Pair excitation-deexcitation coherent states

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A class of coherent states defined in terms of the excitation and deexcitation of pairs of photons is studied with reference to its nonclassical and other quantum-statistical properties. These states supplement the other well-known two-mode states such as Caves-Schumaker states and pair coherent states and can be produced by dissipative processes involving emission and absorption of photons in pairs.

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I. INTRODUCTION

In nonlinear optics one deals very often with processes where photons are either created or destroyed in pairs. Such processes can generally be described by an effective Hamiltonian of the type

$$H=\zeta ab+\zeta^* a^\dagger b^\dagger , \tag{1}$$

where a and b are the annihilation operators for the two modes. It is therefore natural to consider states produced by this H

$$|\psi\rangle \equiv \exp[i(\zeta ab+\zeta^* a^\dagger b^\dagger)]|\alpha,\beta\rangle , \tag{2}$$

where $|\alpha,\beta\rangle$ is the coherent state associated with the two modes. These states have been extensively studied [1–4] and lead to a number of nonclassical properties of the underling fields. There is, however, another class of nonclassical states, called pair coherent states [5,6], equally important in problems involving pairs of photons. These are defined to be the eigenstates of the pair destruction operator ab subject to the condition that $a^\dagger a - b^\dagger b$ is conserved. Dissipative nonlinear optical processes give rise to pair coherent states [7]. The connection between (2) and pair coherent states is discussed in Refs. [8,9]. Note that the importance of the states associated with the annihilation operator [10] a and the linear combination of a and $a^\dagger$ is well known [3,11]. Both are quite useful. It is thus natural to ask, What are the eigenstates of the operator that is a linear combination [11–13] of annihilation and creation operators for pairs of photons? Thus in this paper we examine the solutions of the eigenvalue problem

$$(\mu ab+\nu a^\dagger b^\dagger)|\psi\rangle = \xi|\psi\rangle \tag{3}$$

and we study in detail various quantum-statistical properties of fields in such states. We will call these states "pair excitation-deexcitation coherent states." The present study, in a sense, exhausts the possible two-mode coherent states.

II. EIGENVALUE PROBLEM

Let us consider pair creation and annihilation operators ab and $a^\dagger b^\dagger$ of the two-mode field. These operators form elements of the SU(1,1) group

$$\begin{aligned} &K_+ = K_1+iK_2=a^\dagger b^\dagger , \quad K_-=ab ,\\ &K_3=\tfrac{1}{2}(a^\dagger a+b^\dagger b+1) ,\\ &[K_3,K_\mp]=\mp K_\mp , \quad [K_1,K_2]=-iK_3 . \end{aligned} \tag{4}$$

The commutator (4) implies that for arbitrary states the variances $\Delta K_1,\Delta K_2$ in the two quadratures K_1,K_2 obey the inequality

$$(\Delta K_1)(\Delta K_2) \ge \tfrac{1}{2}|\langle K_3\rangle| . \tag{5}$$

In view of (5) the SU(1,1) squeezing is defined by [14]

$$\Delta K_1^2 < \tfrac{1}{2}|\langle K_3\rangle| \quad \text{or} \quad \Delta K_2^2 < \tfrac{1}{2}|\langle K_3\rangle| . \tag{6}$$

We consider a class of states $|\psi\rangle$ for which the equality in in (5) holds, i.e.,

$$\Delta K_1 \Delta K_2 = \tfrac{1}{2}|\langle K_3\rangle| . \tag{7}$$

The pair coherent states [5] correspond to the special case of the condition (7), given by

$$\Delta K_1^2 = \Delta K_2^2 = \tfrac{1}{2}|\langle K_3\rangle| , \tag{8}$$

i.e., pair coherent states are the minimum-uncertainty states with equal uncertainties in the two quadratures of K_-. From general consideration of the equality sign in (7) it follows that $|\psi\rangle$ should satisfy the eigenvalue problem

$$(K_1 - i\eta K_2)|\psi\rangle = \alpha|\psi\rangle , \tag{9}$$

i.e.,

$$[K_-(1+\eta)+K_+(1-\eta)]|\psi\rangle = 2\alpha|\psi\rangle . \tag{10}$$

Thus the states $|\psi\rangle$ are the eigenstates of an operator that is a linear combination of the excitation (K_+) and the deexcitation (K_-) operators of the SU(1,1) group. The limit $\eta\to 1$ corresponds to the pair coherent state case mentioned above. We call these states $|\psi\rangle$ pair excitation-deexcitation coherent states.

