Universal superpositions of coherent states and self-similar potentials

Vyacheslav Spiridonov*
Centre de Recherches Mathématiques, Université de Montréal,
Case Postale 6128, succursale Centre-ville, Montréal, Québec, Canada H3C 3J7
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A variety of coherent states of the harmonic oscillator is considered. It is formed by a particular superposition of canonical coherent states. In the simplest case, these superpositions are eigenfunctions of the annihilation operator $A = P(d/dx + x)/\sqrt{2}$, where P is the parity operator. Such A arises naturally in the $q \to -1$ limit for a symmetry operator of a specific self-similar potential obeying the q-Weyl algebra $AA^\dagger - q^2 A^\dagger A = 1$. Coherent states for this and other reflectionless potentials whose discrete spectra consist of N geometric series are analyzed. In the harmonic oscillator limit, the surviving part of these states takes the form of orthonormal superpositions of N canonical coherent states $|\epsilon^k \alpha\rangle$, $k = 0, 1, \ldots, N-1$, where ϵ is a primitive Nth root of unity, $\epsilon^N = 1$. A class of q-coherent states related to the bilateral q-hypergeometric series and Ramanujan-type integrals is described. It includes an unusual set of coherent states of the free nonrelativistic particle, which is interpreted as a q-algebraic system without a discrete spectrum. A special degenerate form of the symmetry algebras of self-similar potentials is found to provide a natural q analog of the Floquet theory. Some properties of the factorization method, which is used throughout the paper, are discussed from the differential Galois theory point of view.

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I. INTRODUCTION

Replacement of the commuting coordinate and momentum variables of a classical point particle by the operators x and p satisfying the Heisenberg commutation relation,

$$[x, p] \equiv xp - px = i\hbar, \tag{1.1}$$

endows this particle with wave characteristics. According to the original definition, coherent states are the states in which corpuscular properties of a quantum particle are seen best. For the harmonic oscillator, such a qualitative motivation happens to be supported by the rich group-theoretical content built in the structure of coherent states, so that the symmetry approach provides their alternative description. As a result, there have appeared several different quantitative definitions of coherent states which are equivalent only for the harmonic oscillator case. Independently of this nonuniqueness of definition, from both the physical and mathematical points of view, coherent states of quantum mechanics are fascinating objects having useful applications in many fields [1–4].

In the present paper we discuss coherent states associated with a specific class of one-dimensional Schrödinger operator potentials found in [5–7]. The general class of these *self-similar* potentials is defined with the help of q-periodic closure [6, 7] of the dressing chain, or the chain of Darboux transformations. The latter transformations are known to be closely related to the factorization method [8, 9]. Symmetries of the self-similar potentials are described by some polynomial operator algebras of order N (N is the period of closure) which play the role of spectrum generating algebras. For $N = 1, 2$ these algebras coincide with known q analogs of the bosonic oscillator and su(1, 1) algebras [6]. The *q-coherent states*, defined as eigenfunctions of symmetry operators which lower the energy, have many interesting properties. In particular, the algebras depend on the parameter q^2, so that $q = \pm 1$ coherent states seem to be equivalent; but this is not so. For the q-Weyl algebra system, the limit $q \to -1$ exists only when the potential is symmetric, and then the corresponding coherent states are described by a particular superposition of canonical coherent states. This superposition has a universal form; it represents an example of the Titulaer-Glauber coherent states [10] which were constructed in [11] from a different idea. Its multimode oscillator analog defines the particular entangled coherent states which have a phase difference equal to $\pi/2$. The name *parity coherent states* is suggested for these and other more general two-term superpositions of coherent states for which the parity operator plays a crucial role in the definition.

For the limiting values of parameters corresponding to the harmonic oscillator potential, the raising and lowering operators of the self-similar potentials' symmetry algebras become equal to powers of bosonic creation and annihilation operators. In this limit, part of the q-coherent states degenerate into orthonormal superpositions of N canonical coherent states $|\epsilon^k \alpha\rangle, k = 0, 1, \ldots, N-1$, where ϵ is a primitive Nth root of unity, $\epsilon^N = 1$. These are natural generalizations of even and odd coherent states [12]. Note that there are q-coherent states which do not survive in this limit. In another degenerate limit, when the potential vanishes but q remains arbitrary, one gets a unique set of nontrivial coherent

*On leave of absence from the Institute for Nuclear Research, Russian Academy of Sciences, Moscow, Russia. Electronic address: spiridonov@lps.umontreal.ca

states of the free particle system.

It is interesting that symmetry operators of the self-similar potentials are defined with the help of a scaling operator — a special form of the squeezing operator. Due to this fact, wave functions of these systems resemble wavelets. From a degenerate form of the symmetry algebra, when pure dilatation by a fixed number q becomes physical symmetry of the Schrödinger equation, one finds a natural q analog of the Floquet theory.

Despite their simplicity, group-theoretical roots of superpositions of canonical coherent states comprise an intrinsic possibility for construction of complicated physical systems whose coherent states share common properties with these superpositions. The author considers the description of the relationship between self-similar potentials and superpositions of coherent states as an important physical result of this paper. However, a wider aim of the work is to discuss exactly solvable potentials and their coherent states from the general functional-analytic point of view on the basis of old and new examples.

The paper is organized as follows. Before moving to the analysis of complicated situations, at the end of this section we give a brief account of the canonical coherent states of the harmonic oscillator. In Sec. II we construct nonstandard coherent states of the same system and consider their relation to the Titulaer-Glauber coherent states. In Sec. III we discuss a universality of the derived superpositions of coherent states. An interesting set of coherent states of the free particle determined by the pantograph equation and its generalizations is considered in Sec. IV. The simplest potentials with q-deformed symmetry algebras and nontrivial discrete spectra are described in Sec. V, where some properties of the associated coherent states are analyzed. In Sec. VI we present a general hierarchy of Schrödinger operators whose discrete spectra consist of N geometric series generated by the specific polynomial quantum algebras. Two particular systems arising from $N = 2$, $q = -1$ and $N = 3$, $q = 1$ closures of the dressing chain are considered in Sec. VII. In Sec. VIII, a q-analog of the Floquet theory is outlined. Section IX contains a discussion of integrable potentials and coherent states from the differential ("quantum") Galois theory point of view. Some concluding remarks are given in Sec. X. The paper has a formal character; we consider mostly theoretical aspects of the chosen (stationary) systems rather than their possible experimental implementations. The present analysis of coherent states for the self-similar potentials arose from the investigation of q-oscillator algebra at roots of unity performed in [13]. The results of this work have been reported by the author in [14, 13], and a part of them has been published in [15].

In practical applications it is convenient to use the coordinate representation of (1.1), where $p = -i\hbar d/dx$. For simplicity we use the system of units where Planck's constant $\hbar$ is equal to 1. In terms of the ladder operators $a^\dagger, a$:

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right), \tag{1.2}$$

relation (1.1) takes the form of the bosonic oscillator, or Weyl algebra:

$$[a, a^\dagger] = 1. \tag{1.3}$$

The number operator $N \equiv a^\dagger a$ satisfies the relations $[N, a^\dagger] = a^\dagger$, $[N, a] = -a$. In appropriate units the Hamiltonian of a harmonic oscillator is equal to N up to a constant term: $2H = \{a^\dagger, a\} = -d^2/dx^2 + x^2$. The energy spectrum and orthonormal eigenstates are

$$H|n\rangle = E_n|n\rangle, \quad E_n = n + 1/2,$$
$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle,$$

where the vacuum state $|0\rangle$ is defined from the equation $a|0\rangle = 0$, $\langle x|0\rangle = \pi^{-1/4}\exp(-x^2/2)$ (we set the phase of this state equal to zero).

Coherent states of the harmonic oscillator may be defined either as eigenstates of the annihilation operator a,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{1.4}$$

or as a result of the application of the displacement operator to the vacuum,

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.5}$$

Both definitions are essentially equivalent and give

$$\psi_\alpha(x) \equiv \langle x|\alpha\rangle = \pi^{-1/4}\exp\left[\frac{\alpha^2 - |\alpha|^2}{2} - \left(\frac{x}{\sqrt{2}} - \alpha\right)^2\right]. \tag{1.6}$$

The states (1.4) are defined up to a phase factor $\exp i\chi(\alpha, \alpha^*)$, where $\chi(\alpha, \alpha^*)$ is an arbitrary real function such that $\chi(0,0) = 0$. Only a special choice of χ corresponds to (1.6). Note that the shift of x by a real constant x_0 is a canonical transformation which is not completely equivalent to the shift of α:

$$\psi_\alpha(x - x_0) = e^{ix_0 \mathrm{Im}\,\alpha/\sqrt{2}}\psi_{\alpha + x_0/\sqrt{2}}(x).$$

Since the bound state wave functions are expressed through the Hermite polynomials $H_n(x)$,

$$\langle x|n\rangle = \frac{H_n(x)}{\sqrt{2^n n!\sqrt{\pi}}}e^{-x^2/2}, \tag{1.7}$$

the relations (1.5) and (1.6) lead to the generating function for these polynomials,

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x) = e^{2xt - t^2}, \qquad t = \frac{\alpha}{\sqrt{2}}.$$

The strongest quantitative measure of differences in the behavior of quantum and classical particles is expressed by the Schrödinger-Robertson uncertainty principle [16]:

$$\Delta \equiv \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 \geq \frac{1}{4}, \tag{1.8}$$

where $\sigma_{bc} = \frac{1}{2}\langle bc + cb\rangle - \langle b\rangle\langle c\rangle$, and the angular brackets denote averaging over an arbitrary normalizable state for which the mean values are well defined, $\langle b\rangle = \langle\psi|b|\psi\rangle$.

Averaging over the coherent states $|\alpha\rangle$ one finds $\langle x^2\rangle = \frac{1}{2} + \langle x\rangle^2$, $\langle p^2\rangle = \frac{1}{2} + \langle p\rangle^2$, $\langle px + xp\rangle = 2\langle x\rangle\langle p\rangle$, where

$$\langle x\rangle = (\alpha + \alpha^*)/\sqrt{2}, \qquad \langle p\rangle = (\alpha - \alpha^*)/i\sqrt{2}.$$

In this case $\Delta = 1/4$. Since $\sigma_{xp} = 0$, this is the lower bound of the Heisenberg uncertainty relation as well: $\sigma_{xx}\sigma_{pp} = 1/4$. Note that the latter equality does not determine uniquely coherent states. One may scale x in (1.6) by a real number and preserve minimality of the product $\sigma_{xx}\sigma_{pp}$. The resulting states are called squeezed states; a wider class of states corresponds to the lower bound of (1.8). It is necessary to impose additional constraints in order to get (1.6) uniquely [16, 17].

The elementary example considered here illustrates three possible definitions of coherent states for an arbitrary system: (1) as eigenfunctions of some symmetry operators lowering the energy, (2) as an orbit of states generated by a chosen group element from a fixed state, and (3) as minimum uncertainty states for some physically significant operators. It is the first definition that we employ in this paper.

II. PARITY INVARIANCE AND SUPERPOSITIONS OF COHERENT STATES

The formulas (1.2)–(1.6) are well known and widely used in quantum physics. First, we would like to find their analogs for a nonstandard realization of the bosonic oscillator algebra. It is easy to see that this algebra has a nontrivial automorphism (i.e., a map onto itself), or canonical transformation associated with the parity operator P:

$$PxP = -x, \qquad PpP = -p, \qquad P^2 = 1, \qquad P^\dagger = P. \tag{2.1}$$

Moreover, the transformation of x and p to the Hermitian variables

$$\tilde{x} \equiv -ipP, \qquad \tilde{p} \equiv ixP$$

is also canonical: $[\tilde{x}, \tilde{p}] = i$. Although this is a quite simple fact, it leads to the nontrivial reshaping of coherent states. Let us define new creation and annihilation operators,

$$A^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)P, \qquad A = \frac{1}{\sqrt{2}}P\left(\frac{d}{dx} + x\right), \tag{2.2}$$

so that $A + A^\dagger = \sqrt{2}\tilde{x}$, $A - A^\dagger = i\sqrt{2}\tilde{p}$. Evidently, the algebra, Hamiltonian, and vacuum state of the harmonic oscillator defined by the operator A coincide with the ones considered in the preceding section. However, there is an essential difference in the structure of energy eigenstates generated by $A^\dagger$:

$$|n\rangle_{new} \equiv \frac{1}{\sqrt{n!}}(A^\dagger)^n|0\rangle = (-1)^{s_n}|n\rangle, \tag{2.3}$$

where the sign factor $(-1)^{s_n}$ has the form

$$(-1)^{s_n} = \begin{cases} 1, & n = 4k,\ 4k+1, \\ -1, & n = 4k+2,\ 4k+3. \end{cases} \qquad k = 0, 1, 2, \ldots \tag{2.4}$$

This follows from the relation $(A^\dagger)^2 = -(a^\dagger)^2$ and the parity invariance of the vacuum.

Denote by $|\alpha\rangle_P$ the eigenstates of A, $A|\alpha\rangle_P = \alpha|\alpha\rangle_P$, or

$$(d/dx + x)\psi_\alpha^P(x) = \sqrt{2}\alpha\psi_\alpha^P(-x), \qquad \psi_\alpha^P(x) = \langle x|\alpha\rangle_P. \tag{2.5}$$

This is not an ordinary differential equation, but it can be easily solved using the relation $A^2 = -a^2$. Picking out the appropriate combination of two linearly independent eigenstates of a^2 with the eigenvalue $-\alpha^2$, we find

$$|\alpha\rangle_P = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|i\alpha\rangle + e^{i\pi/4}|-i\alpha\rangle\right), \tag{2.6}$$

$${}_P\langle\beta|\alpha\rangle_P = \langle\beta|\alpha\rangle = \exp(\beta^*\alpha - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2),$$

where $|\alpha\rangle$ are the canonical coherent states. In the coordinate representation one has

$$\psi_\alpha^P(x) = \frac{\sqrt{2}}{\pi^{1/4}}\exp\left(\frac{\alpha^2 - |\alpha|^2 - x^2}{2}\right)\cos\left(\sqrt{2}\alpha x - \frac{\pi}{4}\right). \tag{2.7}$$

We call (2.7) the *parity coherent states*, because the parity symmetry plays a central role in their definition. As is argued below, analogous states can be constructed for an arbitrary symmetric potential. Since $|\alpha\rangle_P$ is defined by (1.5) with A, $A^\dagger$ instead of a, $a^\dagger$, one can derive the following generating relation for the Hermite polynomials:

$$\sum_{n=0}^{\infty}\frac{(-1)^{s_n}t^n}{n!}H_n(x) = \sqrt{2}e^{t^2}\cos\left(2tx - \frac{\pi}{4}\right).$$

The $|\alpha\rangle_P$'s are not minimal uncertainty states for the variables x and p when $\alpha \neq 0$:

$$\langle x\rangle = \frac{\alpha + \alpha^*}{\sqrt{2}}e^{-2|\alpha|^2}, \qquad \langle p\rangle = \frac{\alpha - \alpha^*}{i\sqrt{2}}e^{-2|\alpha|^2},$$

$$\sigma_{xx} = [1 - (\alpha - \alpha^*)^2 - (\alpha + \alpha^*)^2 e^{-4|\alpha|^2}]/2, \tag{2.8}$$

$$\sigma_{pp} = [1 + (\alpha + \alpha^*)^2 + (\alpha - \alpha^*)^2 e^{-4|\alpha|^2}]/2,$$

$$\sigma_{xp} = (\alpha^{*2} - \alpha^2)(1 + e^{-4|\alpha|^2})/2i,$$

so that

$$\Delta = \frac{1}{4}\{1+\rho[1-(1+\rho)e^{-\rho}]\}, \qquad \rho \equiv 4|\alpha|^2,$$

and the minimum is reached only for $\rho = 0$. However, by construction itself, the parity coherent states minimize the product of uncertainties in the new canonical variables $\tilde{x}$ and $\tilde{p}$.

The minimum of σ_{xx} (2.8) is reached for $\alpha = \alpha^* = \pm 1/2$: $\sigma_{xx}\big|_{min} = (1-e^{-1})/2 \approx 0.32$, i.e., there is a squeezing for small $|\alpha|$. However, the states $|\alpha\rangle_P$ differ from the squeezed states [17] arising as eigenstates of the annihilation operator after the canonical transformation

$$A = S^\dagger a S = \cosh|z|\; a + \frac{z}{|z|}\sinh|z|\; a^\dagger,$$

generated by the unitary operator

$$S(z) = e^{(za^{\dagger 2} - z^* a^2)/2}. \tag{2.9}$$

If one multiplies this A by the parity operator from the left, then eigenfunctions of the resulting operator will be given again by superposition (2.6), but now with the $|\alpha\rangle$'s on the right-hand side (rhs) being replaced by squeezed states. Note that in the Bargmann-Fock representation of the harmonic oscillator algebra, when $A = Pd/dz$, $A^\dagger = zP$, one has $\psi_\alpha^P(z) \propto \cos(\alpha z - \pi/4)$.

It is easy to construct analogs of (2.6) for the two-oscillator algebra: $[a_j, a_k^\dagger] = \delta_{jk}$, $[a_j, a_k] = 0$, $j,k = 1,2$. Again, the combinations $A_j = Pa_j$, where P is an operator which inverts both space coordinate axes, satisfy the same algebra. Such a transformation affects only the sign of the energy eigenfunctions. Eigenstates of the A_j operators, $A_j|\alpha_1,\alpha_2\rangle_P = \alpha_j|\alpha_1,\alpha_2\rangle_P$, are given by the superposition

$$|\alpha_1,\alpha_2\rangle_P = \frac{1}{\sqrt{2}}\Big(e^{-i\pi/4}|i\alpha_1\rangle_1|i\alpha_2\rangle_2 + e^{i\pi/4}|-i\alpha_1\rangle_1|-i\alpha_2\rangle_2\Big), \tag{2.10}$$

where $|\alpha\rangle_j$ are the canonical coherent states of the jth degree of freedom. This formula is obtained by choosing an appropriate linear combination of the a_j^2 operator eigenstates. Generalization of (2.10) to an arbitrary number of oscillators is obvious.

