat{\chi}_i \rangle^2$, of the squeezed thermal state [37,39] are given by

$$\langle (\Delta\hat{\chi}_1)^2 \rangle = \frac{1}{4}(2\bar{n}_{\text{th}} + 1)e^{-2s} \tag{2.26}$$

and

$$\langle (\Delta\hat{\chi}_2)^2 \rangle = \frac{1}{4}(2\bar{n}_{\text{th}} + 1)e^{2s} , \tag{2.27}$$

where $\bar{n}_{\text{th}}$ is the average thermal photon number for the thermal field without squeezing.

Using the definition (2.25) we find the following quadrature variances of the superposition field:

$$\langle (\Delta\hat{\chi}_i)^2 \rangle = R\langle (\Delta\hat{\chi}_i)^2 \rangle_{\text{sq}} + T\langle (\Delta\hat{\chi}_i)^2 \rangle_{\text{s}} , \quad i = 1, 2 . \tag{2.28}$$

The quadrature variance for the output field is the weighted sum of the quadrature variances for the input fields. As seen in Eqs. (2.26) and (2.27), one of the quadrature variances of the squeezed thermal field can be reduced below the vacuum limit for $s \gg 1$, which means that it is possible to reduce the quadrature noise of the signal by superposing it with the phase-sensitive noise.

It should be noted here that the above calculation based on the convolution (2.12) is realistic while the direct use of the simplified convolution (1.4) will lead to a wrong result. If we start from (1.4) instead of (2.12), we will arrive at Eq. (2.15) with $r = t = 1$, which leads, with the aid of Eqs. (2.21) and (2.23), to

$$\langle (\Delta\chi_i)^2 \rangle = \langle (\Delta\chi_i)^2 \rangle_{\text{sq}} + \langle (\Delta\chi_i)^2 \rangle_{\text{s}} - \tfrac{1}{4} , \tag{2.29}$$

where $-1/4$ comes from the quadrature variance of the vacuum field. If we assume that both signal and noise fields are in a strongly squeezed state having nearly zero quadrature variance, then the quadrature variance of the superposed field should be negative, which does not make sense. This difficulty is due to the assumption that two fields can be superposed without any modification. The difficulty is removed by adopting the convolution law (2.12), which describes the realistic superposition of two fields via a beam splitter.

The photon number variance $\langle (\Delta\hat{n})^2 \rangle$ for the output field is

$$\begin{aligned}\langle (\Delta\hat{n})^2 \rangle &= R^2\langle (\Delta\hat{n})^2 \rangle_{\text{sq}} + T^2\langle (\Delta\hat{n})^2 \rangle_{\text{s}} \\ &\quad + TR\big[2\langle \hat{a}^\dagger\hat{a} \rangle_{\text{sq}}\langle \hat{a}^\dagger\hat{a} \rangle_{\text{s}} + \langle \hat{a}^2 \rangle_{\text{sq}}\langle (\hat{a}^\dagger)^2 \rangle_{\text{s}} \\ &\quad + \langle \hat{a}^2 \rangle_{\text{s}}\langle (\hat{a}^\dagger)^2 \rangle_{\text{sq}}\big] .\end{aligned} \tag{2.30}$$

This variance is determined not only by the weighted sum of those for the input fields but also by the cross term shown in the third term of Eq. (2.30). This is because the input fields are superposed coherently by the beam splitter; the photon statistics are thus determined by the square of the coherent sum of the amplitudes of the two fields.

III. PHASE-SENSITIVE ATTENUATION

Phase-sensitive reservoirs, based on the establishment of squeezed light, have been studied extensively in recent years [20–22], and in this section the phase-sensitive reservoir is modeled by an infinite array of beam splitters. Each beam splitter provides a degree of freedom, so the reservoir has infinitely many degrees of freedom. As shown in Fig. 2, a signal passes through an infinite number of beam splitters whose transmittivity is nearly unity. The noise fields $\hat{\rho}_j$ are assumed to be produced by independent noise sources, and each $\hat{\rho}_j$ is coupled by

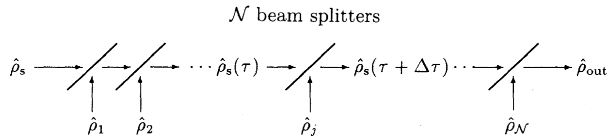

FIG. 2. The phase-sensitive reservoir modeled as an array containing an infinite number of beam splitters. The signal is injected from left and the independent squeezed fields (all with the same properties) are injected into the other ports. The transmittivity is considered to be nearly unity.

the jth beam splitter. In this paper we assume a homogeneous attenuator for simplicity, so the $\hat\rho_j$ all have the same noise property.

A similar study, in the Heisenberg picture, has been done for the signal transmitted through a lossy fiber coupled to the vacuum in a phase-insensitive manner [7,8], and here we are extending the study to couple the signal with the phase-sensitive reservoir in order to describe the dynamics of the signal field in the Schrödinger picture.