III. SOLUTION TO THE EIGENVALUE PROBLEM (9) IN THE COHERENT STATE REPRESENTATION

Let us rewrite the eigenvalue problem (9) as

$$(\mu ab+\nu a^\dagger b^\dagger)|\psi\rangle=\xi|\psi\rangle\ ,\quad \mu^2-\nu^2=1\ , \tag{11}$$

where we have imposed the additional condition on the real parameters μ,ν. These states are degenerate. The degeneracy can be lifted by assuming that the pairs are either created or destroyed together. So we consider the conservation law

$$(a^\dagger a-b^\dagger b)|\psi\rangle=0\ . \tag{12}$$

We use the coherent state representation to solve the eigenvalue problem (11). Let us define the projections of $|\psi\rangle$ onto coherent states

$$|\psi\rangle=\frac{1}{\pi^2}\int\psi(\alpha,\beta)|\alpha,\beta\rangle d^2\alpha\, d^2\beta\ , \tag{13}$$

where

$$\psi(\alpha,\beta)=\langle\alpha,\beta|\psi\rangle\ . \tag{14}$$

In order to compute (14) it is useful to use the unnormalized coherent state $|\bar\alpha,\bar\beta\rangle$ defined by

$$|\bar\alpha,\bar\beta\rangle=\exp(\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2)|\alpha,\beta\rangle \tag{15}$$

and hence we work with the function

$$\phi=\langle\psi|\bar\alpha,\bar\beta\rangle \tag{16}$$

so that the required solutions $\psi(\alpha,\beta)$ are given by

$$\psi(\alpha,\beta)=\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2)\phi^*(\alpha,\beta)\ . \tag{17}$$

The eigenvalue problem can be converted into a differential equation by using the relations [10]

$$ab|\bar\alpha,\bar\beta\rangle=\alpha\beta|\bar\alpha,\bar\beta\rangle\ ,\quad a^\dagger b^\dagger|\bar\alpha,\bar\beta\rangle=\frac{\partial^2}{\partial\alpha\partial\beta}|\bar\alpha,\bar\beta\rangle\ . \tag{18}$$

We project (11) on $|\bar\alpha,\bar\beta\rangle$ to get

$$\xi^*\langle\psi|\bar\alpha,\bar\beta\rangle=\langle\psi|\mu a^\dagger b^\dagger+\nu ab|\bar\alpha,\bar\beta\rangle\ . \tag{19}$$

Using (18) in (19), we get

$$\xi^*\phi(\alpha,\beta)=\nu\alpha\beta\phi(\alpha,\beta)+\mu\frac{\partial^2}{\partial\alpha\partial\beta}\phi(\alpha,\beta)\ . \tag{20}$$

In the coherent state representation the conservation law (12) becomes

$$\alpha\frac{\partial}{\partial\alpha}\phi(\alpha,\beta)-\beta\frac{\partial}{\partial\beta}\phi(\alpha,\beta)=0\ . \tag{21}$$

Equation (21) implies that $\phi(\alpha,\beta)$ must be a function of the product $\alpha\beta$, i.e.,

$$\phi(\alpha,\beta)=\phi(\alpha\beta)\ . \tag{22}$$

Using (20) we can derive a second-order differential equation in terms of the complex variable z defined by

$$z=\alpha\beta$$

with the result

$$\phi''+\frac{1}{z}\phi'+\frac{\nu}{\mu}\phi=\frac{\xi^*}{\mu z}\phi\ , \tag{23}$$

where the prime denotes a derivative with respect to z. Equation (23) has solutions [15] in terms of the degenerate hypergeometric functions. The result for ϕ up to a normalization constant is given by

$$\phi(\alpha\beta)=e^{-\sqrt{-\nu/\mu}\alpha\beta}\,{}_1F_1\left[\frac{1}{2}+\frac{\xi^*}{2\sqrt{-\mu\nu}},1;2\sqrt{-\nu/\mu}\alpha\beta\right]\ . \tag{24}$$

Hence the coherent state representation $\psi(\alpha,\beta)$ for $|\psi\rangle$ defined by (13) will be

$$\begin{aligned}|\psi\rangle=\frac{1}{\pi^2}\int&\exp[-|\alpha|^2/2-|\beta|^2/2-(\sqrt{-\nu/\mu}\alpha\beta)^*]\\&\times{}_1F_1\left[\frac{1}{2}+\frac{\xi}{2(\sqrt{-\mu\nu})^*}\ ,\ 1,2(\sqrt{-\nu/\mu}\alpha\beta)^*\right]\\&\times|\alpha,\beta\rangle d^2\alpha\, d^2\beta\ .\end{aligned} \tag{25}$$