The unitary operator U that transforms (1.2) into (2.2) (or its multidimensional analog) is easily found due to the relation between the operators P and N in Hilbert space: $P = (-1)^N = \exp i\pi N$. It is

$$U = e^{\pi i N(N-1)/2}, \qquad A = U^\dagger a U, \quad A^\dagger = U^\dagger a^\dagger U.$$

This is a special case of the canonical transformations generated by the operator

$$U = e^{-i\theta(N)}, \qquad U^\dagger a U = e^{i\varphi(N)}a,$$

$$\varphi(N) = \theta(N) - \theta(N+1), \tag{2.11}$$

where $\theta(N)$ is a nonsingular function (such transformations were discussed in [18, 19] without relating them to superpositions of coherent states). In the classical case, when $H_{cl} = \alpha^*\alpha$, $\alpha = (x+ip)/\sqrt{2}$, relations (2.11) correspond to a rotation of α through an angle depending on energy: $U^\dagger \alpha U = e^{i\varphi(\alpha^*\alpha)}\alpha$, which is obviously a symmetry of the system.

Applying the operator $U^\dagger$ (2.11) to $|\alpha\rangle$, one finds eigenstates of the operator $A = e^{i\varphi(N)}a$:

$$|\text{coh}\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\theta(n)}|n\rangle. \tag{2.12}$$

These are the generalized coherent states of Titulaer and Glauber [10]. The states (2.6) belong to their particular subclass characterized by the periodicity condition

$$\theta(n+M) = \theta(n), \tag{2.13}$$

imposed upon the function $\theta(n)$ [in our example $M = 4$; cf. (2.4)]. In this case, one has [20, 21]

$$|\text{coh}\rangle = \sum_{k=0}^{M-1} C_k|\epsilon^k\alpha\rangle, \qquad \epsilon = e^{2\pi i/M}, \quad \epsilon^M = 1,$$

i.e., $|\text{coh}\rangle$ is a superposition of M coherent states with the parametrizing variable α modulated by the powers of a primitive Mth root of unity. The states $|\epsilon^k\alpha\rangle$ are linearly independent, so that they can be orthonormalized. For the simplest $M = 2$ case these orthonormal superpositions take the form of even and odd coherent states [12],

$$|\alpha_\pm\rangle = \frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{2 \pm 2e^{-2|\alpha|^2}}}.$$

The general root-of-unity analogs of these states $|\alpha_l\rangle$, $l = 0,1,\ldots,M-1$, $\langle\alpha_l|\alpha_m\rangle = \delta_{lm}$, have the following form:

$$\begin{aligned}|\alpha_l\rangle &= C_l(\alpha)\sum_{m=0}^{M-1}\epsilon^{-lm}|\epsilon^m\alpha\rangle \\ &= MC_l(\alpha)e^{-|\alpha|^2/2}\sum_{k=0}^{\infty}\frac{\alpha^{Mk+l}}{\sqrt{(Mk+l)!}} \\ &\quad\times|Mk+l\rangle,\end{aligned} \tag{2.14}$$

$$|C_l(\alpha)|^2 = \frac{e^{|\alpha|^2}}{M}\left(\sum_{m=0}^{M-1}\epsilon^{-lm}\exp(\epsilon^m|\alpha|^2)\right)^{-1}.$$

It is easy to derive these formulas using the kernel of the finite-dimensional Fourier transformation F,

$$(F)_{lm} \equiv \frac{1}{\sqrt{M}}\epsilon^{lm}, \qquad F^\dagger F = FF^\dagger = 1,$$

$$F^t = F, \qquad (F^2)_{lm} = \delta_{l0}\delta_{m0} + \delta_{M-l,m}, \qquad F^4 = 1.$$

The states (2.14) do not belong to the class (2.12) because for arbitrary values of α they cover only a part of the Hilbert space,

$$\int d^2\alpha\; |\alpha_l\rangle\langle\alpha_l| = \pi_l, \qquad d^2\alpha \equiv d(\text{Re}\;\alpha)d(\text{Im}\;\alpha)/\pi,$$

$$\pi_l\pi_m = \pi_m\delta_{lm}, \qquad \sum_{l=0}^{M-1}\pi_l = 1,$$

whereas the Titulaer-Glauber coherent states form an overcomplete set of states. The possibility of splitting the Hilbert space of the harmonic oscillator into an arbitrary number of orthogonal subspaces is related to the fact that the projection operators π_l are conserved,

$$\pi_l = \frac{1}{M}\sum_{m=0}^{M-1} \epsilon^{m(N-l)}, \qquad [H, \pi_l] = 0.$$

The distinguished property of the parity operator is that it is the only symmetry of separate kinetic and potential terms of the harmonic oscillator Hamiltonian. Moreover, the definition $Pf(x) = f(-x)$ works for any function independently of its membership in the Hilbert space (for complex x this is just the rotation of the complex plane by π). It is only for the functions which can be expanded over the normalizable states $|n\rangle$ that the operator $\exp i\pi N$ coincides with parity, e.g., for the nonnormalizable eigenfunctions of the Hamiltonian the action of this operator is formally equivalent to multiplication by some phase factor which is not related to the transformation $x \to -x$. Therefore it is not clear what are the analogs of the operators $\exp i\theta(N)$ beyond the Hilbert space context.

Although the finite-dimensional truncation of the Titulaer-Glauber coherent states was discovered a long time ago [20, 21], only recently in [11] have Yurke and Stoler derived explicitly the superposition (2.6), with $i\alpha$ being replaced by α. In their approach it emerged as a result of a fixed period time evolution of the standard coherent states governed by a specific Hamiltonian $\propto N^k$, k even. The squeezing and other properties of these and more general finite-term superpositions of coherent states have been analyzed in [22–24]. The states (2.10) belong to the class of so-called two-particle entangled coherent states [22, 25, 26], which have the intrinsic property of nonfactorizability into the product of one-particle states. For the construction of multimode analogs of the even and odd coherent states, see [27]. Recent interest in superpositions of macroscopically distinguishable quantum states (Schrödinger cat states) like (2.6) or the entangled states like (2.10) is inspired by possibilities to create them with the help of optical techniques. This in turn provides interesting experimental tests of the basic principles of quantum mechanics (see, e.g., [11, 22, 25, 26, 28] and references therein).

The general set of coherent states constructed with the help of canonical transformations based upon the parity operator has the following form. Consider the unitary operator V,

$$V = \cos\varphi + iP\sin\varphi, \qquad V^\dagger V = VV^\dagger = 1, \quad (2.15)$$

where φ is an arbitrary parameter, and construct the Weyl algebra generators analogous to (2.2):

$$A = Va = aV^\dagger, \qquad A^\dagger = a^\dagger V^\dagger = Va^\dagger, \quad (2.16)$$

$$[A, A^\dagger] = [a, a^\dagger] = 1, \qquad A^2 = a^2, \qquad (A^\dagger)^2 = (a^\dagger)^2.$$

The eigenfunctions of the operator A, or parity coherent states, are easily found:

$$\psi_\alpha^P(x) = \frac{1}{2}\left[(1 + e^{-i\varphi})\psi_\alpha(x) + (1 - e^{-i\varphi})\psi_{-\alpha}(x)\right], \quad (2.17)$$

with $\psi_\alpha(x)$ defined in (1.6). Calculating the uncertainties of x and p in these states, we obtain

$$\Delta(\varphi) = \frac{1}{4}\left\{1 + \rho\sin^2\varphi[1 - (1+\rho)e^{-\rho}]\right\}, \quad \rho = 4|\alpha|^2.$$

It is seen that the choice $\varphi = \pi/2$, which corresponds to the Yurke-Stoler coherent states, is extremal — for it the value of Δ is maximally deviated from the standard coherent states case $\varphi = 0$, when $\Delta = 1/4$.

Let us discuss the Titulaer-Glauber states when the phases $\theta(n)$ satisfy the following q-periodicity condition:

$$\theta(n + M) = q\theta(n), \quad (2.18)$$

which is a simple q deformation of the condition (2.13). When $q \to 1$ one can renormalize θ, $\theta = \tilde{\theta} - \phi/(1-q)$, and get $\tilde{\theta}(n+M) = \tilde{\theta}(n) + \phi$, which is a quasiperiodicity condition. However, the effect of such a shift by ϕ is equivalent to the multiplication of α by the factor $\exp i\phi/M$, which is harmless for the representation of $|\mathrm{coh}\rangle$ as a finite-term superposition of canonical coherent states.

For generic values of q there is no split of $|\alpha\rangle$ onto a superposition of a finite number of coherent states. For $M = 1$ one has $\theta(n) = \phi q^n$, and

$$|\mathrm{coh}\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\phi q^n}|n\rangle = \sum_{k=0}^{\infty}\frac{(i\phi)^k}{k!}|q^k\alpha\rangle. \quad (2.19)$$

However, when the parameter q is a primitive Mth root of unity, $q^M = 1$, the sum (2.19) is truncated:

$$|\mathrm{coh}\rangle = \sum_{l=0}^{M-1} B_l(\phi)|q^l\alpha\rangle, \quad B_l(\phi) = \frac{1}{M}\sum_{m=0}^{M-1} q^{-lm}e^{iq^m\phi}.$$

The coefficients B_l are defined by the finite sums of exponentials which were already encountered in the calculation of normalization constants for the orthonormal states $|\alpha_l\rangle$ (2.14). We would like to note that the q deformation (2.18) is not related to the q-coherent states to be discussed below, although the discrete energy spectrum of the corresponding Hamiltonians is found from the similar formula $E_{n+M} = q^2 E_n$.

III. UNIVERSALITY OF SUPERPOSITIONS OF COHERENT STATES

Let us discuss whether the superpositions of a finite number of coherent states considered above are characteristic only for the harmonic oscillator case or if their form carries universal character applicable to any system. Unfortunately, there is no completely satisfactory definition of coherent states $|\alpha\rangle$ for an arbitrary Hamiltonian

$$H = -\frac{d^2}{dx^2} + u(x), \tag{3.1}$$

even if the potential $u(x)$ is an analytical or infinitely differentiable function of x (i.e., when the random, singular, and potentials from C^k, $k < \infty$, are excluded). Denote by $|\lambda\rangle$ the physical eigenfunctions of the Hamiltonian:

$$H|\lambda\rangle = E(\lambda)|\lambda\rangle,$$

where λ is some index labeling the spectrum; $\lambda \equiv n = 0, 1, \ldots$ for the discrete eigenvalues [the ordering $E(n) < E(n+1)$ is assumed], and for the continuous spectrum λ may be thought of as some continuous variable such that the spectrum $E(\lambda)$ is monotonically covered by variation of λ, say, from λ_0 to ∞. We assume that the continuous spectrum states are normalized by the condition $\langle\lambda|\sigma\rangle \propto \delta(\lambda - \sigma)$. From the completeness of generalized eigenfunctions of H in Hilbert space,

$$\sum_{n=0}^{n_b-1} |n\rangle\langle n| + \int_{\lambda_0}^{\infty} d\lambda |\lambda\rangle\langle\lambda| = 1, \tag{3.2}$$

where n_b is the number of bound states, it follows that for any definition of coherent states $|\alpha\rangle$, these states can be expanded over $|n\rangle$ and $|\lambda\rangle$:

$$|\alpha\rangle = \sum_{n=0}^{n_b-1} c_n(\alpha)\, |n\rangle + \int_{\lambda_0}^{\infty} d\lambda\, c_\lambda(\alpha)\, |\lambda\rangle. \tag{3.3}$$

The continuous spectrum exists whenever, for $x \to \infty$ (or $-\infty$), the potential is not bounded from below or it is bounded from above. In these cases one cannot exclude in general the second term in expansion (3.3). Such possibilities are rarely discussed in the literature; some models of coherent states built only from the continuous spectrum states are described in the next three sections.

Suppose for a moment that the Hamiltonian H has only a discrete spectrum, i.e., $n_b = \lambda_0 = \infty$. Then, following many existing examples [1], it is natural to assume that coherent states $|\alpha\rangle$ represent a generating function for stationary energy states of the form

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n c_n(|\alpha|)|n\rangle. \tag{3.4}$$

Since the coefficients c_n depend only on the modulus of α, these states are complete,

$$\int d^2\alpha\rho(|\alpha|)|\alpha\rangle\langle\alpha| = \sum_{n,m=0}^{\infty} |n\rangle\langle m| \times \int d^2\alpha\rho(|\alpha|)\alpha^{*m}\alpha^n c_n c_m^* = 1,$$

provided there exists a measure density $\rho(|\alpha|)$ which satisfies the relations

$$\int_0^\infty dr\rho(r)r^{2n+1}|c_n(r)|^2 = \frac{1}{2}$$

for arbitrary $n = 0, 1, \ldots$. When the variables of $c_n(r)$ separate, $c_n(r) = c_n g(r)$, which is typical for the ladder operator approach, this is a moment problem. Theoretically it is possible that $\rho(r)$ is defined nonuniquely, in which case there are many physically distinguishable representations of observables in the coherent state basis.

An interesting definition of coherent states based on the uncertainty principle was suggested in [29] for a wide class of potentials. It uses the fact that for a classical particle moving in a convex potential one can make a noncanonical nonlinear change of phase space variables such that in the new "coordinates" the particle's dynamics is described by the harmonic oscillator equations of motion. After quantization, the minimum uncertainty states of these harmonic motion "position" and "momentum" operators are called coherent states. By definition, this procedure is tied to the quasiclassical approximation. Its general group-theoretical meaning is not clear to the author; probably the approximate dynamical symmetry approach of [18] can be useful in this context. Another constructive definition of coherent states for generic discrete spectrum systems has been suggested by Klauder [30]. In this approach, the requirement that time evolution of coherent states be equivalent to the change $\alpha \to e^{i\omega t}\alpha$ is taken as a basic property. Such states do not spread, but for them one has more complicated expansions than (3.4).

In the following we utilize a different definition of coherent states; namely, we assume that they are defined as eigenstates of some lowering operator A, i.e., the operator which maps a part of physical solutions of a given stationary Schrödinger equation to the physical ones with lower energy. Even for systems with purely discrete spectra this requirement does not necessarily mean that A annihilates the ground state or that A is the lowering operator mapping the discrete spectrum eigenfunction $|n\rangle$ to the closest from below state $|n-1\rangle$ (it may jump over some physical states, which is always so for the continuous spectrum). For example, similarly to the situation in [6, 7], A may play the role of both lowering and raising operators for different ranges of energy.

We use the factorization method [8, 9] as a tool for searching for such a symmetry operator A. Let us factorize the Hamiltonian (3.1), i.e., represent it as a product of two first-order differential operators conjugated formally to each other:

$$H = a^\dagger a + E_0, \quad a = d/dx + f(x),$$
$$a^\dagger = -d/dx + f(x), \tag{3.5}$$

where E_0 is some constant. The potential $u(x)$ and superpotential $f(x)$ are related by the Riccati equation, $u(x) = f^2(x) - f'(x) + E_0$. The zero mode of a,

$$a\psi_0(x) = 0, \qquad \psi_0(x) \propto e^{-\int^x f(y)dy},$$

is the generalized eigenfunction of the Hamiltonian with the eigenvalue E_0. When this function is normalizable and nodeless, E_0 is the ground state energy. If a is a symmetry operator, i.e., if it maps physical eigenstates of the Hamiltonian H onto themselves, then coherent states could be defined as eigenstates of a, but there may be nonuniqueness even in this simple situation. Indeed, the factorization involves an arbitrary unitary operator T:

$$A^\dagger A = a^\dagger a, \quad A^\dagger = a^\dagger T, \quad A = T^{-1} a, \quad T^\dagger = T^{-1}. \tag{3.6}$$

Actually, one may write $A^\dagger = a^\dagger C$, $A = Da$, where C and D are two operators satisfying $CD = 1$. The operators $A^\dagger$ and A are conjugated to each other if $C = D^\dagger$, i.e., when D is an isometric operator [31]. Only the additional requirement $DD^\dagger = 1$ makes D unitary; we restrict ourselves to this case.

Coherent states of potentials for which $A = T^{-1}a$ is a symmetry operator, for some unitary operator T, are thus defined by the equation

$$\psi_0(x) \frac{d}{dx} \frac{\psi_\alpha(x)}{\psi_0(x)} = \alpha T \psi_\alpha(x), \tag{3.7}$$

where $\psi_0(x)$ is an eigenstate of the Hamiltonian (not necessarily a physical one). The natural extension of this definition involves on the lhs of (3.7) a differential operator of the Nth order. In that case $A^\dagger A$ is equal to an order N polynomial of a Hamiltonian, i.e., one has a generalized factorization scheme, which will be described in Sec. VI.

Suppose now that the expansion (3.4) takes place (it does not mean that there is no continuous spectrum; there may be accumulation points such that the set $|n\rangle$ is closed under the action of A). Then, one can introduce the formal number operator N satisfying $N|n\rangle = n|n\rangle$. Acting upon such $|\alpha\rangle$ by the unitary operator $U^\dagger = e^{i\theta(N)}$, one gets the Titulaer-Glauber-type coherent states for the chosen class of potentials. The periodicity condition $\theta(n+M) = \theta(n)$ leads again to the finite-term superpositions of $|\epsilon^k\alpha\rangle$. Therefore the form (but not the normalization constants) of the superpositions $|\alpha_k\rangle$ (2.14) is universal — for any H they perform a split of discrete spectrum Hilbert subspace onto orthogonal components. Note, however, that the symmetry properties of these superpositions may be different; in particular, the general even and odd coherent states are not eigenstates of the parity operator. For instance, for the shifted harmonic oscillator potential $u(x) = (x - x_0)^2$ one has

$$\langle x|\alpha_\pm\rangle \propto e^{-(x-x_0)^2/2} \left(e^{\sqrt{2}(x-x_0)\alpha} \pm e^{-\sqrt{2}(x-x_0)\alpha} \right), \tag{3.8}$$

which are eigenstates of the Hermitian conserved charge $P \exp(2x_0 d/dx)$. For the general asymmetric potential, $|\alpha_\pm\rangle$ are eigenstates of the abstract operator $\exp i\pi N$, which does not have a simple form in the coordinate representation.