A. Fokker-Planck equation

In this section we derive the Fokker-Planck equation for the phase-sensitive reservoir for the Q function. The total duration of time when the field is coupled with the lossy channel is denoted by T, the total number of the beam splitters by $\mathcal{N}$, and the interval between the adjacent beam splitters by $\Delta\tau$. The beam splitters are first taken to be discrete components, but their number

$$\mathcal{N} = T/\Delta\tau \tag{3.1}$$

is later taken to infinity in order to model a continuous attenuating reservoir. Under the assumption that the reflectivity is very small for the beam splitter, Eq. (2.17) is written as

$$Q(\alpha) \approx (1+R)\int d^2\alpha\; Q_a(\beta)\; Q_b\left(\frac{\alpha - r\beta}{t}\right) . \tag{3.2}$$

To calculate the effects of attenuation, we need an expression for the output signal operator in terms of the input operators. To simulate an attenuator, we consider the beam splitters forming a continuous array by taking the limits

$$\mathcal{N}\to\infty, \quad \Delta\tau\to 0, \quad \text{and} \quad R\to 0 . \tag{3.3}$$

These limits cannot be taken independently: from Eq. (3.1), $\mathcal{N}\Delta\tau$ should be kept constant. Also, the total energy loss within T is described by $1-\exp(-\kappa T)$, where κ is the attenuation coefficient, and this loss should be equivalent to the beam splitter loss so that

$$\begin{aligned}(1-R)^{\mathcal{N}} &= e^{-\kappa T} \quad \to \mathcal{N}\ln(1-R)\\ &= -\kappa\mathcal{N}\Delta\tau \quad \to R\approx\kappa\Delta\tau ,\end{aligned} \tag{3.4}$$

where the approximation was made under the assumption that R is small.

Let us define $Q(\tau;\alpha)$ as the Q function of the signal field incident on the beam splitter at time τ, $Q_{sq}(\beta)$ as the Q function for the phase-sensitive noise added to the signal at the beam splitter, and $Q(\tau+\Delta\tau;\alpha)$ as the Q function for the signal leaving from the beam splitter. The squeezed thermal fields produced by the independent stationary sources act as phase-sensitive noise in our model, so the $\hat\rho_j$ have the same Q function, which is denoted by $Q_{sq}(\beta)$ in the following.

From Eq. (3.2), we obtain the relation

$$Q(\tau+\Delta\tau;\alpha) = (1+R)\int d^2\beta\; Q\left(\tau;\frac{\alpha-r\beta}{t}\right) Q_{sq}(\beta) , \tag{3.5}$$

and $(\alpha - r\beta)/t$ is expanded as

$$\begin{aligned}\frac{\alpha-r\beta}{t} = \left(\alpha-\sqrt{R}\,\beta\right)\frac{1}{\sqrt{1-R}} &\approx \alpha + \frac{R}{2}\,\alpha - \sqrt{R}\,\beta\\ &\equiv \alpha+\Delta\alpha ,\end{aligned} \tag{3.6}$$

where up to the first order terms of R are kept. The usual Taylor expansion for a real function having a complex argument is used to expand $Q\left(\tau;\frac{\alpha-r\beta}{t}\right)$,

$$\begin{aligned}Q\left(\tau;\frac{\alpha-r\beta}{t}\right) &= Q(\tau;\alpha+\Delta\alpha)\\ &= Q(\tau;\alpha) + \frac{\partial Q}{\partial\alpha_1}\frac{R\alpha_1}{2} + \frac{\partial Q}{\partial\alpha_2}\frac{R\alpha_2}{2}\\ &\quad + \frac{1}{2}\frac{\partial^2 Q}{\partial\alpha_1^2}R\beta_1^2 + \frac{1}{2}\frac{\partial^2 Q}{\partial\alpha_2^2}R\beta_2^2\\ &\quad + \frac{\partial^2 Q}{\partial\alpha_1\partial\alpha_2}R\beta_1\beta_2 ,\end{aligned} \tag{3.7}$$

where, again, up to the first order terms of R have been kept and the real and imaginary parts of α and β are, respectively, denoted by α_1, α_2 and β_1, β_2. The function Q is the simplified notation of the function $Q(\tau;\alpha)$.

Substituting Eq. (3.7) into Eq. (3.9), we obtain

$$\begin{aligned}Q(\tau+\Delta\tau;\alpha) = Q(\tau;\alpha) + \frac{R}{2}\int d^2\beta\; Q_{sq}(\beta)\Big[\frac{\partial}{\partial\alpha_1}(\alpha_1 Q)\\ + \frac{\partial}{\partial\alpha_2}(\alpha_2 Q) + \frac{\partial^2 Q}{\partial\alpha_1^2}\beta_1^2 + \frac{\partial^2 Q}{\partial\alpha_2^2}\beta_2^2\\ + 2\frac{\partial^2 Q}{\partial\alpha_1\partial\alpha_2}\beta_1\beta_2\Big] ,\end{aligned} \tag{3.8}$$

which, with Eq. (3.4), leads to

$$\begin{aligned}\frac{dQ(\tau;\alpha)}{d\tau} &= \frac{\kappa}{2}\left[\frac{\partial}{\partial\alpha_1}(\alpha_1 Q) + \frac{\partial}{\partial\alpha_2}(\alpha_2 Q)\right]\int d^2\beta\; Q_{sq}(\beta)\\ &\quad + \frac{\kappa}{2}\Big[\frac{\partial^2 Q}{\partial\alpha_1^2}\int d^2\beta\;\beta_1^2 Q_{sq}(\beta)\\ &\quad + \frac{\partial^2 Q}{\partial\alpha_2^2}\int d^2\beta\;\beta_2^2 Q_{sq}(\beta)\Big] .\end{aligned} \tag{3.9}$$