IV. QUASIDISTRIBUTION FOR THE FIELD

We consider the behavior of the quasiprobability distribution $Q(\alpha,\beta)$ defined by

$$Q(\alpha,\beta)=\frac{1}{\pi^2}|\langle\alpha,\beta|\psi\rangle|^2\ , \tag{26}$$

i.e.,

$$Q(\alpha,\beta)=\frac{1}{\pi^2}e^{-|\alpha|^2-|\beta|^2}|\phi(\alpha\beta)|^2\ , \tag{27}$$

which on using (24) reduces to

$$\begin{aligned}Q(\alpha,\beta)&\\&=e^{-|\alpha|^2-|\beta|^2}\\&\quad\times\left|e^{-\sqrt{\nu/\mu}\alpha\beta}\,{}_1F_1\left[\frac{1}{2}+\frac{\xi^*}{2\sqrt{-\mu\nu}}\ ,1,2\sqrt{\nu/\mu}\alpha\beta\right]\right|^2\ .\end{aligned} \tag{28}$$

The relation (28) is valid up to a normalization constant that can be fixed by

$$\int\int Q(\alpha,\beta)d^2\alpha\, d^2\beta=1\ . \tag{29}$$

V. SPECIAL CASES

A. Photon pair excitation-deexcitation vacuum $\xi=0$

The vacuum state corresponds to the eigenvalue equation

$$(\mu ab+\nu a^\dagger b^\dagger)|\psi\rangle=0\ . \tag{30}$$

Solutions of (30) are given by Eq. (23) with $\xi=0$, i.e.,

$$\phi''+\frac{1}{z}\phi'+\frac{\nu}{\mu}\phi=0\ ,\quad z=\alpha\beta\ . \tag{31}$$

Solving this using standard methods [16], we obtain

$$\phi(\alpha\beta)=J_0(\sqrt{\nu/\mu}\alpha\beta) , \tag{32}$$

J_0 being the Bessel function of first kind. Using this in (13) we get the photon pair excitation-deexcitation vacuum state solution given by

$$|\psi\rangle_{\text{vacuum}}=\sum_k \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\left[\frac{\nu}{\mu}\right]^k|2k,2k\rangle . \tag{33}$$

Thus the vacuum state is made of even photon numbers with a fast converging probability distribution. The states (33) resemble the direct product of the squeezed vacuum states [3] for the modes a and b with the same squeezing parameter provided in the direct product we retain terms with equal number of photons in the modes a and b.

B. Pair coherent states

In the limit $\nu\rightarrow 0$, the eigenvalue equation (23) in the coherent state representation reduces to

$$\phi''+\frac{1}{\zeta}\phi'=-\frac{\phi}{4\zeta} , \tag{34}$$

where we have introduced a new complex variable ζ

$$\zeta=-\frac{4\xi^*}{\mu}\alpha\beta . \tag{35}$$

Equation (34) has solutions [17] given by

$$\phi(\alpha\beta)=I_0\left[\left[\frac{4\xi^*}{\mu}\alpha\beta\right]^{1/2}\right] . \tag{36}$$

On substituting (36) in (13) and on integration we retrieve the pair coherent state solutions [5] given by

$$|\psi\rangle_{\text{PCS}}=N\sum\frac{1}{n!}\left[\frac{\xi}{\mu}\right]^n|n,n\rangle , \tag{37}$$

where N is the normalization constant given by

$$N=[I_0(2|\xi/\mu|)]^{-1/2} . \tag{38}$$

C. Special values of ξ

We next consider a special class of eigensolutions corresponding to

$$\xi^*=\sqrt{-\mu\nu} . \tag{39}$$

Using (39) in (24) and using the relation [18]

$${}_1F_1(\alpha,\alpha,z)=e^z \tag{40}$$

we get

$$\psi(\alpha,\beta)=\exp[-|\alpha|^2/2-|\beta|^2/2+(\sqrt{-\nu/\mu})^*\alpha^*\beta^*] . \tag{41}$$

On using (41) and (13) and on simplifications we get

$$|\psi\rangle=N\sum_k(\sqrt{-\nu/\mu})^{*k}|k,k\rangle , \tag{42}$$

where N is the normalization constant given by

$$N=\sqrt{1-(\nu/\mu)} . \tag{43}$$

Thus, in the limit (39) our states reduce to the two-mode squeezed vacuum.

VI. NUMERICAL RESULTS

The two-mode correlated states are characterized by the strength and the nature of the correlations present in them. Manifestations of these correlations are studied through various quantities of experimental significance such as squeezing, photon-number correlations, quadrature correlations, photon-number distribution, Cauchy-Schwarz inequality, Q parameter, and quadrature distributions.