Similarly, one can always define upon the discrete spectrum an analog of the Yurke-Stoler coherent states, but they will be eigenstates of the operator Pa, an analog of (2.2), only in the case when $Pa = -aP$, which assumes that the potentials are symmetric, $u(-x) = u(x)$ (if one has $Pa = aP$, then the symmetric and antisymmetric eigenfunctions of a diagonalize simultaneously the operator Pa). We come thus to the conclusion that the notion of "parity coherent states" is less universal than that of even and odd coherent states and their higher root-of-unity generalizations defined by the abstract formula (2.14). The special name for the states (2.17), and similar ones, was introduced in order to distinguish Titulaer-Glauber states with the $M = 2$ periodic phases for symmetric and asymmetric potentials (in the first case there is no need for the abstract operator N).

Single-valuedness of the expansion (3.4) under the rotation $\alpha \to e^{2\pi i}\alpha$ is the key property allowing to build finite-term superpositions of coherent states (2.14). In general such a property does not hold. In the next section we consider a model where only the continuous spectrum piece is present in (3.3) with $c_\lambda(\alpha) \propto \alpha^{-\ln\lambda/\ln q^2}$. In this case the change $\alpha \to \epsilon^k\alpha$, $\epsilon^M = 1$ does not provide a split of Hilbert space onto the finite number of orthogonal components.

IV. COHERENT STATES OF THE FREE PARTICLE

The above definition of coherent states (3.7) is applicable to systems with continuous energy spectra. Consider the simplest possible case of zero potential, $H = -d^2/dx^2$, for which the generalized eigenfunction of lowest energy is a simple constant. Solutions of the Heisenberg equations of motion $x = p_0 t + x_0$, $p = p_0$, where t is the time variable and p_0, x_0 are operators at $t = 0$, are identical with the classical ones. If this coincidence would be taken as the basis for the definition of coherent states, then any normalizable state of the free particle has to be considered as coherent. However, despite the same form for the equations of motion, even for the $\langle p_0\rangle = 0$ case, when the classical particle stays at the point x_0, the quantum particle tries to occupy the whole space: $\sigma_{xx} \propto \langle p_0^2\rangle t^2$, $t \to \infty$, and nonspreading wave packets do not exist.

The simplest factorization of the free-particle Hamiltonian is obvious and we have $a = d/dx$. Let us find eigenfunctions of $A = T^\dagger d/dx$ when T is a unitary operator performing an affine transformation, or the parity operator. If $T = 1$, then $\psi_\alpha(x) \propto e^{\alpha x}$, which are bounded (but unnormalizable) functions for purely imaginary α: $\alpha = ip$, $-\infty < p < \infty$, so that $\psi_{ip}(x)$ coincide with the momentum eigenstates. If T is a translation by the h operator, then

$$d\psi_\alpha(x)/dx = \alpha\psi_\alpha(x+h)$$

and again there are simple solutions of the form $\psi_\alpha(x) \propto e^{ipx}$, but now α has a real part: $\alpha = ipe^{-iph}$. If $T = P$, the parity operator, then one gets superposition (2.6), $\psi_\alpha^P(x) \propto \cos(\alpha x - \pi/4)$, α real. Since these states are not physically realizable (they belong to the continuous spectrum), their "coherence" is formal. This is related to the fact that the chosen symmetry operators A are integrals of motion commuting with the Hamiltonian. As seen from the considerations given below, such A's appear from real lowering operators in a special limit such that, in fact, all Hamiltonian eigenstates acquire a flavor of coherent states (this is reminiscent of the approach of [2], where coherent states are defined as eigenstates of some combinations of integrals of motion). Examples of symmetry algebras for which the Hamiltonian eigenstates might be counted among the coherent states are given below. In the corresponding cases the operator A may be simultaneously the lowering and raising operator, and an integral of motion for different parts of the spectrum.

Let T be the scaling operator:

$$Tf(x) = \sqrt{|q|}f(qx), \tag{4.1}$$

where q is some real parameter, $0 < q^2 < 1$. For positive q, T is just the squeezing operator in a special form: $T = S(z = \ln q) = q^{(a^{\dagger 2} - a^2)/2}$, where $a^\dagger, a$ are given by (1.2); and for negative q it is a product of the same S and the parity operator: $T = SP$. For simplicity we assume in this section that q is positive. Now the operator $A = T^\dagger a$ is not an integral of motion, but the *raising* operator for the $\lambda > 0$ solutions of the Schrödinger equation

$$H\psi(x) = -\psi''(x) = \lambda\psi(x). \tag{4.2}$$

The commutation relations

$$AA^\dagger = q^2 A^\dagger A, \qquad AH = q^2 HA, \qquad HA^\dagger = q^2 A^\dagger H$$

look similar to some of the defining relations of quantum groups [32, 33]. In fact, they are the progenitors of q-deformed oscillator algebra (see below). Eigenfunctions of the operator A are determined by the differential equation with deviating argument,

$$\psi'_\alpha(x) = \alpha\sqrt{q}\psi_\alpha(qx), \qquad 0 < q < 1. \tag{4.3}$$

The initial value problem for (4.3) is qualitatively different from that for the ordinary differential equations because the initial conditions now have to be fixed on the interval $[qx_0, x_0]$. When x_0 is a fixed point of the scaling transformation, i.e., $x_0 = 0$ or ∞, this interval shrinks to one point. When $x_0 = \infty$, one fixes solutions by taking the asymptotic form of $\psi_\alpha(x)$ from some class of permitted functions [34]. For $x_0 = 0$ it is natural to impose the initial condition $\psi_\alpha(0) = \gamma < \infty$. Then Eq. (4.3) has the unique analytical solution

$$\psi_\alpha(x) = \gamma \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{n!} (\alpha\sqrt{q}x)^n, \tag{4.4}$$

which is an entire function of x for any finite $|\alpha|$. However, as shown in [34] (see also [35]), the $x \to \infty$ asymptotics of any solution of (4.3) is dominated by the factor $\exp(-\ln^2 x/\ln q^2)$, i.e., all solutions grow at infinity so that the functions $\psi_\alpha(x)$ do not describe physical states. Actually, this could be expected from the fact that we were diagonalizing the raising operator.

Consider eigenfunctions of the lowering operator $A^\dagger$,

$$A^\dagger\psi_\alpha(x) = \alpha\psi_\alpha(x), \qquad A^\dagger = -\frac{d}{dx}T,$$

or

$$\psi'_\alpha(x) = -\alpha q^{-3/2}\psi_\alpha(q^{-1}x), \qquad 0 < q < 1. \tag{4.5}$$

The formal series solution of this equation with the boundary condition $\psi(0) = \gamma$ looks similar to (4.4), with q being replaced by q^{-1}. But the radius of convergence of this series is equal to zero, i.e., there are no solutions analytical at zero. This does not mean that there are no solutions at all; in [34] it was shown that, in fact, there are infinitely many nonanalytical solutions of (4.5) from C^∞ satisfying boundary condition $\psi(0) = \gamma$. Moreover, for any α there are solutions with the $|x| \to \infty$ asymptotics $\propto \exp(\ln^2 |x|/\ln q^2)$, i.e., there are functions $\psi_\alpha(x)$ which *are normalizable.* One can expand such $\psi_\alpha(x)$ over the basis of Hamiltonian eigenfunctions

$$\psi_\alpha(x) = \int_0^\infty dp\, e^{ipx}\phi_\alpha(p), \tag{4.6}$$

which is the positive momentum part of the standard Fourier integral. Similarly one can consider the expansion over e^{-ipx} (for negative q the two regions of p should be considered simultaneously). Substituting this expression into (4.5) and solving the corresponding finite-difference equation for the form factor $\phi_\alpha(p)$, we find

$$\psi_\alpha(x) = \int_0^\infty dp\, e^{ipx}h(p)\exp\frac{\ln^2 p/i\alpha}{2\ln q}, \tag{4.7}$$

where $h(p)$ is an arbitrary function periodic on the logarithmic scale, $h(qp) = h(p)$, normalized by the condition

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = 2\pi \int_0^\infty dp|h(p)|^2 \times \exp\frac{\ln^2 p/|\alpha| - (\arg i\alpha)^2}{\ln q} = 1.$$

Due to the freedom in choice of $h(p)$ these states can take various forms.

Let us take, for example, the following $h(p)$:

$$h(p) = a \sum_{n=-\infty}^{\infty} q^n \delta(p - bq^n),$$

which does not correspond to the normalizable $\psi_\alpha(x)$. Despite the formality of the consideration in this case, we obtain the free particle's "coherent states" in the form of a Dirichlet series (cf. [36, 35]),

$$\psi_\alpha(x) = a\exp\frac{\ln^2 b/i\alpha}{2\ln q} \times \sum_{n=-\infty}^{\infty} q^{n(n+2)/2}\left(\frac{b}{i\alpha}\right)^n e^{ibq^n x}, \tag{4.8}$$

which defines a function bounded for any real x and $0 < |\alpha| < \infty$. This is an example of the bounded but non-normalizable solution of (4.5). The derived expansion could be interpreted as a q-Fourier series for $\psi_\alpha(x)$ because the sum goes over the trigonometric functions whose argument is modulated by the powers of q [after renormalizations, $\sum c_n \exp(i\tilde{x}q^n)$ may become the standard Fourier series in the limit $q \to 1$ due to the relation $(q^n - 1)/(q - 1) \to n$; in our case this gives a divergent series]. This situation does not seem to be related to the q-Fourier transformation considered in [37]; it resembles more the wavelet transform [38] or the generalized Taylor expansions for atomic functions [39].

In the above factorization of the free-particle Hamiltonian we took $E_0 = 0$. A richer situation arises when $E_0 = -\beta^2$, where β is a nonzero real number. This gives $H = a^\dagger a - \beta^2$, $a = d/dx + \beta$. Evidently a is an integral of motion, but from the properties of affine transformation operator T,

$$Tf(x) = \sqrt{q} f(qx + l), \qquad 0 < q < 1, \qquad -\infty < l < \infty, \tag{4.9}$$

where q and l are fixed parameters, it can be seen that $A = T^{-1}a$ is the raising operator for positive energy states. In order to save space we shall describe the whole hierarchy of such symmetry operators at once. Let us introduce the operators

$$A = T^{-1} \prod_{k=1}^{N} \left(\frac{d}{dx} + \beta_k \right),$$
$$A^\dagger = \prod_{k=1}^{N} \left(-\frac{d}{dx} + \beta_k \right) T, \tag{4.10}$$

where β_k are N arbitrary real positive constants. It is easy to check that $A^\dagger$ and A satisfy the following nonlinear algebraic relations:

$$A^\dagger A = \prod_{k=1}^{N} (H + \beta_k^2), \qquad AA^\dagger = \prod_{k=1}^{N} (q^2 H + \beta_k^2), \tag{4.11}$$

$$HA^\dagger = q^2 A^\dagger H, \qquad AH = q^2 HA. \tag{4.12}$$

For $N = 1$ these relations define a q analog of the Weyl algebra, or q-oscillator algebra (see, e.g., [40–42]),

$$AA^\dagger - q^2 A^\dagger A = \omega, \qquad \omega = \beta_1^2(1 - q^2). \tag{4.13}$$

For $N = 2$ one gets a q analog of the su(1,1) algebra in the form considered, e.g., in [43, 44]. For $N > 2$ one has polynomial quantum algebras [6, 7]. Note that in the limit $q \to 1$ one does not get nontrivial algebras since the operators A and $A^\dagger$ start to commute, still being the differential-difference operators for $l \neq 0$. For $q \neq 1$ it is possible to set $l = 0$ by going to the reference frame where a fixed point of the affine transformation is taken as the zero point; we shall assume this choice below.

Discrete series representations of the derived algebra are constructed by the action of the operators A and $A^\dagger$ upon the Hamiltonian eigenstates with $\lambda \neq 0$. The lowest weight series have the form

$$A\psi_k^{(0)}(x) = 0, \quad \psi_k^{(0)}(x) \propto e^{-\beta_k x}, \quad k = 1, 2, \ldots, N,$$
$$\psi_k^{(n)}(x) \propto (A^\dagger)^n \psi_k^{(0)}(x),$$
$$H\psi_k^{(n)}(x) = -\beta_k^2 q^{2n} \psi_k^{(n)}(x), \quad n = 0, 1, \ldots, \infty.$$

The wave functions $\psi_k^{(n)}(x)$ are not physical; their eigenvalues accumulate near the $\lambda = 0$ point from below. One can take the β_k in (4.10) to be purely imaginary numbers. Then $\psi_k^{(n)}(x)$ describe continuous spectrum states whose eigenvalues accumulate near the zero *from above*. In this case $A^\dagger$ is not a Hermitian conjugate of A but still one can use representations of the algebra (4.11), (4.12). The highest weight representations appear as follows (in the same notations):

$$A^\dagger\psi_k^{(0)}(x) = 0, \qquad \psi_k^{(0)}(x) \propto e^{\beta_k q^{-1} x},$$
$$\psi_k^{(n)}(x) \propto A^n \psi_k^{(0)}(x),$$
$$H\psi_k^{(n)}(x) = -\beta_k^2 q^{-2n-2} \psi_k^{(n)}(x).$$

The eigenvalues of these states are unbounded from below for real β_k and they go to infinity for imaginary β_k. In any case operator A is the lowering operator for negative λ eigenfunctions, but it raises the energy of continuous spectrum states.

Physical eigenstates of the Hamiltonian $H = -d^2/dx^2$ have the form

$$\psi_\lambda^\pm(x) = \frac{1}{\sqrt{2\pi}} \exp(\pm i\sqrt{\lambda} x),$$
$$\int_{-\infty}^{\infty} dx\, \psi_\lambda^{\sigma *}(x) \psi_{\lambda'}^{\sigma'}(x) = \delta_{\sigma\sigma'} \delta(\sqrt{\lambda} - \sqrt{\lambda'}),$$

where $\sigma, \sigma' = \pm$. The algebra generators act upon them in a simple way,

$$A\psi_\lambda^\pm(x) = q^{-1/2} \prod_{k=1}^{N} \left(\pm i\sqrt{\lambda} + \beta_k \right) \psi_{\lambda q^{-2}}^\pm(x), \tag{4.14}$$

$$A^\dagger\psi_\lambda^\pm(x) = q^{1/2} \prod_{k=1}^{N} \left(\mp iq\sqrt{\lambda} + \beta_k \right) \psi_{\lambda q^2}^\pm(x). \tag{4.15}$$

We conclude that the free particle's Hilbert space provides a unitary realization of the quite complicated symmetry algebras. In the following sections we describe a generalization of this construction to nontrivial potentials along the lines of [6, 7].

Consider coherent states of the above algebras defined as eigenstates of symmetry operators lowering the energy. Let us analyze first eigenstates of the "annihilation" operator A, $A\psi_\alpha(x) = \alpha\psi_\alpha(x)$. For the q-oscillator algebra such a definition has been considered, e.g., in [40, 45–49, 37] and many other recent papers. In our model these coherent states for $N = 1$ are defined by the differential-delay equation

$$\psi'_\alpha(x) = \alpha\sqrt{q}\psi_\alpha(qx) - \beta_1\psi_\alpha(x), \tag{4.16}$$

known in the literature as the pantograph equation [34, 35]. Note that the initial value problem for this equation is highly nontrivial. Using the results of the detailed analysis of (4.16) given in [34], it is possible to see that solutions $\psi_\alpha(x)$ having finite fixed values at $x = 0$ are normalizable near the $x = \infty$ point when $|\alpha| < \beta_1$ but for $x \to -\infty$ they diverge exponentially fast. A similar situation holds for the general $N > 1$ symmetry algebras. We conclude that there are no physically acceptable coherent states of this type in the free-particle model. This is caused by the absence of a discrete spectrum. In the more complicated realizations of (4.11), (4.12) such states do exist.

A different type of coherent states for the q-Weyl algebra has been constructed in [15]. These coherent states are defined as eigenstates of the operator $A^\dagger$, which lowers the energy of the $\lambda > 0$ states, $A^\dagger\psi_\alpha(x) = \alpha\psi_\alpha(x)$. For the free-particle realization of the q-oscillator algebra these states are determined again by the pantograph equation, but now the dilation parameter is bigger than 1:

$$\psi'_\alpha(x) = -\alpha q^{-3/2}\psi_\alpha(q^{-1}x) + \beta_1 q^{-1}\psi_\alpha(x). \tag{4.17}$$

This time there are infinitely many normalizable functions $\psi_\alpha(x)$ for $|\alpha| > \beta_1$, decreasing as x^κ, $q^\kappa = \alpha/q^{1/2}\beta_1$, for $|x| \to \infty$. All of them have finite values at $x = 0$ but they are not analytical near this point (i.e., Taylor series expansion does not converge). For arbitrary N, coherent states of this type are defined by the generalized pantograph equation [35]

$$\prod_{k=1}^{N}\left(-\frac{d}{dx} + \beta_k\right)\psi_\alpha(qx) = \alpha q^{-1/2}\psi_\alpha(x), \tag{4.18}$$

whose solutions are not expressible in terms of the classical special functions. Expansion of the normalizable solutions in the basis of Hamiltonian eigenfunctions is given by the integral

$$\psi^{(s)}_{\alpha\pm}(x) = \frac{C(\alpha)}{\sqrt{4\pi}}\int_0^\infty \frac{\lambda^{d_s}e^{\pm i\sqrt{\lambda}x}d\lambda}{\prod_{k=1}^{N}(\pm iq\sqrt{\lambda}/\beta_k;q)_\infty}, \tag{4.19}$$

$$d_s = \frac{\ln\rho/\alpha q^{3/2} + 2\pi i s}{\ln q^2}, \qquad \rho \equiv \beta_1\cdots\beta_N,$$

where $s = 0, \pm1, \ldots$ is an integer enumerating linearly independent states and it is assumed that $0 \le \arg\alpha < 2\pi$. In (4.19) and below we use the standard notations for q products [50],

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k), \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

For positive integer n one has

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k),$$

$$(a;q)_{-n} = \frac{(-q/a)^n q^{n(n-1)/2}}{(q/a;q)_n}.$$

The normalization constant $C(\alpha)$ is given by the integral

$$|C(\alpha)|^{-2} = \int_0^\infty \frac{\lambda^\tau d\lambda}{\prod_{k=1}^{N}(-\lambda q^2/\beta_k^2;q^2)_\infty}, \qquad \tau = \frac{\ln\rho/|\alpha|q}{\ln q}. \tag{4.20}$$

This constant is finite for $|\alpha| > \rho$, which is the region of definition of the coherent states. For any N the states (4.19) have the common qualitative feature of nonanalyticity near the $x = 0$ point. The origin of this property and possible physical consequences deserve further investigation.