To finish the calculation we need to have the precise form of the Q function for the squeezed thermal field.

Recalling that the Q function is proportional to the average value of the density operator in a coherent state [37]; that is,

$$Q_{\rm sq}(\beta) = \frac{1}{\pi}\langle\beta|\hat{\rho}_{\rm sq}|\beta\rangle \ , \qquad (3.10)$$

we use the definition of the density operator (2.19) for the squeezed thermal field and find $Q_{\rm sq}(\beta)$ to be

$$Q_{\rm sq}(\beta) = \frac{1/\pi}{\sqrt{(1+N+M)(1+N-M)}} \times \exp\left[-\frac{\beta_1^2}{1+N-M} - \frac{\beta_2^2}{1+N+M}\right] , \qquad (3.11)$$

where

$$N = \sinh^2 s + \bar{n}_{\rm th}\cosh 2s \qquad (3.12)$$

is the mean photon number for the squeezed thermal field and

$$M = \tfrac{1}{2}(2\bar{n}_{\rm th}+1)\sinh 2s \qquad (3.13)$$

is the sum of both the quadrature variances given by Eqs. (2.26) and (2.27).

Substituting the simple Gaussian integration of $Q_{\rm sq}(\beta)$ into Eq. (3.9), we obtain the Fokker-Planck equation for the field coupled to a phase-sensitive attenuation reservoir [24],

$$\frac{dQ(\tau,\alpha)}{d\tau} = \frac{\kappa}{2}\left[\frac{\partial}{\partial\alpha_1}\alpha_1 + \frac{\partial}{\partial\alpha_2}\alpha_2 + \frac{1}{2}(1+N+M)\frac{\partial^2}{\partial\alpha_1^2} + \frac{1}{2}(1+N-M)\frac{\partial^2}{\partial\alpha_2^2}\right]Q(\tau,\alpha) \ . \qquad (3.14)$$

For the phase-insensitive case, this reduces to the usual Fokker-Planck equation [20,40] with $N = \bar{n}_{\rm th}$ and $M = 0$. The Fokker-Planck equation is relatively simple and observables can be calculated as correlations of the quasiprobability function. The Fokker-Planck equation has usually been derived, with the help of the correspondence principle [41], from the master equation for the density operator [14].

B. Evolution of attenuated signal

In this section the Fokker-Planck equation (3.14) is solved for an arbitrary signal attenuated in the phase-sensitive reservoir. The density operator for an arbitrary input state of the single-mode field, written in the coherent state basis, takes the form

$$\hat{\rho}_s = \int d^2\mu d^2\nu P(\mu,\nu)\frac{|\mu\rangle\langle\nu^*|}{\langle\nu^*|\mu\rangle} \ , \qquad (3.15)$$

where $|\mu\rangle$ and $|\nu\rangle$ are coherent states and $P(\mu,\nu)$ the positive P function. The Q function corresponding to the input field density operator (3.15) then becomes a weighted integral of Gaussian functions,

$$Q_s(\alpha) = \frac{1}{\pi}\int d^2\mu d^2\nu P(\mu,\nu) \times \exp[-(\alpha_1-A)^2-(\alpha_2-B)^2] \ , \qquad (3.16)$$

where

$$A = \tfrac{1}{2}(\mu+\nu), \qquad B = \frac{1}{2i}(\mu-\nu) \ . \qquad (3.17)$$

It has been recently shown by Peřinová *et al.* [40] that if the initial Q function of the quantum system is (complex) Gaussian, then the solution of the Fokker-Planck equation (3.14) is also Gaussian with time-dependent parameters. The Q function (3.16) is a weighted integral of complex Gaussian functions, so one can use Peřinová's argument to obtain the time evolution of the input state (3.15). The quasiprobability function $Q_{\rm att}(\alpha)$ for the output signal can be written in the form

$$Q_{\rm att}(\alpha,\tau) = \frac{1/\pi}{\sqrt{[N(\tau)+1]^2 - M(\tau)^2}}\int d^2\mu d^2\nu P(\mu,\nu) \times \exp\left\{-\frac{[\alpha_1 - A(\tau)]^2}{1+N(\tau)+M(\tau)} - \frac{[\alpha_2 - B(\tau)]^2}{1+N(\tau)-M(\tau)}\right\} , \qquad (3.18)$$