We use a direct numerical method to get insight into the nonclassical nature of these states. We expand the state $|\psi\rangle$ in terms of the Fock states as

$$|\psi\rangle=\sum_{n,m}C_{nm}|n,m\rangle . \tag{44}$$

However, the conservation law (12) suggests that $C_{nm}=0$ unless $n=m$, i.e., $C_{nm}=C_n\delta_{nm}$. Thus we rewrite (44) as

$$|\psi\rangle=\sum_n C_n|n,n\rangle , \tag{45}$$

where C_n's satisfy the recurrence relation

$$\mu(n+1)C_{n+1}+\nu nC_{n-1}-\xi C_n=0 , \quad n=0,1,\ldots . \tag{46}$$

The parameters μ, ν, and ξ are to be such that the series converges. For $\xi=0$ and $C_1=0$ this leads to

$$|\psi\rangle=\sum_n C_{2n}|2n,2n\rangle , \quad \xi=0 , \tag{47}$$

in agreement with the analytical solution (33). For nonzero ξ the recursion relations are solved numerically. The coefficient C_0 is fixed by the renormalization condition

$$\sum_n|C_n|^2=1 . \tag{48}$$

A. Photon statistics

Photon-number distributions $P_n=|C_n(\xi,\nu)|^2$ are plotted in Fig. 1 for a range of ξ values. For a given ξ, they display oscillatory behavior similar to that reported earlier in the literature [19]. Photon-number distributions are sensitive to the phase of the field. The average photon number is clearly seen to be a phase-dependent quantity. In Fig. 2 we plot the average photon number as a function of ξ and ν for $\theta=0$. The average photon number for the vacuum state is also plotted in Fig. 2 (inset) and is seen to saturate with increasing ν.

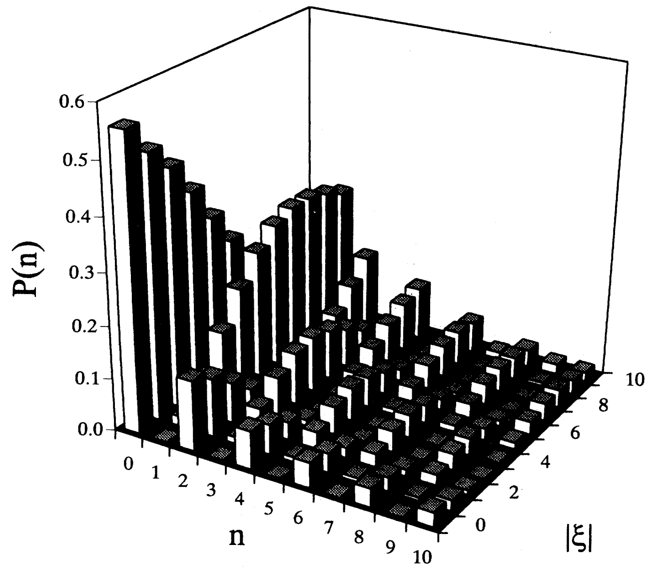

FIG. 1. Photon-number distribution $P(n)$ as a function of $|\xi|$ for $\theta=0, \nu=5$ ($\xi=|\xi|e^{-i\theta}$).

B. Amplitude and phase correlations in the pair excitation-deexcitation coherent states

The form of the eigenvalue problem (11) subject to the condition (12) suggests the existence of a correlated behavior of photons in these states. In particular one has

$$\langle (a^\dagger a - b^\dagger b)^2 \rangle = 0 \ . \tag{49}$$

The correlation coefficient between the two modes can be defined by

$$C_{ab} = \frac{\langle a^\dagger a b^\dagger b \rangle - \langle a^\dagger a \rangle \langle b^\dagger b \rangle}{\langle a^\dagger a \rangle \langle b^\dagger b \rangle} \ . \tag{50}$$

The correlation coefficient C_{ab} is plotted in Fig. 3 as a function of ξ and ν. The correlation coefficient C_{ab} is related to the Q parameter

$$Q = \frac{\langle a^{\dagger 2} a^2 \rangle - \langle a^\dagger a \rangle^2}{\langle a^\dagger a \rangle} \tag{51}$$

via the relation

$$Q = \langle a^\dagger a \rangle C_{ab} - 1 \ . \tag{52}$$

Note that

$$\langle a^\dagger a b^\dagger b \rangle = \sum_n |C_n|^2 n^2 = \langle (a^\dagger a)^2 \rangle \ . \tag{53}$$

From the behavior of the correlation coefficient C_{ab} we find that as the parameter ν increases, the nature of the photon statistics of the field switches over from sub-Poissonian to super-Poissonian.

Correlations between the quadrature phases (i.e., $\langle ab \rangle \neq 0$) give rise to the squeezing in a mode that is a linear combination of a and b modes. Let us consider the superposition modes defined by

$$\begin{aligned} C_1 &= \frac{1}{\sqrt{2}}(a + a^\dagger - b - b^\dagger) \ , \\ C_2 &= \frac{1}{\sqrt{2}}(a + a^\dagger + b + b^\dagger) \ . \end{aligned} \tag{54}$$

In Fig. 4 we plot the squeezing parameter $S = 4[\langle : C_1^2 : \rangle - \langle C_1 \rangle^2]$. We find that the superposition mode squeezing S gradually disappears as the parameter ν increases.