One can look also for solutions of (4.18) in the form of series similar to (4.8). For example, for $N = 1$ one can write

$$\psi_\alpha(x) \propto \sum_{n=-\infty}^{\infty}\left(\frac{\beta_1\sqrt{q}}{\alpha}\right)^n (i\theta q/\beta_1;q)_n e^{i\theta q^n x}, \tag{4.21}$$

where θ is an arbitrary constant, $q < \theta \le 1$. These nonnormalizable solutions are bounded for any x provided $|\alpha| > q^{1/2}\beta_1$; however, it is not clear whether they should be taken into account in order for coherent states to be complete.

It is worth mentioning that the pantograph equation appears in various problems. It has been encountered in a description of the light absorption by interstellar matter [51], the collection of current by the pantograph of an electric locomotive [52], some number theory problem, etc. (for a list of applications, see [35]). Here we have described another physical application of this equation as the one determining the free-particle coherent states within the ladder-operator definition context.

Two linearly independent $\lambda = 0$ solutions of (4.2) can be represented in the form

$$\psi_1(x) = 1, \qquad \psi_2(x) = x + \frac{q}{1-q}\sum_{k=1}^{N}\frac{1}{\beta_k}. \tag{4.22}$$

Only the first one is bounded and belongs to the continuous spectrum of the free particle. Formally upon $\psi_j(x)$, both operators A and $A^\dagger$ are diagonalized simultaneously:

$$A\psi_j(x) = \rho q^{1/2-j}\psi_j(x), \qquad A^\dagger\psi_j(x) = \rho q^{j-1/2}\psi_j(x). \tag{4.23}$$

These are the simplest models of the c-number "condensate" representations of quantum algebras discussed in [13, 53].

In a similar manner one can consider the free particle on the half-line. In order to make the Hamiltonian self-adjoint it is necessary to impose boundary conditions at $x = 0$. The scaling operator defines physical symmetry only in the special cases $\psi(0) = 0$ or $\psi'(0) = 0$ [54]. Then one can define again coherent states as eigenstates of the lowering symmetry operators, but we shall not consider them here.

A curious fact is that for $N = 1$, $q = i$, $\beta_1 = 1/\sqrt{2}$, one arrives at a simple differential-difference operator realization of the fermionic oscillator algebra. Indeed, it is not difficult to check that upon the states

$$A|0\rangle = 0, \quad |0\rangle \propto e^{-x/\sqrt{2}}, \qquad |1\rangle \equiv A^\dagger|0\rangle \propto e^{-ix/\sqrt{2}} \tag{4.24}$$

the relations $A^\dagger|1\rangle = 0$ and $A|1\rangle = |0\rangle$ are satisfied, which means that $AA^\dagger + A^\dagger A = 1$, $A^2 = (A^\dagger)^2 = 0$. It is not clear whether this formal two-dimensional representation has a unitary setting.

The scaling operator T also provides an interesting possibility of a nonstandard realization of the ordinary bosonic oscillator algebra. Let $Tf(x,y) = f(qx,qy)$ and $z = x + iy$; then the relations

$$[A, A^\dagger] = 1, \qquad A = T^{-1}\frac{d}{dz}, \qquad A^\dagger = zT \tag{4.25}$$

define some deformation of the Bargmann-Fock realization (when q is complex the operator T scales and rotates z; we assume that $0 < q < 1$). The measure density $\rho(|z|^2)$ in the scalar product

$$\langle\psi_1|\psi_2\rangle = \int d^2z\rho(|z|^2)\overline{\psi_1(z)}\psi_2(z) \tag{4.26}$$

is found from the requirement for $A^\dagger$ to be a conjugate of A. This gives the equation

$$\frac{d\rho(t)}{dt} = -q^{-4}\rho\left(\frac{t}{q^2}\right), \qquad t = |z|^2, \tag{4.27}$$

which we have just encountered in (4.5). Its solution satisfying $\rho(0) =$ const can be represented in the form

$$\rho(t) = \int_0^\infty dp\, h(p)e^{-pt}\exp\frac{\ln^2 pq}{4\ln q}, \qquad h(q^2p) = h(p).$$

Acting by $A^\dagger$ upon the zero mode of A, $A\psi_0(z) = 0$, $\psi_0(z) =$const, one finds the Fock space basis vectors:

$$\psi_n(z) = C_n(q)z^n \propto (zT)^n 1. \tag{4.28}$$

The normalization constants C_n have the form

$$|C_n|^2 = \frac{q^{n(n-1)}}{n!\sqrt{4\pi\ln 1/q}}\left(\sum_{k=-\infty}^{\infty}(-1)^k h_k \exp\frac{\pi^2k^2}{\ln q}\right)^{-1}, \tag{4.29}$$

where h_k are arbitrary constants appearing due to the nonuniqueness of the measure. More precisely, h_k are the coefficients of the Fourier expansion of an arbitrary periodic function entering the measure $h(p)$,

$$h(p) = \sum_{k=-\infty}^{\infty} h_k \exp\frac{\pi ik\ln p}{\ln q}.$$

The relation between h_k and C_n shows that the moment problem for this measure [i.e., determination of $\rho(t)$ from a given C_n] does not have a unique solution. Physical applications of the described realization of the Heisenberg-Weyl algebra are not known.

V. COHERENT STATES OF THE q-DEFORMED HARMONIC OSCILLATOR POTENTIAL

In Sec. II we gave a simple derivation of the Yurke-Stoler states (2.6) on the basis of a canonical transformation associated with the parity operator. Actually, it was inspired by the analysis of coherent states for the q-oscillator algebra,

$$AA^\dagger - q^2A^\dagger A = \omega, \qquad [A,\omega] = [A^\dagger,\omega] = 0, \tag{5.1}$$

in the realization described in [6]. Let us consider this system in more detail. One can check that the pair of formal operators

$$A = T^{-1}\left[d/dx + f(x)\right], \qquad A^\dagger = \left[-d/dx + f(x)\right]T, \tag{5.2}$$

where T is the scaling operator (4.1), satisfy (5.1) provided $f(x)$ is a solution of the equation

$$\frac{d}{dx}\left[f(x) + qf(qx)\right] + f^2(x) - q^2f^2(qx) = \omega, \tag{5.3}$$

derived in [5] as the simplest self-similar reduction of the dressing chain for the Schrödinger equation. The Hamiltonian of this system,

$$H = A^\dagger A - \nu = -d^2/dx^2 + u(x),$$
$$u(x) = f^2(x) - f'(x) - \nu, \qquad \nu \equiv \omega/(1-q^2), \tag{5.4}$$

satisfies the relations

$$AH = q^2HA, \qquad HA^\dagger = q^2A^\dagger H. \tag{5.5}$$

For $q = 1$ one has $f(x) = \omega x/2$, i.e., the standard harmonic oscillator. Suppose that $f(-x) = -f(x)$, which corresponds to the symmetric potential. Then, in the limit $\omega \to 0$ the solution $f(x)$ analytical at $x = 0$ vanishes due to the initial condition $f(0) = 0$, and we get the zero potential model considered in the preceding section with the factorization constant $E_0 = 0$ [the $E_0 \neq 0$ case corresponds to the symmetric solution $f(x) = \sqrt{\nu}$].

The analysis of [13] shows that for complex values of x and q, a solution of (5.3) analytical near $x = 0$ exists and it is unique provided $|q| < 1$ or q is a primitive root of unity of odd degree, $q^{2k+1} = 1$. For $q^{2k} = 1$ a solution may exist only for special initial conditions which, however, do not guarantee uniqueness. If $|q| = 1$ but $q^n \neq 1$ then $f(x) = \pm\sqrt{\nu}$ are the only analytical solutions known to the author. For $0 < |q| < 1$ the function $f(x)$ cannot be expressed in terms of known special functions. When $q^{2k+1} = 1$, and in the restricted case of $q^{2k} = 1$, the problem is solved in terms of particular hyperelliptic functions characterized by the presence of additional symmetries of the lattice of periods. Since in the simplest cases, $q^3 = 1$, $q^4 = 1$, q is just the modular parameter of elliptic functions, the function $f(x)$ (and its generalizations to be described below) comprises hidden "second" q-deformation properties of hyperelliptic, or finite-gap potentials.

In [55] the system (5.1)–(5.5) has been derived from a special quantization of a simple model of classical mechanics (a particle in a finite-depth potential) characterized by a quadratic Poisson algebra. In this picture Eq. (5.3) has the form

$$\hbar\frac{d}{dx}\left[f(x) + e^{\hbar\eta}f(e^{\hbar\eta}x)\right] + f^2(x) - e^{2\hbar\eta}f^2(e^{\hbar\eta}x) = c(1 - e^{2\hbar\eta}), \tag{5.6}$$

where $\hbar$ is Planck's constant, $q = e^{\hbar\eta}$, and η and c are parameters of the classical potential. Expanding this equation over $\hbar$ one finds successively the classical, quasiclassical, and so on approximations to the exact solution $f(x)$ (note that in this approach the potential is an infinite series over $\hbar$, there is a large nonuniqueness, etc.

Let x and q^2 be real, $\omega > 0$, and $f(x)$ antisymmetric, $f(-x) = -f(x)$. For $q^2 > 1$ the function $f(x)$ has singularities [6] so that $A^\dagger$ is not conjugated to A and the realization of (5.1) is not unitary. For $0 < q^2 < 1$ the function $f(x)$ is bounded and has only one zero [5]. These are the crucial properties sufficient for $A^\dagger$ and A to be well-defined operators in the Hilbert space. In particular, the zero mode of A, $A|0\rangle = 0$, $\langle x|0\rangle \propto \exp[-\int^x f(y)dy]$, is normalizable and describes the ground state of Hamiltonian H. As a result the whole spectrum of H is found from unitary representations of the algebra (5.1).

It is not difficult to see from (5.5) that the spectrum may consist of three parts: a discrete one, describing bound states accumulating near the zero energy level from below, a continuous part going from zero to infinity and corresponding to scattering states, and, finally, a zero energy piece. The discrete spectrum is described by the lowest weight discrete series generated by $A^\dagger$ from the vacuum state $|0\rangle$:

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{\omega^n [n]!}}|0\rangle, \quad A|0\rangle = 0, \quad \langle n|m\rangle = \delta_{nm},$$

$$[n]! = [n][n-1]!, \quad [0]! = 1,$$
$$[n] = (1-q^{2n})/(1-q^2),$$

$$A^\dagger|n\rangle = \omega^{1/2}\sqrt{\frac{1-q^{2(n+1)}}{1-q^2}}|n+1\rangle,$$
$$A|n\rangle = \omega^{1/2}\sqrt{\frac{1-q^{2n}}{1-q^2}}|n-1\rangle,$$

$$H|n\rangle = E_n|n\rangle, \qquad E_n = -\,\nu q^{2n}. \tag{5.7}$$

It consists of one geometric series. Since zero modes of A are determined by the first-order differential equation, it follows that (5.7) are the only physical states for $E < 0$. Indeed, suppose that we missed one physical state $|E\rangle$ with energy $E < 0$. Acting by powers of A upon $|E\rangle$ we get a sequence of states of lower energies. Since the potential is bounded this series should be truncated, which is possible only if $|E\rangle$ is annihilated by some power of A, i.e., if the state $|E\rangle$ belongs to the series (5.7).

The same argument shows that the states with positive energy, $E > 0$, appear in the form of a geometric series infinite in both directions,

$$H|n\rangle_\lambda = \lambda q^{2n}|n\rangle_\lambda, \qquad n = 0, \pm 1, \pm 2, \dots , \tag{5.8}$$
$$A|n\rangle_\lambda = \sqrt{\nu + \lambda q^{2n}}|n-1\rangle_\lambda,$$
$$A^\dagger|n\rangle_\lambda = \sqrt{\nu + \lambda q^{2(n+1)}}|n+1\rangle_\lambda,$$

where $\lambda > 0$ is an arbitrarily chosen eigenvalue of H. (This representation of the q-Weyl algebra was discussed in [42, 13]; cf. also [56]; note that it is not defined for $q \to 1$.) Theoretically these states could be normalizable, but in our case this is not so — they belong to the continuous spectrum. Indeed, from (5.7), (5.8) it follows that the scaled potential $q^2 u(qx)$ has the same spectrum as $u(x)$ except for the lowest state with the energy $E_0 = -\nu$. Similarly, $q^{2k}u(q^k x)$ does not have the k lowest states. Taking the limit $k \to \infty$, one gets a system without a negative spectrum, whereas the states (5.8) are not washed away. From the boundedness of the initial potential it follows that $q^{2k}u(q^k x) \to 0$ for $k \to \infty$ because $q^2 < 1$. This means that our potential is reflectionless, being obtained by a special infinite step dressing of zero potential. But for zero potential the series (5.8) with $\omega = 0$ (removal of the levels is equivalent to rescaling $\omega \to q^{2k}\omega \to 0$) correspond to the continuous spectrum (see the preceding section). A rigorous proof of the absence of positive energy bound states requires an estimate of the asymptotics of the potential [57]. The $|x| \to \infty$ asymptotics of (5.4) proposed in [58] decreases sufficiently fast in order to guarantee the absence of such exotic states, $u(x) \to h(x)/x^2 + O(1/x^3)$, where $h(qx) = h(x)$ is a bounded function.

Upon the zero modes of the Hamiltonian the operators A and $A^\dagger$ behave like integrals of motion, i.e., they commute with H. Since the relations $A^\dagger A = AA^\dagger = \nu$ can be satisfied by an arbitrary invertable matrix, the dimension of this representation is not restricted. It may be either infinite-dimensional under additional requirements [42] or just one-dimensional. In the latter case the creation and annihilation operators degenerate into complex numbers [13, 53],

$$H|\mathrm{cl}\rangle = 0, \quad A|\mathrm{cl}\rangle = \sqrt{\nu}e^{-i\theta}|\mathrm{cl}\rangle,$$

$$A^\dagger|\mathrm{cl}\rangle = \sqrt{\nu}e^{i\theta}|\mathrm{cl}\rangle, \tag{5.9}$$

where we assume that the state $|\mathrm{cl}\rangle$ is normalizable. Note that the limit $q \to 1$ is not defined. In our case these "classical" states correspond to the boundary between discrete and continuous spectra. As we shall show below, in the q-oscillator model (5.2) with $f(0) = 0$ the corresponding wave functions are not bounded and their eigenvalues differ from those in (5.9).

Remark. Actually, the existence of the nonzero c-number representations is not very rare for quantum algebras. For example, for the Cartesian version of the $sl_q(2)$ algebra

$$q^{-1}J_1J_2 - qJ_2J_1 = J_3, \quad q^{-1}J_2J_3 - qJ_3J_2 = J_1,$$

$$q^{-1}J_3J_1 - qJ_1J_3 = J_2,$$

describing dynamical symmetries of some discrete reflectionless potentials [59], one can set $J_k = 1/(q^{-1} - q)$ and the algebra is satisfied. Such exotic representations are often skipped in discussions of applications of quantum algebras.

As we have discussed already, coherent states of the first type of the algebra (5.1) are defined as eigenfunctions of the annihilation operator A. These states are built from the lowest weight discrete series representation (5.7) (see, e.g., [40]):

$$A|\alpha,q\rangle=\alpha|\alpha,q\rangle,\qquad |\alpha,q\rangle=C(\alpha)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\omega^n[n]!}}|n\rangle, \tag{5.10}$$

where

$$|C(\alpha)|^{-2}\equiv e_{q^2}(z)=\sum_{n=0}^{\infty}\frac{z^n}{(q^2;q^2)_n}=\frac{1}{(z;q^2)_\infty},$$

$$z=\frac{|\alpha|^2(1-q^2)}{\omega},$$

is a q analog of the exponential function, or ${}_1\varphi_0(0;q^2,z)$ basic hypergeometric function. The general q series of such type is defined as follows [50]:

$${}_r\varphi_s\begin{pmatrix}a_1,\ a_2,\ \dots,\ a_r\\ b_1,\ b_2,\ \dots,\ b_s\end{pmatrix};q,z\Bigg)=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n}{(q;q)_n(b_1;q)_n\dots(b_s;q)_n}[(-1)^nq^{n(n-1)/2}]^{1+s-r}z^n,$$

where r and s are arbitrary positive integers, and $a_1,\dots,a_r,b_1,\dots,b_s$ are free parameters. The states (5.10) are normalizable only for $|\alpha|^2<\nu$. Since the Hilbert space of our model is larger than the discrete spectrum Fock space spanned by $|n\rangle$, the states (5.10) *are not complete*. There remain two parts corresponding to zero and positive energy eigenvalues. Although the latter are not normalizable, one cannot discard them. Expansion of the eigenfunctions of A over the fixed energy states should contain in general an integral over the continuous spectrum. This situation differs drastically from the $q=1$ algebra case, where the Fock space was complete.

Note that the definition (5.10) works for $q^2>1$ as well since A remains to be the lowering operator for positive energy states. Then, the states $|\alpha\rangle$ are normalizable for arbitrary values of α,

$$|C(\alpha)|^{-2}\equiv E_{q^{-2}}(z)=\sum_{n=0}^{\infty}\frac{z^nq^{-n(n-1)}}{(q^{-2};q^{-2})_n}=(-z;q^{-2})_\infty,$$

$$z=\frac{|\alpha|^2(1-q^{-2})}{\omega}, \tag{5.11}$$

where $E_{q^{-2}}(z)$ is another analog of the exponential function, or ${}_0\varphi_0(q^{-2},-z)$ basic hypergeometric function. In this case coherent states are complete because the Hamiltonian has only a discrete spectrum, as in [40].