where the attenuated signal is represented by

$$A(t) = A\exp(-\kappa\tau/2) \quad \text{and} \quad B(t) = B\exp(-\kappa\tau/2) \ , \qquad (3.19)$$

and the noise parameters are

$$N(\tau) = N[1-\exp(-\kappa\tau)]$$

and

$$M(\tau) = M[1-\exp(-\kappa\tau)] \ . \qquad (3.20)$$

The inverse Fourier transform of this Q function gives the characteristic function $C_{\rm att}^{(q)}$ as a product of two characteristic functions,

$$C_{\rm att}^{(q)}(\xi,\tau) = C_s^{(q)}(\xi,\tau)C_{\rm sq}^{(q)}(\xi,\tau) \ , \qquad (3.21)$$

where $C_s^{(q)}(\xi,\tau)$ is the characteristic function for the attenuated signal without noise [the characteristic function $C_s^{(q)}(\xi,\tau=0)$ is the inverse Fourier transform of $Q_s(\alpha)$ in Eq. (3.16)],

$$C_s^{(q)}(\xi,\tau) = C_s^{(q)}(\sqrt{e^{-\kappa\tau}}\,\xi) \ , \qquad (3.22)$$

and $C_{\rm sq}^{(q)}(\xi,\tau)$ is the characteristic function, given by Eq. (3.11), for the time-dependent squeezed thermal field with the parameters (3.20),

$$C_{\rm sq}^{(q)}(\xi,\tau) = \exp[-(N+M+1)(1-e^{-\kappa\tau}) \times\xi_1^2 - (N-M+1)(1-e^{-\kappa\tau})\xi_2^2] = C_{\rm sq}^{(q)}\left(\sqrt{1-e^{-\kappa\tau}}\,\xi\right) \ , \qquad (3.23)$$

where the real and imaginary parts of ξ are, respectively, denoted by ξ_1 and ξ_2. We have shown that the factorization is extended to the dynamics of the signal in the phase-sensitive reservoir. Because the characteristic function for the signal influenced by the phase-sensitive reservoir is factorized, the convolution law (2.17) can be applied. Thus coupling of the signal with the phase-sensitive reservoir brings about the superposition of the squeezed thermal field on the attenuated signal. We can also easily find the properties of the output field by using the relation (2.23).

In the case of attenuation we take $r=\sqrt{1-e^{-\kappa\tau}}$ and $t=\sqrt{e^{-\kappa\tau}}$ to calculate the moments specified by (2.23). The quadrature variance for the output field is

$$\langle(\Delta\hat{\chi}_1)^2\rangle_{\rm att}=\langle(\Delta\hat{\chi}_1)^2\rangle_s e^{-\kappa\tau}+\langle(\Delta\hat{\chi}_1)^2\rangle_{\rm sq}(1-e^{-\kappa\tau})\,, \qquad (3.24)$$

where $\langle(\Delta\hat{\chi}_1)^2\rangle_s$ is the quadrature variance for the signal and the quadrature variance (2.26) for the squeezed thermal state is written as

$$\langle(\Delta\hat{\chi}_1)^2\rangle_{\rm sq}=\tfrac{1}{4}[2(N-M)+1]\,, \qquad (3.25)$$

with the definitions (3.12) and (3.13). From (3.24) we clearly see that while the properties of the signal decay exponentially, those of the reservoir become pronounced.

The above description is similar to those discussed by Fearn [42] and Leonhardt [16] in the sense that the evolution is described by exponential decay of the original signal field with the noise inflow from the reservoir. The longer the signal couples to the reservoir, the more the signal is dissipated and the more noise flows into it. Fearn [42] described this dynamics in the Heisenberg picture by a beam splitter having a decaying transmittivity, and Leonhardt described the dynamics of the density operator of the signal in the Schrödinger picture. In the present paper the evolution of the signal is described using the Schrödinger picture in quite a simple way — as shown in Eqs. (3.21)–(3.24) — and we extend the analysis to describe (phase-sensitive) amplifiers.

IV. PHASE-SENSITIVE AMPLIFICATION

An amplification process is inevitably accompanied by the increase of the quantum noise in the system [21]. In other words, the amplification degrades an optical signal and rapidly destroys quantum features that may have been associated with the signal. The noise transferred from the amplifier to the system can be associated either with quadrature fluctuations or with photon-number fluctuations, and the nature of the amplifier affects the physical properties of the amplified states of light. In particular, for an arbitrarily squeezed input the phase-insensitive amplifier provides a squeezed output only for a gain smaller than two [33]. To overcome this cloning limit, phase-sensitive amplifiers with reduced quadrature fluctuations have been proposed. The phase-sensitive amplifier is conceptually based on the establishment of squeezed light and enables a squeezed input to remain squeezed after amplification with a gain greater than two [22]. The phase-sensitive amplifier is a nonclassical amplifier that selectively preserves the phase information during the amplification process.