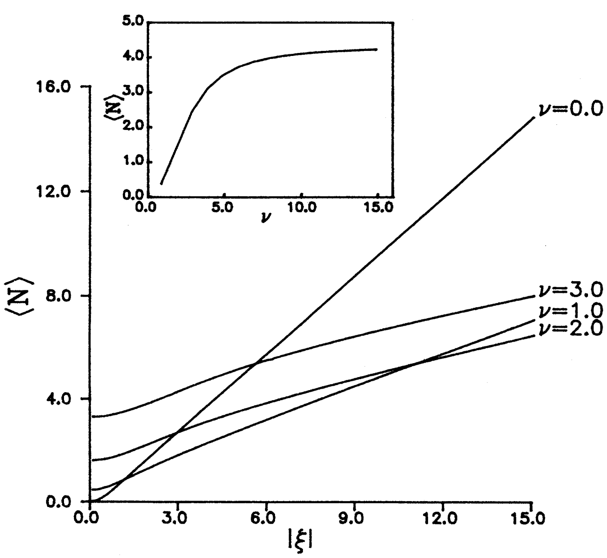

FIG. 2. Average photon number $\langle N \rangle$ as a function of $|\xi|$ and ν. The inset shows the number for the pair excitation-deexcitation vacuum state as a function of ν.

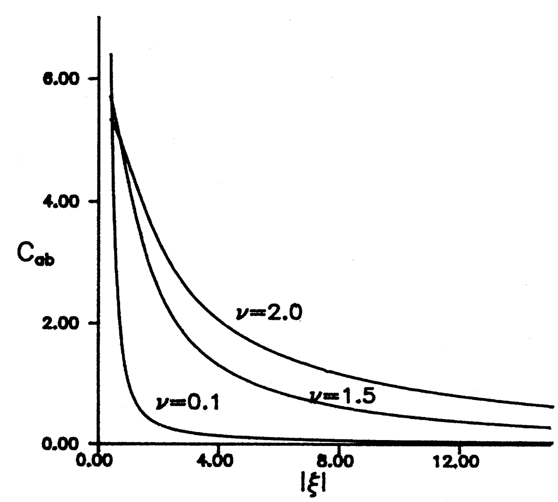

FIG. 3. Correlation between the modes a and b as a function of $|\xi|$ for different values of ν.

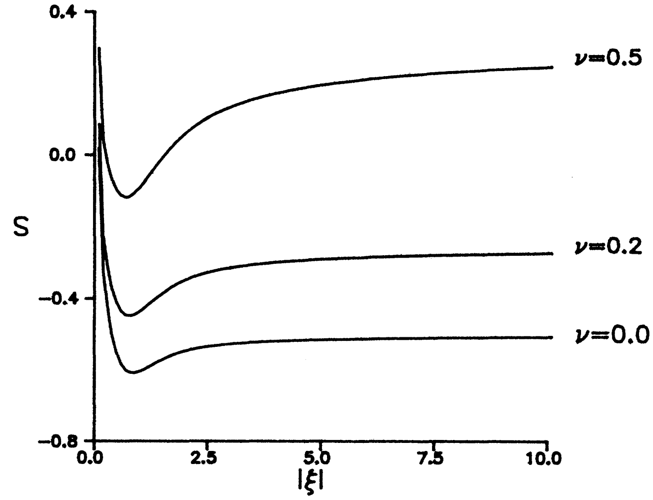

FIG. 4. Superposition mode squeezing S as a function of $|\xi|$ and ν.

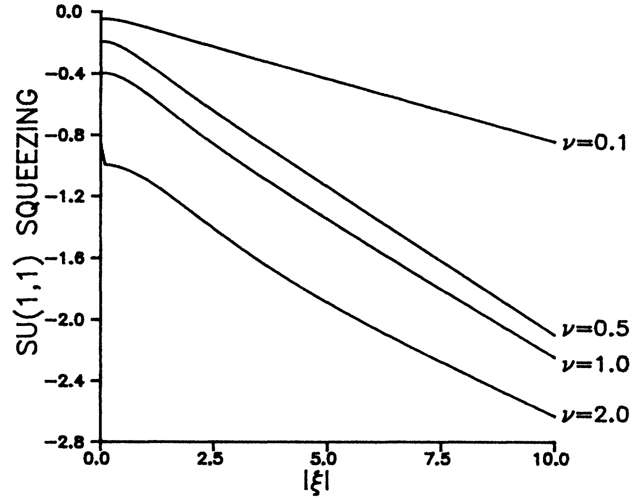

FIG. 5. SU(1,1) squeezing as a function of $|\xi|$ and ν.