For $q^2<1$, coherent states formed by the positive energy states should be defined as eigenstates of the operator $A^\dagger$,

$$A^\dagger|\alpha,q\rangle=\alpha|\alpha,q\rangle, \tag{5.12}$$

since for $E>0$, not A but $A^\dagger$ lowers the energy [13]. Suppose that for some λ the states of the series (5.8) are normalizable (this is not so in our case, but in principle it is possible). Then we find

$$|\alpha,q\rangle_\lambda=C(\alpha)\left[|0\rangle_\lambda+\sum_{n=1}^{\infty}\left(\frac{\nu^{n/2}}{\alpha^n}(-\lambda q^2/\nu;q^2)_n^{1/2}|n\rangle_\lambda+\frac{\alpha^nq^{n(n-1)/2}}{\lambda^{n/2}(-\nu/\lambda;q^2)_n^{1/2}}|-n\rangle_\lambda\right)\right]. \tag{5.13}$$

The normalization constant $C(\alpha)$ is related to the bilateral basic hypergeometric series ${}_0\psi_1$,

$$|C(\alpha)|^{-2}={}_0\psi_1(b;q^2,z),\qquad b=-\nu/\lambda,\quad z=-|\alpha|^2/\lambda.$$

The general bilateral q-hypergeometric series is defined as follows [50]:

$${}_r\psi_s\begin{pmatrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{pmatrix};q,z\Bigg)=\sum_{n=-\infty}^{\infty}\frac{(a_1;q)_n\dots(a_r;q)_n}{(b_1;q)_n\dots(b_s;q)_n}\times\left[(-1)^nq^{n(n-1)/2}\right]^{s-r}z^n.$$

Using the Ramanujan sum for the ${}_1\psi_1$ series [50] one can express $C(\alpha)$ in terms of the infinite products

$$|C(\alpha)|^{-2}=\frac{(q^2;q^2)_\infty(-\lambda q^2/|\alpha|^2;q^2)_\infty(-|\alpha|^2/\lambda;q^2)_\infty}{(-\nu/\lambda;q^2)_\infty(\nu/|\alpha|^2;q^2)_\infty}.$$

The states (5.13) are normalizable when $|\alpha|^2>\nu$, i.e., α should lie outside of the region where the states (5.10) were defined.

When positive eigenvalues of the abstract Hamiltonian H, $H|\lambda\rangle=\lambda|\lambda\rangle$, form a continuous spectrum, the states $|\alpha,q\rangle$ are defined by the integral over $|\lambda\rangle$,

$$|\alpha,q\rangle=\int_0^\infty d\lambda\,\frac{\lambda^d\chi(\lambda,\alpha)\,|\lambda\rangle}{\sqrt{(-\lambda q^2/\nu;q^2)_\infty}},\qquad d=\frac{\ln\sqrt{\nu}/q^2\alpha}{\ln q^2}, \tag{5.14}$$

where $\chi(\lambda)$ is an arbitrary function periodic on the logarithmic scale, $\chi(q^2\lambda)=\chi(\lambda)$. In principle the continuous spectrum may have an infinity of gaps, each type of gap appearing in the form of a geometric series. Here we assume that it fills the whole interval $0<\lambda<\infty$. From the requirement for A and $A^\dagger$ to be Hermitian conjugates of each other, with the action (which is defined only up to an arbitrary phase factor depending on λ)

$$A|\lambda\rangle=\sqrt{\nu+\lambda}\,|\lambda q^{-2}\rangle,\qquad A^\dagger|\lambda\rangle=\sqrt{\nu+\lambda q^2}\,|\lambda q^2\rangle,$$

one finds the form of the scalar product,

$$\langle\lambda|\sigma\rangle=\lambda\delta(\lambda-\sigma). \tag{5.15}$$

In our case there are two sets of states (5.14) because the continuous spectrum is doubly degenerate. Expanding the function $\chi(\lambda)$ into a Fourier series, we obtain an infinite number of linearly independent coherent states of the form

$$|\alpha, q\rangle_s = C(\alpha) \int_0^\infty \frac{d\lambda\, \lambda^{d_s}\, |\lambda\rangle}{(-\lambda q^2/\nu; q^2)_\infty^{1/2}},$$

$$d_s = d + \frac{2\pi i s}{\ln q^2}, \quad s = 0, \pm 1, \ldots, \tag{5.16}$$

which are normalizable for $|\alpha|^2 > \nu$. The normalization constant is calculated exactly, being determined by the special case of the Ramanujan q-beta integral [50],

$$|C(\alpha)|^{-2} = \int_0^\infty \frac{d\lambda\, \lambda^\tau}{(-\lambda q^2/\nu; q^2)_\infty} = -\frac{\pi}{\sin \pi\tau} \frac{(q^{-2\tau}; q^2)_\infty}{(q^2; q^2)_\infty} \left(\frac{\nu}{q^2}\right)^{\tau+1},$$

where $\tau = 2\text{Re}\, d+1$. The fact that there exists an infinity of normalizable states (5.16) leads to an interesting effect in the model of a q oscillator interacting with a classical current [60].

Consider the structure of coherent states of the first type (5.10) in the realization (5.2), (5.3). Denote $\psi_\alpha(x) = \langle x|\alpha, q\rangle$. These wave functions are defined by the equation

$$[d/dx + f(x)]\psi_\alpha(x) = \alpha\sqrt{|q|}\psi_\alpha(qx), \tag{5.17}$$

where $f(x)$ is a solution of (5.3). Unfortunately, a full description of the properties of $\psi_\alpha(x)$ is not accessible at present. Nevertheless, several crucial points can be brought into view.

Suppose $0 < q < 1$; then in the $x \to \infty$ limit we get the equation

$$\psi'_\alpha(x) = \alpha\sqrt{q}\psi_\alpha(qx) - \sqrt{\nu}\psi_\alpha(x), \tag{5.18}$$

which is exactly the pantograph equation encountered in the free-particle model. Using the corresponding analysis, we conclude that the $x \to \infty$ asymptotics of coherent states is

$$\psi_\alpha(x) = \frac{h(x,\alpha)}{x^\kappa}, \qquad \kappa = \frac{\ln \alpha\sqrt{q/\nu}}{\ln q}, \tag{5.19}$$

where $h(x)$ is some function satisfying $h(qx) = h(x)$. A similar asymptotics of $\psi_\alpha(x)$ holds for $x \to -\infty$ because in this limit $f(x) \to -\sqrt{\nu}$. Therefore coherent states are normalizable near infinity provided Re $\kappa > 1/2$, or $|\alpha|^2 < \nu$, consistent with the previous considerations. Due to the arbitrariness of $h(x,\alpha)$, asymptotics (5.19) corresponds to a countable set of solutions of (5.18) for fixed α, whereas it is possible to construct only one coherent state by superposing discrete spectrum states. Probably it is the requirement for analyticity at $x = 0$ that determines the latter solution uniquely. Then the nonanalytical solutions should not be normalizable near this point. For Re $\kappa < 0$ the functions $\psi_\alpha(x)$ are not bounded, and so not physical. The significance of the bounded at infinity but not normalizable solutions of (5.17), appearing in the range $0 \leq$ Re $\kappa \leq 1/2$, is not yet clear. This problem is related to the analysis of the completeness of coherent states, which is beyond the scope of the present work.

Consider the zero modes of the Hamiltonian for the solution of (5.3) determined by the condition $f(0) = 0$. Denoting $\psi_{\text{cl}}(x) = \langle x|\text{cl}\rangle$, we have

$$\psi''_{\text{cl}}(x) = u(x)\psi_{\text{cl}}(x). \tag{5.20}$$

The first several terms of the Taylor expansion of the functions $f(x)$ and $u(x)$ are easily found:

$$f(x) = \frac{\omega x}{1+q^2} + \frac{(q^2-1)\omega^2 x^3}{3(1+q^2)(1+q^4)} + O(x^5),$$

$$u(x) = \frac{2\omega}{q^4-1} + \frac{2\omega^2 x^2}{(1+q^2)^2(1+q^4)} + O(x^4).$$

Looking for $\psi_{\text{cl}}(x)$ in the form of a Taylor series, we find the first three terms of the odd and even wave functions:

$$\psi_{\text{cl}}^{even}(x) = 1 + \frac{\omega x^2}{q^4-1} + \frac{(1-q^2+q^4)\omega^2 x^4}{3(1+q^4)(1-q^4)^2} + O(x^6),$$

$$\psi_{\text{cl}}^{odd}(x) = x + \frac{\omega x^3}{3(q^4-1)} + \frac{(2-3q^2+2q^4)\omega^2 x^5}{15(1+q^4)(1-q^4)^2} + O(x^7).$$

Since the operator A maps the space of zero modes onto itself, there should be at least one eigenfunction of A. Taking the linear combination of the above solutions we find that there are two such functions,

$$A\psi_{\text{cl}}^\pm = \pm i\sqrt{\nu}q^{-1}\psi_{\text{cl}}^\pm,$$
$$\psi_{\text{cl}}^\pm(x) = \psi_{\text{cl}}^{even}(x) \pm i\sqrt{\nu}q^{-1/2}\psi_{\text{cl}}^{odd}(x). \tag{5.21}$$

Evidently $\psi_{\text{cl}}^\pm$ are eigenfunctions of the operator $A^\dagger$ as well, $A^\dagger\psi_{\text{cl}}^\pm = \mp iq\sqrt{\nu}\psi_{\text{cl}}^\pm$. The eigenvalues of A and $A^\dagger$ are not complex conjugate to each other. Hence, the "condensate" representations of the q-oscillator algebra do not belong to the discrete spectrum; actually, $\psi_{\text{cl}}^\pm$ are not even bounded. This can be verified by the direct estimation of the asymptotics of these functions using the fact that they satisfy Eq. (5.17) with $\alpha = \pm i\sqrt{\nu}/q$. We, however, choose a slightly different approach. Consider the eigenvalue equation for $A^\dagger$,

$$[-d/dx + f(x)]\sqrt{q}\psi_{\text{cl}}^\pm(qx) = \mp iq\sqrt{\nu}\psi_{\text{cl}}^\pm(x). \tag{5.22}$$

Eliminating the derivative part from (5.17) and (5.22), we get

$$[f(x) + q^{-1}f(q^{-1}x)]\psi_{\text{cl}}^\pm(x) = \pm i\sqrt{\nu}q^{-1/2}\left[\psi_{\text{cl}}^\pm(qx) - \psi_{\text{cl}}^\pm(q^{-1}x)\right], \tag{5.23}$$

i.e., zero energy wave functions satisfy the second-order purely finite-difference equation. Since for $x \to \infty$, $f(x) \to \sqrt{\nu}$, the leading asymptotics of $\psi_{\text{cl}}^\pm$ are found as solutions of the *free* finite-difference Schrödinger equation, which are

$$\psi_{\rm cl}^{\pm}(x) = \chi^{\pm}(x)e^{ik_{\pm}\ln x}, \qquad \chi^{\pm}(qx) = \chi^{\pm}(x). \tag{5.24}$$

Substituting this ansatz into (5.18) with $\alpha = \pm i\sqrt{\nu}/q$, one finds $2ik_{\pm} = 1 \mp i\pi/\ln q$. The derived asymptotics of $\psi_{\rm cl}^{\pm}(x)$ look like Bloch wave functions for a particle in a periodic potential with the coordinate variable being $\ln x$. Note that the quasimomenta $k_{\pm}$ are complex, which forces the wave function to increase at infinity as $\sqrt{x}$, unlike the $u(x) = 0$ case when a symmetric constant wave function was bounded. A more detailed consideration of such nonstandard implementation of the Bloch theorem is given in Sec. VIII.

Let us discuss briefly the $-1 < q < 0$ case. Due to the parity symmetry one has the relation: $A^2\big|_{q<0} = -A^2\big|_{q>0}$. It means that the $q > 0$ coherent states $|i\alpha, q\rangle$ and $|-i\alpha, q\rangle$ provide independent eigenstates of $A^2\big|_{q<0}$. The parity transformation changes only the sign of α. Therefore, the eigenstates of A for $q < 0$ are given by the parity coherent states (2.6) where on the rhs one has q-coherent states for $q > 0$. Moreover, the precise realization (2.2) and superposition (2.6) appear in the $q \to -1$ limit of the q-oscillator system under consideration [13]. Indeed, the general solution of Eq. (5.3) for $q = -1$ is $f(x) = \omega x/2$, and the variation of q from 1 to -1 performs a transition from the canonical coherent states to the parity coherent states (2.6). Note, however, that if one takes the solution of (5.3) satisfying asymmetric initial condition $f(0) \neq 0$, then the $q \to -1$ limit simply does not exist and for $-1 < q < 0$ the parity operator does not help in the analysis.

Coherent states of the second type are defined as eigenstates of the $A^{\dagger}$ operator (5.12):

$$[-d/dx + f(x)]\sqrt{q}\psi_{\alpha}(qx) = \alpha\psi_{\alpha}(x), \tag{5.25}$$

where we use the same notations as in (5.17). In the $x \to \infty$ limit one gets again the pantograph equation, but with the scaling parameter q^{-1}:

$$\psi_{\alpha}'(x) = -\alpha q^{-3/2}\psi_{\alpha}(q^{-1}x) + q^{-1}\sqrt{\nu}\psi(x). \tag{5.26}$$

From the preceding section we know that this equation has solutions with asymptotics,

$$\psi_{\alpha}(x) \to \frac{h(x,\alpha)}{x^{\kappa}}, h(qx) = h(x), \quad \kappa = \frac{\ln\sqrt{\nu q}/\alpha}{\ln q}.$$

These coherent states are normalizable provided Re $\kappa > 1/2$, or $|\alpha|^2 > \nu$. Their expansion over the continuous spectrum states was presented in the abstract form (5.16). For Re $\kappa < 0$, or $|\alpha|^2 < q\nu$, the functions $\psi_{\alpha}(x)$ are not bounded. For $0 \leq$ Re $\kappa \leq 1/2$ we get again wave functions that are bounded at $|x| \to \infty$ but unnormalizable. It looks like they do not have an expansion over the eigenstates of the Hamiltonian and the reason for this needs clarification. In the considered models there are no normalizable coherent states for the circle $|\alpha|^2 = \nu$. In principle it is possible that zero modes of a Hamiltonian H are normalizable, in which case it is natural to count them among the coherent states (otherwise there will be no completeness).

VI. GENERAL CLASS OF SELF-SIMILAR POTENTIALS AND THEIR COHERENT STATES

It is well known [61] that the one-dimensional Schrödinger equation

$$H\psi(x) = -\psi''(x) + u(x)\psi(x) = \lambda\psi(x), \tag{6.1}$$

has an important nonquantum mechanical application in the theory of nonlinear evolution equations. In particular, the Korteweg–de Vries (KdV) equation can be solved with the help of the inverse scattering method for two classes of initial conditions $u(x, t = 0)$. The first one consists of the potentials $u(x)$ satisfying the restriction $\int_{-\infty}^{\infty}(1 + |x|)|u(x)|dx < \infty$, which guarantees that the number of bound states is finite. Reflectionless potentials with N discrete eigenvalues are the simplest examples from this family. Since they generate N soliton solutions of the KdV equation they are called the soliton potentials. The second class is related to nonsingular periodic (or quasiperiodic) functions, $u(x + l) = u(x)$, characterized by the presence of N gaps of finite width in the spectrum of (6.1). These finite-gap (hyperelliptic) potentials [62] are reduced to the solitonic ones in the limit $l \to \infty$. They can be thought of as superpositions of an infinite number of solitons ("periodic solitons"), but there is no scattering problem for such objects, i.e., the solitary character of the ingredient waves is lost. There is a third relatively simple class of self-similar solutions of the KdV equation related to the Painlevé transcendents requiring a different treatment.

Recently, reflectionless potentials with an infinite number of discrete levels have been systematically considered in [63, 5, 6, 64, 65, 13, 58]. For $x \to \infty$ such potentials decrease slowly and the standard inverse scattering method does not produce constructive results. These potentials deserve to be named the infinite soliton ones, since they do not reflect and may be approximated with some accuracy by the N-soliton potentials ($N < \infty$). Moreover, this class absorbs the finite-gap potentials which emerge for special limiting values of parameters. The related problem of the approximation of confining and band spectrum potentials with the help of reflectionless potentials was discussed earlier [66, 67].

Let us describe briefly the factorization method [8, 9] that allows us to find the particular subclass of infinite-soliton potentials characterized by the q-deformed symmetry algebras. This method was invented in quantum mechanics by Schrödinger; it is deeply related to the Darboux (Bäcklund, dressing, etc.) transformations for linear differential equations. Within this approach, one takes a set of Hamiltonians,

$$L_j = -d^2/dx^2 + u_j(x), \qquad j = 0, \pm 1, \pm 2, \ldots, \tag{6.2}$$

and represents them as products of the first-order differential operators,

$$A_j^+ = -\frac{d}{dx} + f_j(x), \qquad A_j^- = \frac{d}{dx} + f_j(x), \tag{6.3}$$

up to some constants λ_j:

$$L_j = A_j^+ A_j^- + \lambda_j, \tag{6.4}$$

i.e., $u_j(x) = f_j^2(x) - f_j'(x) + \lambda_j$. Then one imposes the following intertwining relations:

$$L_j A_j^+ = A_j^+ L_{j+1}, \qquad A_j^- L_j = L_{j+1} A_j^-, \tag{6.5}$$

which constrain the difference in spectral properties of L_j and L_{j+1} and are equivalent to the equations

$$A_{j+1}^+ A_{j+1}^- + \lambda_{j+1} = A_j^- A_j^+ + \lambda_j. \tag{6.6}$$

The same results are obtained if one starts from the linear equations

$$L_j \psi_j = \lambda \psi_j, \qquad \psi_{j+1} = A_j^- \psi_j. \tag{6.7}$$

The compatibility condition of (6.7) is given by (6.5). Resolving the latter relations, one finds λ_j as the integration constants such that the factorization (6.4) takes place. These are the basic ingredients of the factorization method which allows us to construct new solvable Schrödinger equations from a given one.

Substitution of (6.3) into (6.6) yields the chain of differential equations:

$$f_j'(x) + f_{j+1}'(x) + f_j^2(x) - f_{j+1}^2(x) = \mu_j,$$

$$\mu_j \equiv \lambda_{j+1} - \lambda_j, \tag{6.8}$$

which is called the dressing chain [5, 68]. Any spectral problem with a known nontrivial discrete spectrum generates some solution of (6.8) such that the ordered discrete eigenvalues are given by the constants λ_j. The factorization method works with the inverse problem — to find such solutions of (6.6) or (6.8) for which the spectrum of the associated Schrödinger operator will be determined automatically. Note that in some cases the spectrum can be found even if λ_j do not belong to it, e.g., such a situation can take place for hyperelliptic potentials [62, 69–71].