A. Fokker-Planck equation

The amplification process can also be modeled by an array of beam splitters similar to that shown in Fig. 2. In their experiment with phase-sensitive amplification, Ou *et al.* have a nondegenerate parametric amplifier where the signal field is amplified and the idler mode is coupled with the squeezed vacuum [27]. As in their experiment a two-mode parametric optical amplifier is modeled here by an amplification beam splitter matrix. For a two-mode parametric amplifier the signal input $\hat{b}$ is transformed into the amplified output $\hat{d}$ with unavoidable noise $\hat{a}^\dagger$,

$$\begin{pmatrix}\hat{d}\\ \hat{c}^\dagger\end{pmatrix}=\begin{pmatrix}\sqrt{g} & i\sqrt{g-1}\\ -i\sqrt{g-1} & \sqrt{g}\end{pmatrix}\begin{pmatrix}\hat{b}\\ \hat{a}^\dagger\end{pmatrix}, \qquad (4.1)$$

where $g>1$ is the infinitesimal amplification factor. We are going to build a beam-splitter-like relation for amplification, and consecutive application of an infinite number of Eq. (4.1) will give the final amplification result. The actual gain G by the amplifier will, thus, be proportional to g. The beam splitter output and input transformation can be written as

$$\begin{pmatrix}\hat{b}\\ \hat{a}\end{pmatrix}=\hat{B}_1\begin{pmatrix}\hat{d}\\ \hat{c}\end{pmatrix}B_1^\dagger=\begin{pmatrix}\sqrt{g}\,\hat{d}-i\sqrt{g-1}\,\hat{c}^\dagger\\ \sqrt{g}\,\hat{c}-i\sqrt{g-1}\,\hat{d}^\dagger\end{pmatrix}, \qquad (4.2)$$

where the unitary amplification beam splitter operator has been introduced in analogy with the two-mode squeezing operator [43],

$$B_1=\exp[i\theta_1(\hat{c}^\dagger\hat{d}^\dagger+\hat{c}\hat{d})]\,, \qquad (4.3)$$

with $\cosh\theta_1=\sqrt{g}$ and $\sinh\theta_1=\sqrt{g-1}$.

To analyze the beam splitter transformation for the amplifier, let us assume we have two coherent states $|\alpha\rangle_a$ and $|\beta\rangle_b$ at the two input ports. Then the output field is

$$\hat{B}_1\hat{D}_a(\alpha)\hat{D}_b(\beta)|0\rangle_a|0\rangle_b=\hat{D}_c(\sqrt{g}\,\alpha)\hat{D}_d(i\sqrt{g-1}\,\alpha^*)\times\hat{D}_c(i\sqrt{g-1}\,\beta^*)\times\hat{D}_d(\sqrt{g}\,\beta)\hat{B}_1|0\rangle_a|0\rangle_b\,, \qquad (4.4)$$

where

$$\hat{B}_1|0\rangle_a|0\rangle_b=\frac{1}{\cosh\theta_1}\sum_n(\tanh\theta_1)^n e^{i\frac{n\pi}{2}}|n\rangle_c|n\rangle_d\,. \qquad (4.5)$$

More generally, the input fields are expressed as a weighted sum of diagonal coherent components,

$$\hat{\rho}_{\rm in}=\int d^2\alpha\, d^2\beta\, P_a(\alpha)P_b(\beta)|\alpha\rangle_{aa}\langle\alpha|\otimes|\beta\rangle_{bb}\langle\beta|\,, \qquad (4.6)$$

where P_a and P_b are, respectively, the GS P representations for modes a and b. Tracing the output field over mode c, we find the output density operator for mode d,

$$\hat{\rho}_d = \int d^2\alpha \, d^2\beta \, P_a(\alpha)P_b(\beta)D_d(\delta)\hat{\rho}_{\rm th}D_d^\dagger(\delta) \; , \qquad (4.7)$$

where

$$\delta = i\sqrt{g-1}\,\alpha^* + \sqrt{g}\,\beta \qquad (4.8)$$

and $\hat{\rho}_{\rm th}$ is the thermal field density operator for the mean photon number $\bar{n} = g-1$. Even when both the signal and the idler fields are in the vacuum state, i.e., $\alpha = \beta = 0$, the amplifier brings noise into the fields. The noise energy comes from the pump.

The density operator for the thermal field [38] can be written as

$$\hat{\rho}_{\rm th} = \int d^2\phi \; P_T(\phi)|\phi\rangle\langle\phi| \; , \qquad (4.9)$$

and its GS P representation is

$$P_T(\phi) = \frac{1/\pi}{g-1}\exp\left(-\frac{|\phi|^2}{g-1}\right) \; . \qquad (4.10)$$

By using Eq. (4.9), we find the density operator (4.7) for output mode d to be

$$\begin{aligned}\hat{\rho}_d &= \int d^2\alpha \, d^2\beta \, d^2\phi \; P_T(\phi)P_a(\alpha)P_b(\beta)|\phi+\delta\rangle\langle\phi+\delta| \\ &= \int d^2\zeta \; P_d(\zeta)|\zeta\rangle\langle\zeta| \; , \end{aligned} \qquad (4.11)$$

where the GS P representation for the output field is defined as convolution of the three P representations

$$\begin{aligned}P_d(\zeta) = \frac{1}{g}\int d^2\phi \, d^2\alpha \; P_a(\alpha)P_b\left(\frac{\zeta-\phi-i\alpha^*\sqrt{g-1}}{\sqrt{g}}\right) \\ \times P_T(\phi) \; . \end{aligned} \qquad (4.12)$$