C. SU(1,1) squeezing

We next consider SU(1,1) squeezing defined by (6). Instead of considering the variances in K_1 or K_2 we can use K_ϕ defined by

$$K_\phi=(e^{i\phi}a^\dagger b^\dagger+e^{-i\phi}ab)/2 . \tag{55}$$

The SU(1,1) squeezing condition now reads

$$(\Delta K_\phi)^2-\tfrac{1}{4}(N_a+N_b+1)<0 . \tag{56}$$

The operators $K_\pm, K_3$ are bilinear in terms of the operators a and b. So SU(1,1) *squeezing can be detected in a nonlinear process in which the output field is a product of the input field modes.* Clearly this can be achieved in sum-frequency generation. The interaction comes about through the second-order polarizability of the nonlinear medium. Squeezing in the quadrature K_ϕ of the input field can be observed by studying the usual quadratures of the output field. In Fig. 5 we plot SU(1,1) squeezing given by the expression on the left-hand side of Eq. (56) for $\phi=0$. For large ξ and ν these states are highly SU(1,1) squeezed. Also the vacuum states display a considerable amount of SU(1,1) squeezing.

D. Pair excitation-deexcitation coherent states in two-photon interferometry

Degree of fourth-order coherence [20] of light is an important measurable quantity that very often has been used to distinguish the predictions of quantum theory from its classical counterparts. These states show violations of the quantum analog of the Cauchy-Schwarz inequality [5] defined as

$$I_0\geq 0 , \tag{57}$$

where

$$I_0=\frac{[\langle b^{\dagger 2}b^2\rangle\langle a^{\dagger 2}a^2\rangle]^{1/2}}{|\langle a^\dagger ab^\dagger b\rangle|}-1 . \tag{58}$$

It was discussed in Ref. [5] that the violations of (57) can be directly measured in a two-photon experiment. Consider the field obtained by the superposition of the two modes. The field at a point r_j is given by

$$\varepsilon_j^{(+)}=[a\exp(i\phi_j)+b\exp(i\psi_j)] . \tag{59}$$

The joint probability P_2 of detecting one photon at r_1 and another at r_2 is given by the fourth-order correlation function

$$P_2=\langle\varepsilon_1^{(-)}\varepsilon_2^{(-)}\varepsilon_2^{(+)}\varepsilon_1^{(+)}\rangle . \tag{60}$$

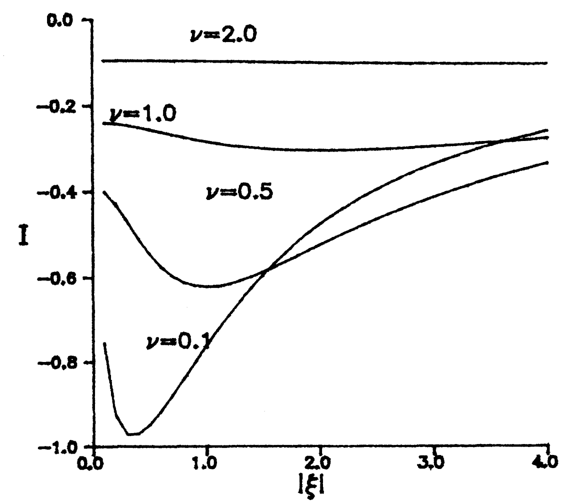

FIG. 6. I_0, defined by Eq. (58), as a function of $|\xi|$ and ν.

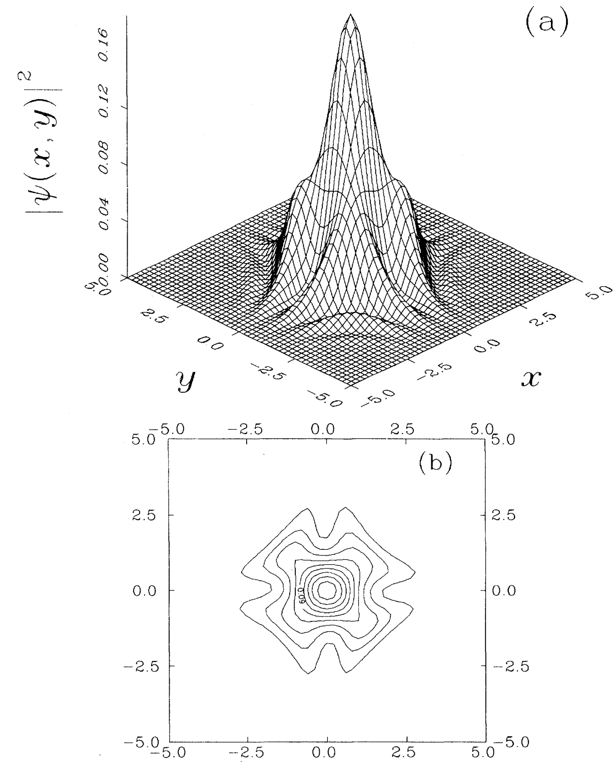

FIG. 7. (a) Position quadrature distribution $|\psi(x,y)|^2$ for the modes $a(x)$ and $b(y)$ for $\nu=1$ and $\theta=0$. (b) Two-dimensional projection of (a).