The variable j was playing, above, the role of a label. It could therefore be removed in favor of other notations, e.g., $f_j \equiv f$, $f_{j+1} \equiv \tilde{f}$, etc. However, it is convenient to think of j as a discrete set of points on a continuous manifold. Then one can consider j (or its function) as a continuous variable and look for solutions of (6.8) in series form $f_j(x) = \sum g_k(x) j^k$. Infeld and Hull [8] have considered such an ansatz and have found that the series contains a finite number of terms iff $f_j(x) = \alpha(x)j + \beta(x) + \gamma(x)/j$, where α, β, γ are some elementary functions. This does not mean that the infinite-series solutions are meaningless; it indicates rather that truncation of the series is related to some simple symmetries [9]. It is difficult to work with formal power series; often even their convergence is not known. Analysis of solutions characterized by separation of variables or by some special dependence on them is essentially simplified. In the Lie theory of differential equations solutions of such type are called the similarity or self-similarity solutions. Unfortunately for the differential-difference equation there is no complete theory of such solutions; a subclass of them can be found using the methods developed for purely differential equations [72]. An additional complication associated with Eq. (6.8) consists of the fact that there are two unknowns, $f_j(x)$ and μ_j, i.e., the system is highly underdetermined.

According to definition, (self-)similarity solutions are the solutions invariant under symmetry transformations of a given equation. The potentials we are interested in appear as fixed points of the combination of an affine transformation of the coordinate, $x \to qx + l$, and a shift along the discrete lattice, $j \to j + N$. Indeed, the change in numeration of solutions by an integer maps solutions of (6.8) to the solutions $f_j(x) \to f_{j+N}(x)$, $\mu_j \to \mu_{j+N}$. The same is true for the affine group, $f_j(x) \to q f_j(qx + l)$, $\mu_j \to q^2 \mu_j$. We may look for the class of solutions invariant under the combination of these transformations:

$$f_{j+N}(x) = q f_j(qx + l), \qquad \mu_{j+N} = q^2 \mu_j. \tag{6.9}$$

These relations define a class of potentials that we are going to analyze below. The simplest reduction of such type, $f_j(x) = q^j f(q^j x)$, $\lambda_j = q^{2j}$, corresponding to $N = 1$, $l = 0$ in (6.9), has been found by Shabat [5]. Actually, the general class of closures of the dressing chain (6.9) has been introduced in a way different from the above [7], namely, from q deformation of the parasupersymmetric quantum mechanics based upon some polynomial algebras [73].

At the operator level, the relations (6.9) lead to the Schrödinger operators with nontrivial q-deformed symmetry algebras. Let us consider the products

$$M_j^+ = A_j^+ A_{j+1}^+ \cdots A_{j+N-1}^+,$$

$$M_j^- = A_{j+N-1}^- \cdots A_{j+1}^- A_j^-, \tag{6.10}$$

which generate the intertwinings

$$L_j M_j^+ = M_j^+ L_{j+N}, \qquad M_j^- L_j = L_{j+N} M_j^-. \tag{6.11}$$

The structure relations complimentary to (6.11) appear as

$$M_j^+ M_j^- = \prod_{k=0}^{N-1} (L_j - \lambda_{j+k}),$$

$$M_j^- M_j^+ = \prod_{k=0}^{N-1} (L_{j+N} - \lambda_{j+k}). \tag{6.12}$$

Equations (6.11) and (6.12) can be rewritten as the higher-order polynomial supersymmetry algebra [74]:

$$\{Q^+, Q^-\} = \prod_{k=0}^{N-1} (K - \lambda_k), \qquad (Q^\pm)^2 = [K, Q^\pm] = 0, \tag{6.13}$$

where

$$Q^+ = \begin{pmatrix} 0 & M_0^+ \\ 0 & 0 \end{pmatrix}, \qquad Q^- = \begin{pmatrix} 0 & 0 \\ M_0^- & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} L_0 & 0 \\ 0 & L_N \end{pmatrix};$$

the index $j = 0$ was fixed for simplicity of notation. The presence of nonlinearity brings essentially new features with respect to the standard supersymmetric quantum mechanics case corresponding to $N = 1$.

The identities (6.11) show that if the operators L_j and L_{j+N} are related to each other through some simple similarity transformation, e.g.,

$$L_{j+N} = q^2 T L_j T^{-1} + \omega, \tag{6.14}$$

where T is some invertable operator, then the combinations

$$B_j^+ \equiv M_j^+ T, \qquad B_j^- \equiv T^{-1} M_j^-,$$

map eigenfunctions of L_j onto themselves, i.e., they describe symmetries of L_j. The form of T is restricted by the requirement for the L_j's to be of the Schrödinger form. The closure (6.9) corresponds to the choice of T as the affine transformation operator, $Tf(x) = \sqrt{|q|} f(qx + l)$. Fixing the indices and removing their irrelevant part ($L \equiv L_0$, $B^\pm \equiv B_0^\pm$), we get the symmetry algebra [6, 7]:

$$LB^+ - q^2 B^+ L = \omega B^+, \qquad B^- L - q^2 L B^- = \omega B^-, \tag{6.15}$$

$$B^+ B^- = \prod_{k=0}^{N-1} (L - \lambda_k), \qquad B^- B^+ = \prod_{k=0}^{N-1} (q^2 L + \omega - \lambda_k). \tag{6.16}$$

After the shift of the zero energy point,

$$H \equiv L - \frac{\omega}{1 - q^2}, \qquad E_k \equiv \lambda_k - \frac{\omega}{1 - q^2},$$

the algebra takes a simpler form:

$$\begin{aligned} HB^\pm &= q^{\pm 2} B^\pm H, \\ B^+ B^- &= \prod_{k=0}^{N-1} (H - E_k), \\ B^- B^+ &= \prod_{k=0}^{N-1} (q^2 H - E_k). \end{aligned} \tag{6.17}$$

This algebra was already met in the discussion of coherent states of the free nonrelativistic particle.

Let us write out explicitly the system of nonlinear differential equations with the deviating argument that one needs to solve in order to find the explicit form of the self-similar potentials:

$$\frac{d}{dx}[f_0(x) + f_1(x)] + f_0^2(x) - f_1^2(x) = \mu_0,$$

$$\frac{d}{dx}[f_1(x) + f_2(x)] + f_1^2(x) - f_2^2(x) = \mu_1,$$

$$\vdots$$

$$\frac{d}{dx}[f_{N-1}(x) + q f_0(qx + l)] + f_{N-1}^2(x) - q^2 f_0^2(qx + l) = \mu_{N-1}. \tag{6.18}$$

Note that the limit $q \to 1$ is not trivial. For the nonzero parameter l we get a realization of the algebra (6.15), (6.16) at $q = 1$, which generalizes the one described in [68]. Below we assume that $l = 0$ (for $q \neq 1$ this corresponds to the fixed point reference frame, but for $q = 1$ this gives a nontrivial simplification). The zero potential model considered in Sec. IV corresponds to a particular solution of this system, $f_i(x) = \beta_j =$ const, which thus determines the simplest self-similar potential.

Consider some examples. The $N = 1$ case describes a q deformation of the harmonic oscillator potential, since for $q = 1$ one has $f(x) = \mu x/2$ and $u(x) \propto x^2$. The $N = 2$, $q = 1$ system coincides with the conformal quantum mechanical model [75],

$$f_{0,1}(x) = \pm\frac{\gamma}{x} + \beta x, \quad \gamma = \frac{\mu_0 - \mu_1}{2(\mu_0 + \mu_1)}, \qquad \beta = \frac{\mu_0 + \mu_1}{4}, \tag{6.19}$$

$$u_{0,1}(x) = \frac{\gamma(\gamma \pm 1)}{x^2} + \beta^2 x^2 - \beta(1 \mp 2\gamma) + \lambda_{0,1}.$$

This is the singular oscillator potential whose physical spectrum consists formally of two arithmetic series, but only one of them is physical due to the boundary condition at $x = 0$. Coherent states of this model defined as eigenstates of the symmetry operator B^- were constructed by Barut and Girardello [76] (for some amendments, see [77]). The $N = 2$, $q = -1$ system obeying the same symmetry algebra will be considered in the next section.

It is natural to call the physics of the $N = 2$, $q^2 \neq 1$ models as the q-deformed conformal quantum mechanics because their symmetry algebra is $\mathrm{su}_q(1,1)$. A general solution of the basic equations (6.18) is not available in a closed form already for $N = 1$. Let us find $f_{0,1}(x)$ as a formal series near $x = 0$. Consider first singular solutions with the pole-type singularity. Such solutions appear to be odd functions with a simple pole at zero:

$$f_0(x) = \frac{a}{x} + \sum_{i=1}^{\infty} b_i x^{2i-1}, \qquad f_1(x) = -\frac{a}{x} + \sum_{i=1}^{\infty} c_i x^{2i-1},$$

$$b_i + c_i = \sum_{j=1}^{i-1} \frac{c_j c_{i-j} - b_j b_{i-j}}{2i - 1 + 2a},$$

$$q^{2i} b_i + c_i = \sum_{j=1}^{i-1} \frac{q^{2i} b_j b_{i-j} - c_j c_{i-j}}{2i - 1 - 2a}, \tag{6.20}$$

where $i = 2, 3, \ldots$ and a is an arbitrary parameter; the coefficients b_1, c_1 have the form

$$b_1 = \frac{1}{1-q^2}\left(\frac{\mu_0}{1+2a} - \frac{\mu_1}{1-2a}\right),$$
$$c_1 = \frac{1}{1-q^2}\left(\frac{\mu_1}{1-2a} - \frac{q^2\mu_0}{1+2a}\right).$$

In general, the series diverge at $q \to 1$, but for the special choice of a one gets the truncated solution (6.19). In the limit $q \to 0$, the function $qf_0(qx)$ does not vanish, $qf_0(qx) \to a/x$, and then the potentials $u_{0,1}$ are obtained by simple dressing of the potential $a(a+1)/x^2$. It is natural to expect that for $0 < q < 1$ there exist $\mu_{0,1}$ such that the series converge for arbitrarily large x. The condition $a(a+1) \geq 3/4$ guarantees that normalizable wave functions and their first derivatives vanish at zero [75]. The spectrum of such a system would arise from only one geometric series (for the same reason that there is only one arithmetic series for the singular oscillator).

Due to the presence of the singularity one can restrict the space to be a half-line, $0 < x < \infty$, and interpret x as a radial coordinate appearing from the separation of variables in a three-dimensional Schrödinger equation. The constant a acquires then an interpretation of orbital momentum of a particle when it takes integer values $a = l$. Then in addition to the standard term $l(l+1)/x^2$ there is a very complicated dependence of the potential on the quantum number l, indicating that in this picture the "q-deformed Laplacian" is a nonlocal operator.

For the solutions that are nonsingular at zero one has $f_{0,1} = \sum_{i=0}^{\infty} b_i^{(0,1)} x^i$, where $b_0^{(0,1)}$ are two arbitrary constants. Again, in general the series diverge at $q \to 1$. A particular choice of initial conditions gives the solution which in this limit corresponds to (6.19) with a coordinate shift. In the $0 < q < 1$ region, the solution that is nonsingular for all real x defines an infinite soliton potential whose spectrum is composed from two independent lowest weight irreducible representations of $\mathrm{su}_q(1,1)$. In the limit $q \to 0$ it shrinks to the smooth two-soliton potential. It is this solution that reduces to the q-oscillator one (6.12)–(6.16) (with q replaced by $q^{1/2}$) after the restrictions $f_1(x) = q^{1/2} f_0(q^{1/2}x)$, $\mu_1 = q\mu_0$. Probably without such a restriction this solution does not have a $q \to 1$ limit. In any case, at $q \to 1$ the spectral series become equidistant, which means that the potentials start to be unbounded at infinity. Obviously a more complete and rigorous analysis of the structure of the solutions of the $N = 2$ equations is necessary.

When $q = 1$, already the $N = 3$ case corresponds to transcendental potentials, namely, the $f_j(x)$ depend now on solutions of the Painlevé-IV (PIV) equation [68]:

$$f_0(x) = \frac{1}{2}\omega x + f(x), \qquad \omega = \mu_0 + \mu_1 + \mu_2, \tag{6.21}$$

$$f_{1,2}(x) = -\frac{1}{2}f(x) \mp \frac{1}{2f(x)}(f'(x) + \mu_1),$$
$$f'' = \frac{f'^2}{2f} + \frac{3}{2}f^3 + 2\omega x f^2 + \left(\tfrac{1}{2}\omega^2 x^2 + \mu_2 - \mu_0\right) f - \frac{\mu_1^2}{2f}. \tag{6.22}$$

Thus, a simple generalization of the harmonic oscillator, characterized by the split of its linear spectrum onto N independent terms, is connected with highly nontrivial ordinary differential equations whose solutions are transcendental over the solutions of linear differential equations with coefficients given by the rational and algebraic functions. For the $N = 3$, $q \neq 1$ system it is not possible to reduce the order of the equation (i.e., no first integral is known). Because in this case there are solutions of (6.18) reducing in the limit $q \to 1$ to the PIV functions, it is natural to refer to this system as the q-deformed PIV equation.

The notion of q periodicity (6.18) and the corresponding algebraic relations are central in this section. Suppose that superpotentials $f_j(x)$ are not singular and that the operator B^- is well defined $[B^{\pm} = (B^{\mp})^{\dagger}]$ and has N normalizable zero modes (the necessary condition for this is $E_k < E_{k+1}$),

$$B^-|l\rangle = 0, \qquad \langle l|m\rangle = \delta_{lm}, \quad l, m = 0, 1, \ldots, N-1. \tag{6.23}$$

Then $|l\rangle$ represent the first N bound states of self-similar potentials and the whole discrete spectrum consists of N independent geometric series:

$$H|n\rangle = E_n|n\rangle, \qquad E_{n+N} = q^2 E_n, \quad E_n < E_{n+1} < 0,$$
$$E_{kN+l} = E_l q^{2k}, \quad l = 0, 1, \ldots, N-1, \quad k = 0, 1, \ldots, \infty, \tag{6.24}$$

$$|Nk + l\rangle = C_k (B^+)^k |l\rangle, \qquad \langle Nk + l|Nk' + l'\rangle = \delta_{kk'}\delta_{ll'}.$$

The normalization constants C_k are determined up to an arbitrary energy-dependent phase factor

$$|C_k|^{-2} = \nu \prod_{s=0}^{N-1} \prod_{m=1}^{k} (1 - q^{2m} E_l / E_s),$$

$$\nu = (-1)^N E_0 E_1 \cdots E_{N-1} > 0,$$

and similar freedom exists in the action of raising and lowering operators,

$$B^+|kN + l\rangle = \prod_{s=0}^{N-1} \sqrt{E_l q^{2(k+1)} - E_s}\,|(k+1)N + l\rangle,$$
$$B^-|kN + l\rangle = \prod_{s=0}^{N-1} \sqrt{E_l q^{2k} - E_s}\,|(k-1)N + l\rangle. \tag{6.25}$$

In the "crystal" limit $q \to 0$ there remain only first N levels and we get the general N-soliton potentials.

There may be intermediate situations, when only some of the zero modes of B^- satisfy necessary boundary conditions (the singular oscillator model is a known example). Then some of the spectral terms disappear; we do not consider such possibilities in detail here. Note that for $q > 1$ the above formulas probably only have formal meaning in the differential Schrödinger equation case due to a too rapid growth of the spectrum. Analysis of the $N = 1$ potential showed that, for $q > 1$, it has singularities, and thus the symmetry operators are not well defined [6]. The finite-difference Schrödinger operators, or the Jacobi matrices, have a richer spectral structure [37, 78, 59]. In particular, the spectrum may consist of a discrete set of points only and be compact; it may grow exponentially fast, and the geometric series may accumulate near zero both from below and from above, etc.

Let us summarize what one can do with the help of Darboux transformations. Starting from the zero potential $u(x) = 0$, and performing N dressing transformations, one can get N-soliton potentials. Starting from the latter and performing an infinite number of dressing transformations in the particular self-similar manner, we get a subclass of infinite soliton systems. Taking the $q \to 1$ limit we derive the Painlevé-type potentials, some of which degenerate into the finite-gap potentials in the limit $\omega \to 0$. All these complicated systems are thus equivalent in some sense to the free Schrödinger equation.

Consider coherent states of the algebra (6.17) defined as eigenstates of the operator B^-,

$$B^-|\alpha\rangle = \alpha^N|\alpha\rangle, \tag{6.26}$$

where for convenience we denote the eigenvalue as the Nth power of α. Since B^- has N linearly independent zero modes, there are N independent states:

$$|\alpha_l\rangle = C_l(\alpha)\sum_{k=0}^{\infty}\frac{\alpha^{kN+l}|kN+l\rangle}{\prod_{s=0}^{N-1}\prod_{m=1}^{k}(E_lq^{2m}-E_s)^{1/2}}, \tag{6.27}$$

$$|C_l(\alpha)|^{-2} = \sum_{k=0}^{\infty}\frac{|\alpha|^{2(kN+l)}}{\prod_{s=0}^{N-1}\prod_{m=1}^{k}(E_lq^{2m}-E_s)} = |\alpha|^{2l}{}_N\varphi_{N-1}\left(\begin{matrix}0,\ldots,0\\ b_1^l,\ldots,b_{N-1}^l\end{matrix};q^2,z\right), \quad z = |\alpha|^{2N}/\nu,$$

$$b_1^l = E_lq^2/E_0,\ \ldots,\ b_{l+1}^l = E_lq^2/E_{l+1},\ \ldots,\ b_{N-1}^l = E_lq^2/E_{N-1}.$$

These coherent states are normalizable for $|\alpha|^{2N} < \nu$. In order to find coherent states corresponding to the limit $q \to 1$ it is necessary to replace the q products by the Pochhammer symbols. The spectrum of the resulting system is found from the formula $E_{n+N} = E_n + \omega$. The hierarchy of potentials with such a linear spectrum was investigated in [68]. Representation theory of the $q = 1$ algebras was considered in [79, 80]. Their possible physical applications were discussed also in [81–83]. In the limit $E_l \to E_0 + l\omega/N$, $l = 0, 1, \ldots, N-1$, one finds the harmonic oscillator such that the states $|\alpha_l\rangle$ constructed above coincide with the generalizations of even and odd coherent states (2.14).