The inverse Fourier transform of the GS P representation gives the characteristic function $C_d^{(p)}$ for the output field in the form of the product of the characteristic functions $C_a^{(p)}$, $C_b^{(p)}$, and $C_T^{(p)}$ for the input modes a and b and the thermal field,

$$C_d^{(p)}(\eta) = C_a^{(p)}(-i\sqrt{g-1}\;\eta)C_b^{(p)}(\sqrt{g}\;\eta)C_T^{(p)}(\eta) \; . \qquad (4.13)$$

We can simplify this relation using the relation between the characteristic functions for the various quasiprobabilities (2.14) and the exact form of the characteristic function for the thermal field from the inverse Fourier transformation of (4.10), yielding

$$C_d^{(q)}(\eta) = C_a^{(p)}(-i\sqrt{g-1}\;\eta)C_b^{(q)}(\sqrt{g}\;\eta) \; . \qquad (4.14)$$

The Fourier transformation of this shows that a modified convolution between Q_b for the signal Q function and P_a for the noise P representation results in Q_d for the Q function of the output field,

$$Q_d(\zeta) = \frac{1}{g}\int d^2\lambda \; P_a(\lambda)Q_b\left(\frac{\zeta - i\sqrt{g-1}\;\lambda}{\sqrt{g}}\right) \; . \qquad (4.15)$$

The convolution relation for amplification differs from that for attenuation (2.17) because of the unavoidable extra noise due to the thermal field (4.9).

Consider an array of $\mathcal{N}$ beam splitters that satisfy the transformation relation (4.2). To simulate an amplifier we will take $\mathcal{N} \to \infty$ and let the infinitesimal amplification factor for each beam splitter be given by $g = 1+\epsilon \approx 1$. After a signal passes through the $\mathcal{N}$ beam splitters, it is amplified by the factor of

$$G = e^{\gamma T} = (1+\epsilon)^{\mathcal{N}} \; , \qquad (4.16)$$

where T is the total transition time of the signal through the beam splitters and γ is the amplification coefficient. By using the Taylor expansion of Q function (4.15) to the second order under the assumption $\epsilon \approx 0$ (we had $r \approx 0$ for attenuation), we obtain the Fokker-Planck equation for amplification,

$$\begin{aligned}\frac{dQ(\tau,\alpha)}{d\tau} = \frac{\gamma}{2}\bigg[&-\frac{\partial}{\partial\alpha_1}\alpha_1 - \frac{\partial}{\partial\alpha_2}\alpha_2 + \frac{1}{2}(N+M)\frac{\partial^2}{\partial\alpha_1^2} \\ &+\frac{1}{2}(N-M)\frac{\partial^2}{\partial\alpha_2^2}\bigg]Q(\tau,\alpha) \; , \end{aligned} \qquad (4.17)$$

where the amplification coefficient $\gamma > 0$. This is identical to the Fokker-Planck equation derived from the master equation [24].

B. Evolution of amplified field

The solution of the Fokker-Planck equation is, as for the attenuation case, straightforward. Solving Eq. (4.17) for the arbitrary signal described by Eq. (3.15), and amplified in the phase-sensitive reservoir, we find that the Q function of the amplified signal can be written as

$$\begin{aligned}Q_{\rm amp}(\alpha,\tau) = \frac{1/\pi}{\sqrt{(N_1(\tau)+1)^2 - M_1(\tau)^2}}\int d^2\mu d^2\nu P(\mu,\nu) \\ \times\exp\left\{-\frac{[\alpha_1 - A(\tau)]^2}{1+N_1(\tau)-M_1(\tau)} \right. \\ \left. -\frac{[\alpha_2 - B(\tau)]^2}{1+N_1(\tau)+M_1(\tau)}\right\} , \end{aligned} \qquad (4.18)$$

where the time dependencies of the amplification parameters are

$$A(t) = A\sqrt{G} = Ae^{\gamma\tau} \quad \text{and} \quad B(t) = B\sqrt{G} \; , \qquad (4.19)$$

and those of the noise parameters are

$$N_1(\tau) = (N+1)(G-1) \quad \text{and} \quad M_1(\tau) = M(G-1) \; . \qquad (4.20)$$

The inverse Fourier transformation of the Q function (4.18) shows that the characteristic function $C_{\rm amp}^{(q)}$ for the Q function of the amplified field is the product of the characteristic function $C_{\rm sq}^{(p)}$ for the P representation of the squeezed thermal field and that $C_{\rm s}^{(q)}$ for the Q function of the amplified signal without noise

$$C^{(q)}_{\rm amp}(\zeta) = C^{(p)}_{\rm sq}(i\sqrt{G-1}\,\zeta)C^{(q)}_{\rm s}(\sqrt{G}\,\zeta)\,. \qquad (4.21)$$

The convolution relation for this relation is then in a form analogous to Eq. (4.15) for the amplification beam splitter superposition of two input fields.