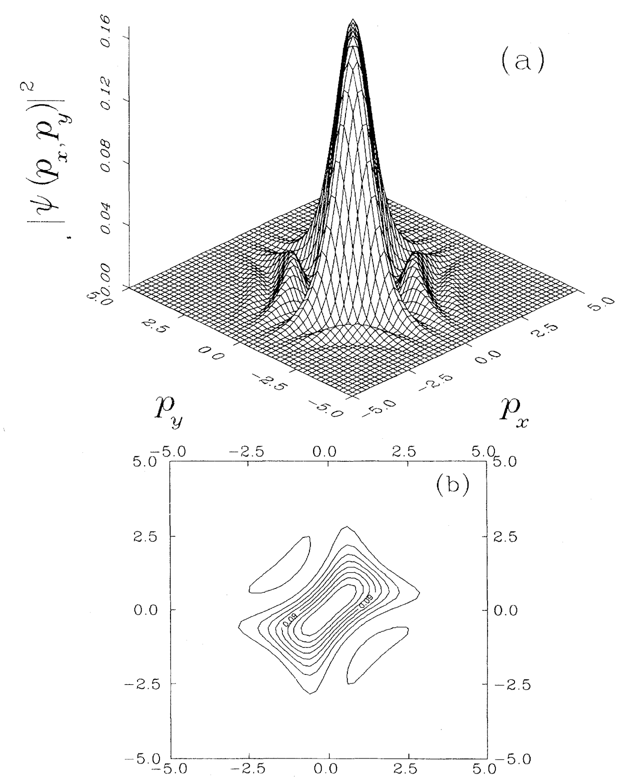

FIG. 8. (a) Momentum space distribution $|\psi(p_x,p_y)|^2$ for the modes $a(p_x)$ and $b(p_y)$ for $\nu=1$ and $\theta=0$. (b) Two-dimensional projection of (a).

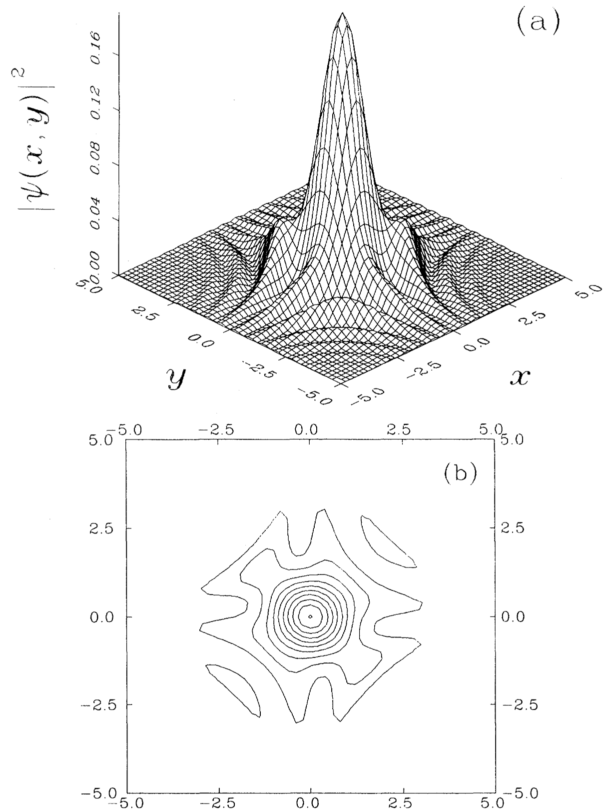

FIG. 9. Same as Fig. 7 but with $\nu=2$.