The positive energy states are described by the following representation:

$$H\,|\lambda\rangle = \lambda\,|\lambda\rangle, \quad \lambda > 0,$$

$$B^+\,|\lambda\rangle = \prod_{s=0}^{N-1}\sqrt{\lambda q^2 - E_s}|\lambda q^2\rangle,$$

$$B^-\,|\lambda\rangle = \prod_{s=0}^{N-1}\sqrt{\lambda - E_s}|\lambda q^{-2}\rangle. \tag{6.28}$$

When the states generated by $B^\pm$ from $|\lambda\rangle$ for some fixed λ are the only eigenstates of the Hamiltonian, generalization of the coherent states (5.13) has the following form:

$$|\alpha\rangle_\lambda = C(\alpha)\left(\sum_{n=0}^{\infty}\frac{\alpha^{nN}q^{Nn(n-1)/2}}{\lambda^{N/2}\prod_{s=0}^{N-1}\sqrt{(E_s/\lambda;q^2)_n}}|\lambda q^{-2n}\rangle + \sum_{n=1}^{\infty}\frac{\nu^{n/2}}{\alpha^{nN}}\prod_{s=0}^{N-1}\sqrt{(\lambda q^2/E_s;q^2)_n}|\lambda q^{2n}\rangle\right), \tag{6.29}$$

where

$$|C(\alpha)|^{-2} = \sum_{n=0}^{\infty}\frac{(-1)^{Nn}q^{Nn(n-1)}z^n}{\prod_{s=0}^{N-1}(E_s/\lambda;q^2)_n} + \sum_{n=1}^{\infty}\frac{\nu^n}{|\alpha|^{2nN}}\prod_{s=0}^{N-1}(\lambda q^2/E_s;q^2)_n = {}_0\psi_N(b_1,\ldots,b_{N-1};q^2,z),$$

$$z = (-1)^N|\alpha|^{2N}/\lambda^N, \quad b_l = E_l/\lambda.$$

These states are normalizable for $|\alpha|^{2N} > \nu$.

Analogously, one can find the countable set of coherent states in the case when positive energies form the continuous spectrum:

$$|\alpha\rangle_s = C(\alpha)\int_0^\infty \frac{d\lambda\ \lambda^{d_s}\ |\lambda\rangle}{\prod_{l=0}^{N-1}\sqrt{(\lambda q^2/E_l;q^2)_\infty}},$$

$$d_s = \frac{\ln\sqrt{\nu}/\alpha^N q^2 + 2\pi i s}{\ln q^2}, \quad s = 0, \pm 1, \dots , \qquad (6.30)$$

where the normalization constant $C(\alpha)$ is given in (4.20) with $\beta_l^2 = -E_l$. Zero modes of the Hamiltonian H may be eigenstates of the operators A and $A^\dagger$ in the same way as in the q-Weyl algebra case; there is no need to describe them again.

We conclude that coherent states of the self-similar potentials are related to the basic hypergeometric series ${}_N\varphi_{N-1}(z)$, ${}_0\psi_N(z)$ or to some integrals over the latter functions. However, we do not know the explicit form of these coherent states because the Hamiltonian eigenstates involve new complicated transcendental functions whose complete analytical structure is not accessible at present. Let us mention in passing that the finite-difference analogs of some of the above self-similar potentials associated with the discrete Schrödinger equation have been described in [78]. The discrete dressing chain in this case happens to define a discrete time Toda lattice [59]. A q-deformed supersymmetric interpretation of the models considered in this and the previous two sections was given in [84, 7, 74].

VII. TWO PARTICULAR EXAMPLES

Consider in more detail the $N = 2$, $q = -1$, and a special subcase of the $N = 3$, $q = 1$ closures of the dressing chain. In the first case, the system of equations determining superpotentials has the form

$$\frac{d}{dx}[f_0(x) + f_1(x)] + f_0^2(x) - f_1^2(x) = \mu_0, \qquad (7.1)$$
$$\frac{d}{dx}[f_1(x) - f_0(-x)] + f_1^2(x) - f_0^2(-x) = \mu_1,$$

where we assume that $\mu_0 + \mu_1 \neq 0$. From the algebraic point of view this case is related to the su(1,1) algebra, which serves as the formal spectrum generating algebra. If one assumes that $f_0(x)$ [or $f_1(x)$] is antisymmetric, $f_0(-x) = -f_0(x)$, then our system is equivalent to the $N = 2$, $q = 1$ case, or to the singular harmonic oscillator model (6.19). The coordinate region of this problem is restricted to the half-axis which forbids the parity operator. Indeed, the wave functions are not single-valued near $x = 0$ for noninteger values of the parameter γ and the action of the parity operator is not well defined. As a result, the eigenvalue equation for the lowering operator $B^- = P(d^2/dx^2 + \cdots)$ is also not well defined (for integer γ, half of the wave functions are singular, the coherent states have fixed parity, and the action of B^- and B^-P may differ only by sign).

It is easy to see that $f_0(x)$ or $f_1(x)$ cannot be symmetric; therefore the nontrivial solutions of (7.1) do not have fixed parity. In spite of its simplicity, the system (7.1) is hard to solve. Let us represent $f_j(x)$ as sums of symmetric and antisymmetric parts:

$$f_{s,a}(x) = \frac{1}{2}[f_0(x) \pm f_0(-x)],$$
$$g_{s,a}(x) = \frac{1}{2}[f_1(x) \pm f_1(-x))] \qquad (7.2)$$

and substitute this splitting into (7.1). Then it is not difficult to find from these equations and their $x \to -x$ partners two "integrals,"

$$f_a(x) + g_a(x) = \sigma x, \qquad \sigma = \frac{1}{2}(\mu_0 + \mu_1), \qquad (7.3)$$

and

$$f_s^2(x) + f_a^2(x) = g_s^2(x) + g_a^2(x) + \tau, \qquad \tau = \frac{1}{2}(\mu_0 - \mu_1). \qquad (7.4)$$

As a result, it is possible to express f_a and g_a via their symmetric partners,

$$f_a(x) = \frac{1}{2\sigma x}\left(g_s^2 - f_s^2 + \sigma^2 x^2 + \tau\right),$$
$$g_a(x) = \frac{1}{2\sigma x}\left(f_s^2 - g_s^2 + \sigma^2 x^2 - \tau\right). \qquad (7.5)$$

Eventually the original system of functional-differential equations is reduced to the form

$$f_s'(x) = 2g_s(x)g_a(x), \qquad g_s'(x) = -2f_s(x)f_a(x). \qquad (7.6)$$

This system of ordinary differential equations is related to a Painlevé-V equation, which follows from the fact that we are considering a subcase of the $N = 4$, $q = 1$ closure analyzed in [85].

For the special choice of initial conditions $f_a(0) = g_a(0) = 0$ one has $f_s^2(0) = g_s^2(0) + \tau$. Then the singularity at $x = 0$ cancels and one obtains a formal nonsingular solution of the equations (7.6). It is possible that this solution defines a smooth potential growing indefinitely when $|x| \to \infty$. Under this asumption, consequences of the presence of the symmetry algebra su(1,1) in this model are different from those for the singular oscillator potential. Namely, the spectrum consists now of two arithmetic series, each being determined by irreducible representations of the su(1,1) algebra. The coherent states $|\alpha_l\rangle$, $l = 0, 1$, or even and odd coherent states, are both physical but they are not eigenstates of the parity operator because the potential is not symmetric.

Another example that we would like to discuss in more detail concerns the $N = 3$, $q = 1$ closure considered in [68] (note that in this paper all Hamiltonians L, admitting Nth-order differential operator B as a symmetry operator satisfying the relation $[L, B] = B$, were characterized). The PIV function appearing in this context has an infinite set of rational solutions emerging for the specific choices of the parameters μ_0, μ_1, μ_2 [86]. Consider the following rational solution:

$$f_0(x) = \frac{x}{2} + \frac{2x}{x^2+1},$$
$$f_1(x) = \frac{x}{2} - \frac{1}{x}, \qquad (7.7)$$
$$f_2(x) = \frac{x}{2} + \frac{1}{x} - \frac{2x}{x^2+1},$$

which corresponds to the constants $\mu_0 = 4,\ \mu_1 = 1,\ \mu_2 = -2$. The symmetry algebra has thus the form

$$[L, B^{\pm}] = \pm 3B^{\pm},$$
$$B^+B^- = L(L-4)(L-5), \tag{7.8}$$
$$B^-B^+ = (L+3)(L-1)(L-2),$$

where the Hamiltonian L is

$$L = -\frac{d^2}{dx^2} + \frac{4}{x^2+1} - \frac{8}{(x^2+1)^2} + \frac{1}{4}x^2 + \frac{3}{2}. \tag{7.9}$$

There are three normalizable eigenstates of the lowering operator B^- corresponding to the lowest energy state, and second and third excited levels. Acting by B^+ upon these three "vacua" one finds the whole spectrum. It consists of three arithmetic series forming the sequence $0, 3, 4, 5, 6, \ldots$; i.e., there is a hole between 0 and 3 after which the spectrum becomes equidistant (the first excited state with the energy 3 is generated by B^+: $|1\rangle \propto B^+|0\rangle$).

The explicit form of the coherent states, $B^-|\alpha\rangle = \alpha^3|\alpha\rangle$, is determined from the equation

$$\chi'''(x) - \frac{6x}{x^2+1}\chi''(x) + \frac{12x^2}{(x^2+1)^2}\chi'(x) = \alpha^3\chi(x), \tag{7.10}$$

where

$$\chi(x) \equiv (x^2+1)e^{x^2/4}\langle x|\alpha\rangle.$$

The author does not know any simple solution of this equation. Each of the three linearly independent solutions defines a particular coherent state. Evidently, they have some common origin with the cubic root-of-unity superpositions of the coherent states (2.14).

The structure of the spectrum hints that there are some additional symmetries in this model, and, indeed, there is a different set of raising and lowering operators satisfying the different algebra found earlier in [82]. It corresponds to a different solution of the dressing chain for the same Hamiltonian (7.9):

$$f_0(x) = -f_2(x) = \frac{x}{2} + \frac{2x}{x^2+1}, \qquad f_1(x) = \frac{x}{2}, \tag{7.11}$$

for which $\mu_0 = 3,\ \mu_1 = -2,\ \mu_2 = 0$. Now the symmetry algebra has the form

$$[L, B^{\pm}] = \pm B^{\pm},$$
$$B^+B^- = L(L-3)(L-1), \tag{7.12}$$
$$B^-B^+ = (L+1)(L-2)L.$$

The operator B^- has only two normalizable zero modes corresponding to the first two levels of the Hamiltonian. The principal important feature of this realization is that the lowest energy state is also a zero mode of the raising operator. As a result, it is the first excited level which serves as the lowest weight vector of the discrete series representation of the above algebra from which the raising operator B^+ creates an infinite tower of states. In this picture the spectrum consists of only one arithmetic series with step 1, and the isolated zero energy state forms a trivial one-dimensional irreducible representation of the algebra. One may thus conclude that the spectrum generating algebra of a given potential, and hence the coherent states, may be nonunique.

The explicit form of the coherent states of the second symmetry algebra are determined by the equation

$$\chi'''(x) - x\left(1 + \frac{6}{x^2+1}\right)\chi''(x) + \left(2 + \frac{8}{x^2+1} - \frac{12}{(x^2+1)^2}\right)\chi'(x) = \alpha^3\chi(x), \tag{7.13}$$

where $\chi(x)$ is defined in the same way as in the first case. Simple solutions of this equation also are not known to the author [87]. There should be only one physically acceptable solution because now the coherent states have a unique expansion over the Hamiltonian eigenstates for $\alpha \neq 0$ (i.e., there is no connection with the root-of-unity superpositions). An analogous situation has been described recently for a different particular Hamiltonian with an equidistant spectrum [88]. It would be interesting to find physical characteristics of the coherent states (7.10), (7.13) and similar ones.

VIII. SCHRÖDINGER OPERATORS WITH DISCRETE SCALING SYMMETRY

Spectral theory of the one-dimensional Schrödinger equation (6.1) for the bounded periodic potentials,

$$u(x+l) = u(x), \tag{8.1}$$

is well known [89]. The Bloch, or Floquet, theorem states that there is at least one wave function $\psi(x)$ which can be represented in the form

$$\psi(x) = e^{ikx}\phi(x), \tag{8.2}$$

where $\phi(x)$ is a periodic function, $\phi(x+l) = \phi(x)$. When $\psi(x)$ is a bounded function it defines a physically acceptable state. This requirement restricts the quasimomentum k to be real.

Consider a class of Schrödinger operators which satisfy the relations

$$T^\dagger H = q^2 H T^\dagger, \qquad HT = q^2 TH, \qquad T^\dagger T = TT^\dagger = 1, \tag{8.3}$$

where T is the unitary fixed affine transformation operator (4.9). The algebra (8.3) corresponds to the degenerate case of (6.17) when the polynomials of H on the rhs are replaced by a constant. It is equivalent to the following constraint upon the potential:

$$q^2 u(qx+l) = u(x), \tag{8.4}$$

which for $q = 1$ is the periodicity condition (8.1). Note, however, that the limit $q \to 1$ is not defined uniquely. After the shift $u(x) \to u(x) + \sigma/(1-q^2)$ the condition (8.4) is reduced in this limit to $u(x+l) = u(x) + \sigma$, which shows that $u(x)$ is the sum of a periodic and Airy potentials. Only the additional requirement $\sigma \to 0$ leads to the standard Bloch-Floquet theory.

For the shifted potential $v(x) = u(x + x^*)$, where x^* is the fixed point, $qx^* + l = x^*$, the constraint (8.4) is converted into $q^2 v(qx) = v(x)$. This means that

$$v(x) = h(x)/x^2, \qquad h(qx) = h(x), \tag{8.5}$$

where $h(x)$ is an arbitrary function obeying the indicated property of periodicity on the logarithmic scale. In the following we restrict ourselves to the case when $h(x)$ is bounded for $0 < x < \infty$, which is a natural coordinate region of the problem. In this case the potential $v(x)$ is singular at $x = 0$ and vanishes in the limit $x \to \infty$. The value of $h(x)$ is not defined at zero unless it is a constant. This means that one cannot soften the singularity by requiring $h(x)$ to vanish at $x = 0$. The formula (8.5) defines a class of potentials whose spectral theory we propose to call the q-Floquet theory.

It is instructive to rewrite the resulting Schrödinger equation using the change of variables $x = \exp y$ and renormalization of the wave function $\psi(x) = \sqrt{x}\chi(y)$:

$$-\frac{d^2\chi(y)}{dy^2} + \left(h(y) + \frac{1}{4}\right)\chi(y) = \lambda e^{2y}\chi(y), \tag{8.6}$$

where $h(y + \ln q) = h(y)$ is now a periodic function. This equation has a form similar to the original one but the normalization condition for bound state wave functions $\chi(y)$,

$$\int_0^\infty |\psi(x)|^2 dx = \int_{-\infty}^\infty e^{2y}|\chi(y)|^2 dy = 1, \tag{8.7}$$

contains a nontrivial factor e^{2y} in the measure, which diverges when $y \to \infty$.

The presence of the nontrivial symmetry constrains the structure of the wave functions. The operator T acts upon the wave functions as follows:

$$\begin{aligned} T\psi_i(x,\lambda) &= \sqrt{q}\psi_i(qx + l, \lambda) \\ &= \sum_{j=1}^{2} c_{ij}\psi_j(x, q^2\lambda), \end{aligned} \tag{8.8}$$

where $\psi_i(x,\lambda)$ are two linearly independent solutions of the Schrödinger equation. On the one hand, this q-difference equation shows that if $\psi(x,\lambda)$ is normalizable for some fixed $\lambda = \lambda_0$, then T does not change its normalizability. On the other hand, the resulting wave function has the eigenvalue $\lambda = q^2\lambda_0$, which thus also belongs to the physical spectrum. Applying the T^{-1} operator one gets a wave function with the eigenvalue $\lambda = q^{-2}\lambda_0$. This simple argument shows that if our Hamiltonian has physical states with negative energy, then it exhibits the "fall onto the center" phenomenon [90, 91]: the energy is not bounded from below.

The class (8.5) unifies periodic potentials with the conformal one corresponding to the special choice $h(x) =$ const [75]. Indeed, the Hamiltonian $H = -d^2/dx^2 + h/x^2$ enters the following conformal symmetry algebra:

$$[H, D] = iH, \qquad [K, D] = -iK, \qquad [H, K] = 2iD, \tag{8.9}$$

where $D = i\{x, d/dx\}/4$ and $K = x^2/4$ are the Hermitian scaling and special conformal transformation generators. In this case dilatation by arbitrary parameter maps solutions of the Schrödinger equation onto the solutions, which means formally that the spectrum of H is purely continuous (such a situation holds only for $h > -1/4$). If $h(x) \neq$ const the discrete spectrum consists of a number of geometric series infinite in both directions, with $\lambda = 0$ and $\lambda = -\infty$ as the accumulation points. Note that (8.5), with some unknown $h(x)$, was suggested as the asymptotic form of the self-similar potential with one geometric series of bound states truncated from below [58]. In the more general case one has N such series [7]; it is a matter of conjecture that in all cases asymptotics of the potentials have a similar form.

From the symmetry point of view we have the following situation. The simple conformal potential has a large group of continuous symmetries which fixes the dynamics completely. We can add an interaction term which does not remove all symmetries, but preserves a discrete part of them. This situation is similar to the case where a free particle is put into a periodic potential — instead of the group of continuous translations one has only a discrete subgroup of it. Note that the generator of the discrete subgroup is now considered as an element of the symmetry algebra characterizing the spectrum — this is one of the ways of building quantum algebras out of Lie algebras.