Similarly to the calculation procedure from Eq. (2.21) to (2.23), the antinormally ordered moments can be calculated from the characteristic function $C^{(q)}$. Using this procedure with the aid of Eq. (4.14), we find the antinormally ordered moments,

$$\begin{aligned}\langle \hat{a}^n(\hat{a}^\dagger)^m\rangle = \sum_{\ell=0}^{m}\sum_{k=0}^{n}\binom{m}{\ell}\binom{n}{k}(i\sqrt{G-1})^\ell \\ \times(-i\sqrt{G-1})^k\langle(\hat{a}^\dagger)^\ell\hat{a}^k\rangle_{\rm sq} \\ \times(\sqrt{G})^{m+n-\ell-k}\langle\hat{a}^{n-k}(\hat{a}^\dagger)^{m-\ell}\rangle_{\rm s}\,. \qquad (4.22)\end{aligned}$$

This relation is then used to calculate the mean photon number,

$$\langle a^\dagger a\rangle_{\rm amp} = (G-1)(\langle a^\dagger a\rangle_{\rm sq}+1) + G\langle a^\dagger a\rangle_{\rm s}\,. \qquad (4.23)$$

The quadrature variance (3.24) is calculated for amplification as

$$\langle(\Delta\hat{\chi}_1)^2\rangle = G\langle(\Delta\hat{\chi}_1)^2\rangle_{\rm s} + \tfrac{1}{4}[2(N+M)+1](G-1)\,. \qquad (4.24)$$

From Eqs. (3.12) and (3.13) we can easily see that $2(M+N)+1$ is always positive, so the quadrature noise indeed increases even during phase-sensitive amplification [26,27].

V. CONCLUSION

We have developed a general theory, in the Schrödinger picture, to describe the superposition of two input fields in a beam splitter. The superposition of the fields is described by convolution of their quasiprobabilities as shown in Eqs. (2.12), (2.17), and (2.18), and is simply described by the product of their characteristic functions as in Eqs. (2.15) and (2.16). The attenuation of a signal field is modeled by using an infinite array of beam splitters each having an infinitesimal reflectivity. To show the evolution of the signal field coupled with the phase-sensitive and phase-insensitive reservoir, we have used the convolution relation to derive the Fokker-Planck equation for this attenuation.

We have also extended this analysis to describe amplification of the signal field. We introduced an amplification beam splitter relation based on the two-mode parametric amplifier model, and we obtained the modified convolution relation (4.14). Using this for the signal field passing through an infinite number of the amplification beam splitters, we have derived the Fokker-Planck equation for amplification of the signal in a phase-sensitive amplifier.

The amplification process is not merely the inverse of the attenuation process. The transformation of the input fields into the output fields is described by the beam splitter operators $\hat{B}$ in Eq. (2.2) for attenuation while the transformation is described by $\hat{B}_1$ in Eq. (4.3) for amplification. The argument θ of the beam splitter operator B is related to the beam splitter reflection r and transmission t coefficients by $r=\sin\theta$ and $t=\cos\theta$, and the argument θ_1 for B_1 is related to the amplifier gain $g=\cosh\theta_1$. From $\cos^2\theta+\sin^2\theta=1$, energy is conserved in the reflection and transmission, which reflects the fact that a usual beam splitter is a passive device. For an amplification beam splitter, however, $\cosh^2\theta_1-\sinh^2\theta_1=1$ holds, and thus energy is not conserved. The vacuum input is transformed into the vacuum output by the usual beam splitter, i.e., $B|0\rangle_a|0\rangle_b = |0\rangle_c|0\rangle_d$, while it is not transformed into the vacuum output by the amplificaion beam splitter, i.e., $B_1|0\rangle_a|0\rangle_b \neq |0\rangle_c|0\rangle_d$. The vacuum input is transformed into a thermal-state output by a phase-insensitive reservoir and into a squeezed thermal-state output by a phase-sensitive reservoir. The fact that the amplification is not a reverse process of the attenuation is also reflected in the difference of the convolution relations, (2.12) and (4.15), and in the difference of the characteristic function relations, (2.15) and (4.14).

The present theory has a potential of application to many different directions. For example, the generalization of the theory to frequency-dependent attenuation and amplification is straightforward by introducing the frequency dependent beam splitter with $t(\omega)$ and $r(\omega)$. Also, although we treated only thermal-state-like reservoirs in the present paper, the generalization to any other type of reservoir (even non-Gaussian) is possible because we can replace $Q_{\rm sq}(\beta)$ with the Q function of any kind of the noise field. As discussed in Sec. II, this theory also gives us a way to describe simultaneous measurement of two quadrature components with an inevitable noise, which is possibly squeezed (phase sensitive).