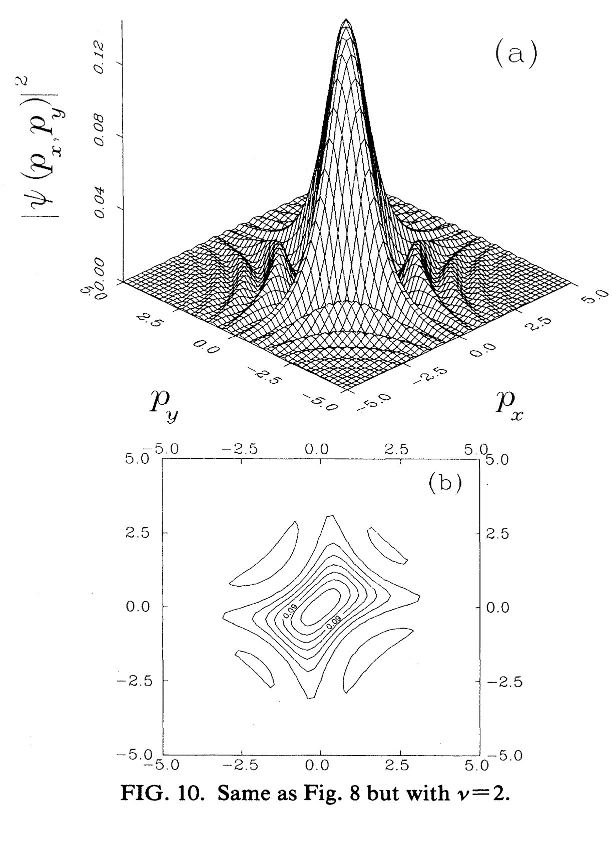

FIG. 10. Same as Fig. 8 but with $\nu=2$.

For the superposed field (59), P_2 is given by

$$P_2=\{\langle a^{\dagger 2}a^2\rangle+\langle b^{\dagger 2}b^2\rangle+2\langle a^\dagger ab^\dagger b\rangle\}(1+\sigma\cos\psi)\ , \tag{61}$$

where

$$\psi=\psi_2-\phi_2+\phi_1-\psi_1 \tag{62}$$

and the modulation index σ is given by

$$\sigma=\frac{2\langle a^\dagger b^\dagger ab\rangle}{[\langle a^{\dagger 2}a^2\rangle+\langle b^{\dagger 2}b^2\rangle+2\langle a^\dagger ab^\dagger b\rangle]}\ . \tag{63}$$

For the states obeying the conservation law (12), we have

$$\sigma=\frac{1}{2+I_0}\ . \tag{64}$$

Thus any violation of the Cauchy-Schwarz inequality gives rise to the modulation index greater than 0.5. For large $|\xi|$ these states show violations of the Cauch-Schwarz inequality with increasing values of the parameter ν as shown in the Fig. 6. Thus it is possible to achieve a relatively large depth of modulation with a source in the pair excitation-deexcitation coherent state.

E. Quadrature distributions

Quadrature distributions in both coordinate and momentum spaces, i.e., $|\psi(x,y)|^2$ and $|\psi(p_x,p_y)|^2$, are plotted for two different values of ν in Figs. 7–10. These distributions show oscillatory behavior. Oscillations are more pronounced at large values of ν. Thus the contribution from the pair creation part of the eigenvalue problem gives rise to the interference effects.

It is believed that the present study exhausts the possible coherent states associated with two-mode systems. These states should be of interest in problems that involve both creation and annihilation of photons in pairs.

[1] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); **31**, 3093 (1985).

[2] A. M. Perelemov, Sov. Phys. Ups. **20**, 703 (1977).

[3] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[4] D. F. Walls, Nature **324**, 210 (1986); A. S. Lane, M. D. Reid, and D. F. Walls, Phys. Rev. A **38**, 788 (1988).

[5] G. S. Agarwal, J. Opt. Soc. Am. B **5**, 1940 (1988).

[6] A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971).

[7] G. S. Agarwal, Phys Rev. Lett. **57**, 827 (1986).

[8] P. Shanta, S. Chaturvedi, V. Srinivasan, G. S. Agarwal, and C. L. Mehta, Phys. Rev. Lett. **72**, 1447 (1994).

[9] G. S. Agarwal, Opt. Commun. **100**, 479 (1993).

[10] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[11] J. A. Bergou, M. Hillery, and D. Yu, Phys. Rev. A **43**, 515 (1991); D. Yu and M. Hillery, Quantum Opt. (to be published).

[12] M. M. Nieto and D. R. Truax, Phys. Rev. Lett. **71**, 2843 (1993).

[13] G. Satya Prakash and G. S. Agarwal, Phys. Rev. A **50**, 4258 (1994).

[14] M. Hillery, Phys. Rev. A **40**, 3147 (1989).

[15] I. S. Gradsteyn and I. M. Ryzhik, in *Tables of Integrals, Series and Products,* edited by Alan Jeffrey (Academic, New York, 1965), p. 1059, Eq. 9.216, no. 1.

[16] I. S. Gradsteyn and I. M. Ryzhik, in *Tables of Integrals, Series and Products* (Ref. [15]), p. 971, Eq. 8.491. no. 1.

[17] I. S. Gradsteyn and I. M. Ryzhik, in *Tables of Integrals, Series and Products* (Ref. [15]), p. 971, Eq. 8.491, no. 9.

[18] I. S. Gradsteyn and I. M. Ryzhik, in *Tables of Integrals, Series and Products* (Ref. [15]), p. 1059, E. 9.215, no. 1.

[19] W. Schleich and J. A. Wheeler, Nature **326**, 574 (1987).

[20] L. Mandel, Phys. Rev. A **28**, 929 (1983); Z. Y. Ou and L. Mandel, Phys. Rev. Lett. **61**, 50 (1988).