Due to the nonanalyticity of the potential at $x = 0$, the accurate spectral analysis requires some rigorous mathematical tools. The main problem is to find restrictions upon $h(x)$ for which the Hamiltonian is self-adjoint, or for which it has self-adjoint extensions. Here we give only a qualitative picture. From (8.5) we see that there are three essentially different regions of the spectrum. Positive energy states form continuous spectrum because the potential goes to zero sufficiently fast. Degeneracy of these states depends on the boundary condition imposed upon the wave functions at $x = 0$. Only for boundary conditions that are invariant under the taken scaling transformation does the operator T represent a physical symmetry.

The second region of energy corresponds to zero modes of the Hamiltonian. It is especially interesting because upon these zero modes T commutes formally with the Hamiltonian. For $\lambda = 0$, (8.6) looks like the standard Schrödinger equation for a periodic potential with the spectral parameter equal to $-1/4$:

$$-\chi''(y) + h(y)\chi(y) = -\frac{1}{4}\chi(y),$$

$$h(y + \ln q) = h(y).$$

If both independent solutions of this equations are bounded, i.e., if the eigenvalue $-1/4$ belongs to the permitted band of $h(y)$, then the original wave function $\psi(x)$ does not describe physical states, being unbounded at infinity. For boundedness of $\psi(x)$ at $x = \infty$ the quasimomentum k of the Bloch eigenfunction $\chi(y) = e^{iky}\phi(y)$, $\phi(y + \ln q) = \phi(y)$, has to have an imaginary part Im $k \geq 1/2$ [for Im $k = 1/2$ one has $\psi(qx) = q^{i\mathrm{Re}\, k}\psi(x)$].

Normalizability of $\psi(x)$ near $x = 0$ imposes the restriction Im $k < 1$. For Im $k = 1$, wave functions $\psi(x)$ are generalized eigenfunctions of T with eigenvalues of modulus 1, as it should be for a unitary operator. The zero energy states of the Hamiltonian H may thus belong to the continuous spectrum when the quasimomentum k of the Bloch function $\chi(y)$ is complex and $1/2 \le \text{Im } k \le 1$.

For any $\lambda < 0$ there is a wave function $\psi(x)$ which is normalizable at infinity, so that it is the boundary condition at $x = 0$ which determines the quantization of the spectral parameter. If $h(x) > 0$ then there are no bound states at all. When squares of the absolute values of both solutions of the Schrödinger equation are integrable near zero, the Hamiltonian is not self-adjoint. Its self-adjoint extensions are fixed by the requirement of a particular dependence of wave functions near zero.

Let us consider the case $h(x) = \text{const}$, which exhibits already basic features of the q-Floquet theory. This model was considered by Case [91] who has shown that when $h < -1/4$ one can choose an orthonormal basis of states such that the energies of bound states form an infinite geometric series in both directions. Such a spectrum is stipulated by the presence in the model of a discrete scaling symmetry. More precisely, the operator D in (8.9) is not a physical symmetry operator any more, but the group element q^{2iD} is — it maps wave functions of the discrete spectrum onto each other. Let us consider in more detail how this situation appears using the Wronskian technique of self-adjoint extensions of singular Schrödinger operators [92]. According to this approach, one takes two solutions of the original equation for some fixed eigenvalue satisfying $\phi_1 = \cos\alpha$, $\phi_1' = \sin\alpha$ and $\phi_2 = \sin\alpha$, $\phi_2' = -\cos\alpha$ at some regular point $x = \delta$ (evidently, $\phi_1\phi_2' - \phi_1'\phi_2 = -1$). Then it is necessary to take arbitrary linear combination $\phi(x) = A\phi_1(x) + B\phi_2(x)$, A, B real constants, and look for solutions of the Schrödinger equation whose Wronskian with $\phi(x)$ vanishes for $x \to 0$:

$$W(\phi, \psi) \equiv \phi(x)\psi'(x) - \phi'(x)\psi(x) \to 0, \qquad x \to 0. \tag{8.10}$$

Physically this means that the particle's behavior near the point of singularity is fixed by the choice of auxiliary function $\phi(x)$. When $\phi(x)$ and its derivative take finite values at $x = 0$, the constraint (8.10) becomes equivalent to the well-known condition $\psi'(0) = c\psi(0)$, where c is a real constant.

For $h < -1/4$ derivatives of all wave functions are singular at $x = 0$, and one needs to use a limiting procedure. Let us take as the auxiliary function the general zero energy eigenfunction

$$\phi(x) = a\sqrt{x}\cos(\sigma \ln x + \theta), \qquad \sigma = \sqrt{|h| - 1/4},$$

where a and θ are some real constants. Substituting it into the above relation one finds

$$\frac{d}{d\ln x}\frac{\psi(x)}{\sqrt{x}} + \sigma\tan(\sigma \ln x + \theta)\frac{\psi(x)}{\sqrt{x}} \to 0, \quad x \to 0.$$

The chosen auxiliary function is homogeneous under the scaling of the coordinate by a specific constant $q = e^{\pi/\sigma}$, $\phi(qx) = -\sqrt{q}\phi(x)$. Therefore, from the scaled form of the condition (8.10),

$$\phi(qx)\psi'(qx) - \phi'(qx)\psi(qx) \propto \phi(x)\frac{d\psi(qx)}{dx} - \frac{d\phi(x)}{dx}\psi(qx) \to 0,$$

it is seen that if $\psi(x)$ satisfies (8.10), the same holds for the scaled wave function $\psi(qx)$, i.e., scaling by q is a physical symmetry of the problem. This fact guarantees that discrete eigenvalues appear in the form of a geometric series. Since the behavior of the general solution of the Schrödinger equation for small x is known, the Wronskian vanishes only if near zero $\psi(x)$ has the form of $\phi(x)$ taken with the same angle θ, which is a free parameter of the self-adjoint extension. The spectrum itself is found from the requirement of normalizability of wave functions with such a property. The described technique of fixing the boundary condition at zero seems to be valid in the arbitrary case.

As was remarked in [7], eigenfunctions of the Schrödinger operators with self-similar potentials resemble discrete wavelets [38] — the functions $\psi(x)$ affine transformations of which, $\psi_{j,n}(x) = 2^{j/2}\psi(2^j x + n)$, generate an orthonormal basis of Hilbert space $L^2(\mathbb{R})$, $\langle\psi_{j,n}|\psi_{l,m}\rangle = \delta_{jl}\delta_{nm}$. In the above q-Floquet theory, we have a partial realization of this construction because the scaling of the coordinate of the wave functions by q creates an orthogonal function. The connection with wavelets originates, of course, from the group of affine transformations, because wavelets can be interpreted as coherent states associated with unitary representations of this group [93] according to one of the definitions mentioned in the Introduction. An interesting fact that differential-delay equations similar to those determining coherent states of self-similar potentials may have solutions with finite support, or atomic solutions [39], indicates also a hidden relation with wavelets.

IX. FACTORIZATION METHOD AND "QUANTUM GALOIS THEORY"

Let us discuss the problem of integrability of a given equation. If this equation is algebraic, $P(x) = \sum_{j=0}^{n} a_j x^j = 0$, with coefficients a_j from some fixed number field k, then its solvability in terms of the radicals is determined by the solvability of its Galois group [the group of permutations of roots of $P(x)$] [94]. One can say that the algebraic equation is exactly solvable when the latter situation takes place, but this criterion is sensitive to the choice of k — there exists a universal field of complex numbers for which any algebraic equation is exactly solvable.

Integrability, or exact solvability of differential equations, is a looser notion. The weakest definition requires the existence of a solution of an initial value problem analytical in a bounded domain. The class of equations integrable in this sense is too large — the field of functions determined by formal power series can be thought of as a universal one since solutions of any differential equation with coefficients from this field are given by functions of the same type. Another extreme definition consists in demanding that solutions belong to differential fields of elementary or classical special functions, i.e., the fields built from the simplest functions and their derivatives. The question of integrability acquires precise meaning when the functions entering the equations and those allowed in the solutions are completely specified. In short, the problem is to understand transcendence of solutions over the given field of functions.

In fact there exists a natural analog of the Galois theory for fields with operators [95, 96], which can be considered physically as a "quantum" (first-quantized) Galois theory. The main notions of it are the Picard-Vessiot (PV) extension and the differential Galois group. Let $u(x)$ in $y''(x) = u(x)y(x)$ belong to the differential field $k = \mathbb{C}(x)$ (field of rational functions). Roughly, the PV extension $M = k\langle y_1, y_2\rangle$ is built from rational functions of independent solutions of this equation and their derivatives with coefficients from k. The group of automorphisms of $k\langle y_1, y_2\rangle$ which commute with the derivative and keep k fixed is called the Galois group $\mathrm{Gal}(M/k)$. For a general nth-order linear ordinary differential equation (ODE) it is isomorphic to the group of invertable matrices $\mathrm{GL}(n, \mathbb{C})$. Roughly, the equation is solvable in terms of Liouvillian functions (exponentials, integrals, algebraics of the functions from k) when $\mathrm{Gal}(M/k)$ is solvable (or, more simply, when all elements of it can be made triangular). This theory is not widely known, although it forms the basis of the intensively used computer programs of symbolic integration of differential equations [97].

Let D be an indeterminate over some number field; then permutations of roots of a polynomial of D preserve this polynomial $(D - x_1)\cdots(D - x_n) = 0 \to (D - x_{i_1})\cdots(D - x_{i_n}) = 0$. In the quantum picture, when D is an operator, e.g., d/dx, there are two analogs of these "classical" permutations — the automorphisms figuring in the definition of the Galois group, which preserve the differential equation, and the permutations $f_i(x) \leftrightarrow f_j(x)$ in $[D - f_1(x)]\cdots[D - f_n(x)]y(x) = 0$. The latter do not, in general, preserve the initial equation, but cyclic permutations of such type play a key role in the factorization method [8, 98], which thus should be considered as one of the ingredients of "quantum" Galois theory.

Consider the free Schrödinger equation with the zero spectral parameter,

$$y'' \equiv (D + 1/x)\,(D - 1/x)\,y = 0 \tag{9.1}$$

over the field of complex numbers $\mathbb{C}$. The PV extension coincides with $\mathbb{C}(x)$ and the Galois group consists of nontrivial triangular matrices. Permutation of operator factors in (9.1) gives the equation

$$(D - 1/x)\,(D + 1/x)\,y = y'' - (2/x^2)y = 0, \tag{9.2}$$

which is different from the initial one, e.g., the potential lies now in $\mathbb{C}(x)$. Its Galois group is trivial (consisting of the unit matrix), because the general solution $y = a/x + bx^2$ belongs to $\mathbb{C}(x)$. In general, the Darboux transformation changes the field of coefficients of the ODE and its Galois group. However, an important feature is preserved in the chosen example; namely, solutions of both equations belong to one and the same field. If one factorizes the second equation using its general solution and permutes the operator factors, the solutions of the third equation,

$$\left(D^2 + \frac{6x(2t - x^3)}{(t + x^3)^2}\right) y(x) = 0, \tag{9.3}$$

belong to $\mathbb{C}(x)$ again. This procedure of building rational potentials out of the zero one with the help of Darboux transformations with zero eigenvalue level has been considered in [99–101]. It can be shown that solutions of all equations built in this way lie in the PV extension of (9.1), i.e., the corresponding Galois groups are always trivial (an analogous situation takes place for repeated Darboux transformations with nonzero eigenvalue level). A similar iso-Galois picture prevails, e.g., for the Bessel functions. It would be interesting to investigate from this point of view the finite-gap potentials for which one has a simple factorization of the Hamiltonian.

It is not difficult to describe a differential field which is preserved by any Darboux transformation. For this, consider spectral problems generated by two Hamiltonians neighboring in the factorization chain

$$[D \pm f(x)][D \mp f(x)]y_{1,2}(x) = \lambda y_{1,2}(x). \tag{9.4}$$

Let $y(x, \lambda)$ be a particular solution of the first equation. Then

$$y_1(x, \lambda) = y(x, \lambda)\left(a_1 + b_1 \int^x \frac{dx'}{y^2(x', \lambda)}\right) \tag{9.5}$$

is a general solution. Consequently, $y_2(x, \lambda) \equiv [D - f(x)]y_1(x, \lambda)$ gives a general solution of the second equation for $\lambda \neq 0$. For $\lambda = 0$ one has

$$y_2(x, 0) = \frac{1}{y(x, 0)}\left(a_2 + b_2 \int^x y^2(x', 0)dx'\right). \tag{9.6}$$

This function belongs to the PV extension of the first equation when the derivative $dy(x, \lambda)/d\lambda$ at $\lambda = 0$ does so. Indeed, one can verify by direct differentiation that

$$\int^x y^2(x', \lambda)dx' = y^2(x, \lambda)\frac{d^2 \ln y(x, \lambda)}{dx d\lambda} + \text{const.} \tag{9.7}$$

This relation shows that Darboux transformations preserve the differential field generated by solutions of a spectral equation and their derivatives over λ for arbitrary values of λ. This is a very wide field, but it exists and may be used in a search for such potentials, $u(x)$, and values of λ that $y(x, \lambda)$ are Liouvillian over the field from which $u(x)$ is taken. It seems that the differential Galois theory, in conjunction with the equivalence up to a change of variables, can be used for a constructive definition of the notion of self-similarity. Indeed, for $u(x) = 0$ and fixed λ the PV extension contains only $\exp(\pm n\sqrt{\lambda}x)$, $n \in \mathbb{N}$, and this hints immediately at the form of the spectrum, $\lambda_n \propto \lambda n^2$, for which Darboux transformations preserve this field. Analogously, defining relations of the self-similar potentials resemble the requirement of preservation of the PV extension after N Darboux transformations.

In a sense, the power of the factorization method (or of Darboux transformations) stems from its relation to the Galois theory. This explains also a large number of its applications. For instance, we mention its effectiveness in numerical calculations [102], in approximation theory [103], in search of bispectral equations [104], in the theory of orthogonal polynomials [78, 59], its connection with symplectic structures [105], etc. In particular, Darboux transformations with zero eigenvalue appeared to play an important role in the construction of equations satisfying the Huygens principle [100, 106].

In conclusion, let us summarize the criteria which allow us to call potentials "exactly solvable." First, such potentials should belong to some sufficiently simple differential field k, e.g., to $\mathbb{C}(x)$, but not to the field of formal power series. This requirement assumes that a full analytical structure of potentials is known. Then one can say that the Schrödinger equation is exactly solvable if the solutions satisfying the taken boundary conditions (i.e., when λ belongs to the spectrum) are given by Liouvillian functions over k. This definition is tied to the differential Galois theory, where one has a simple test of such solvability. Note that in this case a change of boundary conditions may change the "solvability" of the equation. A weaker definition of exact solvability refers to the availability of global structure of solutions [107], i.e., it demands knowledge of various asymptotics of solutions, which often may be found even if the solution is not Liouvillian. The latter requirement is natural for spectral problems, since in order to satisfy boundary conditions one should be able to connect solutions at various distances. In this respect, presently only classical special functions provide a completely satisfactory bank of information.

The situation with self-similar potentials is instructive. It is difficult to specify the field of functions to which they belong [in particular, to find analytic properties of $f_j(x)$], but once it is done the normalizable wave functions are given by Liouvillian functions. The discrete spectrum of these potentials corresponds thus to the exactly solvable problem in the differential Galois theory sense. Analysis of the continuous spectrum states, or of the solutions satisfying different boundary conditions, requires information which is not available at present, i.e., the rank of special functions for solutions of Schrödinger equations with self-similar potentials is not yet established.

Coherent states provide a basis of the Hilbert space of states of quantum particles different from that determined by the Hamiltonian eigenstates. It would be interesting to understand their role from the differential Galois theory point of view. For the standard harmonic oscillator case the picture is simple. The second-order differential (Schrödinger) equation is replaced by a first-order one whose solutions provide an overcomplete set of Hilbert space vectors. Certainly this provides a simplification and "minimization" of the problem — the nonphysical eigenfunctions of the Hamiltonian are removed in this procedure. Such minimization of the types of functions relevant to the given physical problem is characteristic for some coherent states from the functional-analytic point of view. This is evident for coherent states defined as orbits of states generated by physical symmetry groups. The ladder operator approach does not obey such a property in general because, starting from $N = 2$, coherent states of the self-similar potentials are determined by differential or differential-delay equations of higher order, which may contain nonphysical solutions as well. Moreover, for $q \neq 1$ solutions of the latter equations contain some functional arbitrariness with respect to the Schrödinger equation.

X. CONCLUSIONS

The conclusions are short. The superpositions of coherent states in the abstract form (2.14), generalizing the even and odd states of [12], are applicable to a very wide variety of systems. The parity coherent states (2.6), (2.17) are less universal — their meaning and the way of derivation are strongly tied to the presence of the parity symmetry. For any parity invariant potential admitting ladder operators $A^\dagger, A$, one can make the canonical transformation $A \to VA,\ A^\dagger \to A^\dagger V$, where V is a unitary operator linear in the parity operator (2.15), which does not change the algebra satisfied by these operators and the Hamiltonian. This is the simplest physical observation of the present work. For the discussion of experimental implementations of superpositions of coherent states we refer to the papers [11, 22, 23, 25, 26, 28] and references therein.

Coherent states of the free particle, constructed in Sec. IV, differ qualitatively from those of the harmonic oscillator. First, they are built from the continuous spectrum states, which does not allow us to form superpositions similar to (2.14). Second, their analytical properties are quite unusual, so that at present it is not even known whether or not they are complete. It would be interesting to investigate the physical characteristics of these states.

In many respects the results of this paper are not complete. This is caused by the complicated structure of self-similar potentials and their limiting cases, namely, by the appearance of Painlevé transcendents and their q analogs. However, the author believes that this is a temporary situation and later many things will be simplified and, more importantly, the appropriate phenomenological applications of the coherent states for self-similar potentials considered in Secs. V and VI will be found. Because of such expectations, the open problems indicated in this paper are interesting primarily from the physical point of view. Among the problems worth further investigation, we mention the detailed analysis of various limiting cases of the type discussed in Sec. VII, the q-Floquet theory, connections with wavelets, etc. On the mathematical side, it is necessary to find general structure of q-special functions behind the self-similar potentials and to understand in a more appropriate way the differential Galois theory origin of the notion of self-similarity.

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