ACKNOWLEDGMENTS

This research has been supported by the Korea Research Foundation through the nondirected research fund project. The authors would like to acknowledge helpful discussions with Dr. Bruno Huttner and Dr. Selvakumar Nair. M.S.K. is grateful to Professor Vladimir Bužek for introducing him to the convolution theory.

[1] H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177 (1983).
[2] R. A. Campos, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A **40**, 1371 (1989).
[3] H. Fearn and R. Loudon, Opt. Commun. **64**, 485 (1987).
[4] B. C. Sanders, Phys. Rev. A **45**, 6811 (1992); **46**, 2966 (1992).
[5] M. J. Collett, R. Loudon, and C. W. Gardiner, J. Mod. Opt. **34**, 881 (1987).
[6] B. Huttner and Y. Ben-Aryeh, Phys. Rev. A **38**, 204 (1988).
[7] N. Imoto, J. R. Jeffers, and R. Loudon, *Quantum Measurements in Optics*, edited by P. Tombesi and

D. F. Walls (Plenum, New York, 1992).
[8] J. R. Jeffers, N. Imoto, and R. Loudon, Phys. Rev. A **47**, 3346 (1993).
[9] Here the concept of superposition of fields differs from that of the so-called Schrödinger cat states [B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13, (1986)]. The Schrödinger cat states represent the superposition of two ket states in ket space, whereas in this paper superposition means physical addition of two waves; as, for example, in Young's double-slit experiment.
[10] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[11] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[12] G. Lachs, Phys. Rev. **138**, B1012 (1965).
[13] J. Peřina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Kluwer, Dordrecht, 1991).
[14] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[15] P. D. Drummond and C. W. Gardiner, J. Phys. A **13** 2353 (1980).
[16] U. Leonhardt, Phys. Rev. A **48**, 3265 (1993).
[17] M. Freyberger and W. Schleich, Phys. Rev. A **47**, 30 (1993).
[18] D. T. Smithey, M. Beck, J. Cooper, and M. G. Raymer, Phys. Rev. A **48**, 3159 (1993); D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. **T48**, 35 (1993).
[19] G. S. Agarwal, J. Mod. Opt. **34**, 909 (1987).
[20] C. W. Gardiner, Phys. Rev. Lett. **56**, 1917 (1986).
[21] C. M. Caves, Phys. Rev. D **26**, 1817 (1982).
[22] M. -A. Dupertuis and S. Stenholm, J. Opt. Soc. Am. B **4**, 1094 (1987); M. -A. Dupertuis, S. M. Barnett, and S. Stenholm, *ibid.* **4**, 1102 (1987).
[23] H. P. Yuen, Phys. Rev. A **13**, 2226(1976); R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).
[24] M. S. Kim and V. Bužek, Phys. Rev. A **47**, 610 (1993); M. S. Kim, K. S. Lee, and V. Bužek, *ibid.* **47**, 4302 (1993).
[25] S. M. Barnett and C. R. Gilson, Phys. Rev. A **40**, 6314 (1989); M. S. Kim, Opt. Commun. **114**, 262 (1995).
[26] M. -A. Dupertuis and S. Stenholm, Phys. Rev. A **37**, 1226 (1988); M. O. Scully and M. S. Zubairy, *ibid.* **66**, 303 (1988).
[27] Z. Y. Ou, S. F. Pereira, and H. J. Kimble, Phys. Rev. Lett. **70** 3239 (1993).
[28] U. M. Titulaer and R. J. Glauber, Phys. Rev. **145**, 1041 (1965).
[29] Y. Aharonov, D. Z. Albert, and C. K. Au, Phys. Rev. Lett. **47**, 1029 (1981).
[30] R. F. O'Connell and A. K. Rajagopal, Phys. Rev. Lett. **48**, 525 (1982).
[31] K. Wódkiewicz, Phys. Rev. Lett. **52**, 1064 (1984).
[32] S. Friberg and L. Mandel, Opt. Commun. **46**, 141 (1983).
[33] R. Loudon and T. J. Shepherd, Opt. Acta **31**, 1243 (1984).
[34] V. Peřinová, J. Křpelka, and J. Peřina, J. Mod. Opt. **33**, 1263 (1986).
[35] G. S. Agarwal and G. Adam, Phys. Rev. A **39**, 6259 (1989); G. Adam, J. Mod. Opt. (to be published).
[36] E. Arthus and J. L. Kelly, Jr., Bell Sys. Tech. J., 725 (1965).
[37] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Phys. Rev. A **40**, 2494 (1989).
[38] R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1983).
[39] P. M. Marian and T. A. Marian, Phys. Rev. A **47**, 4474 (1993); **47**, 4487 (1993).
[40] V. Peřinová, A. Lukš, and P. Szlachetka, J. Mod. Opt. **36**, 1435 (1989).
[41] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1970).
[42] H. Fearn, Quantum Opt. **2**, 103 (1990).
[43] S. M. Barnett and P. L. Knight, J. Mod. Opt. **34**, 841 (1987).