Eigenvalues of anharmonic oscillators and the perturbed Coulomb problem in N-dimensional space

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(Received 28 February 1994; revised manuscript received 17 February 1995)

The eigenvalues of the potentials $V_1(r)=-a/r+br+cr^2$ and $V_2(r)=\mu r^2+\lambda r^4+\eta r^6$ are obtained in N-dimensional space by the infinite Hill determinant method for a wide range of values of the parameters. We discuss the explicit dependence of these two potentials in higher-dimensional space. By using the formalism of supersymmetric quantum mechanics, it is shown that exact solutions of these potentials exist when the parameters satisfy certain constraints.

PACS number(s): 03.65.Ca, 03.65.Ge, 02.90.+p

I. INTRODUCTION

The problem of quantum anharmonic oscillators has been the subject of much discussion for decades, both from an analytical and a numerical point of view, because of its important applications in quantum-field theory [1,2], molecular physics [3], and solid-state and statistical physics [4,5]. Various methods [6–20] have been used successfully for the one-dimensional anharmonic oscillators with various types of anharmonicities. Relatively less attention has been given to the anharmonic oscillators in higher-dimensional space because of the presence of angular-momentum states that make the problem more complicated.

Recently it has been shown that there are many interesting features of the anharmonic oscillators and the perturbed Coulomb problems in higher-dimensional space [21–28]. As a result of studies in the far infrared and microwave regions, there has been considerable interest in the analysis of two- and three-dimensional anharmonic oscillators [21]. The classical limits of the two-dimensional and the three-dimensional hydrogen atom have been studied [24] by using the coherent states. The connection between the three-dimensional hydrogen atom and the four-dimensional or two-dimensional harmonic oscillators has been established by various authors [25,26]. Vasan, Seetharaman, and Sushama [27,28] have considered the problem of quantum anharmonic oscillators on the basis of a radial generalization of the JWKB quantization rule and derived analytical expressions for the energy levels. Their results are poor for the ground state and for the higher anharmonicities. In view of this it is necessary to study the quantum anharmonic oscillators and the perturbed Coulomb problem in general N-dimensional space.

It is well known that the Hill determinant method produces excellent results for both polynomial and nonpolynomial potentials [10,18,29]. To compute the eigenvalues of a one-dimensional polynomial anharmonic oscillator by the Hill determinant method we [18] previously used variational minimization in conjunction with the operator method of Feranchuk and Komarov [30] to arrive at the optimal convergence parameter. However, the determinants did not have the convenient banded structure necessary for a straightforward recursive evaluation. Agrawal and Varma [31] subsequently improved our method by using vector recursion relations for calculating the successive approximants of the Hill determinant. In this paper we apply the Hill determinant method to the sextic anharmonic and perturbed Coulomb problems in N-dimensional space for any angular-momentum state. Due to the simple structure of both the potentials, the Hill determinants now have a convenient banded structure and satisfy the recurrence relations that are used to evaluate determinants of any order. We obtain highly accurate numerical values for eigenvalues of N-dimensional anharmonic oscillators and the perturbed Coulomb problem and compare our results with those given by Vasan, Seetharaman, and Sushama [27]. We have computed the eigenvalues for very large values of the parameters. Most of the approximation methods are valid when the parameters are small. Thus, our results with large values of the parameters may be used to test the efficacy of any approximation method.

The perturbed Coulomb

$$V_1(r)=-a/r+br+cr^2 \tag{1}$$

and the sextic anharmonic oscillator problems

$$V_2(r)=\mu r^2+\lambda r^4+\eta r^6 \tag{2}$$

are related in higher-dimensional space and these connections are checked by employing the Hill determinant method. The analyticity of the Schrödinger energy levels for a class of confining potentials (1) have been studied by Dutta and Mukherjee [32] by using Kato-Rellich perturbation theory for linear operations. The confinement potential (1) is also used for calculation of $q\bar{q}$ bound-state masses [33]. Killingbeck [34] has calculated the energy eigenvalues of the potential (1) by using hypervirial relations. Exact solutions [35] of the potentials V_1 and V_2 are obtained by a number of authors in three-dimensional space when the parameters satisfy certain relations.

Next we study the N-dimensional anharmonic oscillators and the perturbed Coulomb problem within the framework of supersymmetric quantum mechanics [36–38]. The ideas of supersymmetric quantum mechanics have been used for the study of atomic systems [38], the evaluation of the eigenvalues of a bistable potential [39], the improvement of the large-N expansion [40], the analysis of all known shape invariant potentials [37], and the development of a more accurate WKB approximation [41]. Dutt, Khare, and Sukhatme [37] have obtained the analytical solutions for shape invariant potentials. We have shown that the supersymmetric quantum mechanics yields exact solutions for a single state only for the quasi-exactly-solvable potentials of type (1) and (2) in N-dimensional space with some constraints on the coupling constants. We use these analytic results to compare our numerical results obtained by the Hill determinant method and find excellent agreement. We have also shown that for a certain choice of parameters, the first column of the Hill determinant vanishes and produces the supersymmetric quasi-exactly-solvable potentials for the N-dimensional anharmonic oscillators and the perturbed Coulomb problem.

II. SCHRÖDINGER EQUATION IN N-DIMENSIONAL SPACE

The radial wave Schrödinger equation for a spherically symmetric potential $V(r)$ in N-dimensional space

$$-\frac{1}{2}\left[\frac{d^2R}{dr^2}+\frac{N-1}{r}\frac{dR}{dr}\right]+\frac{l(l+N-2)}{2r^2}R=[E-V(r)]R \quad (3)$$

is transformed to

$$-\frac{d^2\chi}{dr^2}+\left[\frac{(M-1)(M-3)}{4r^2}+2V(r)\right]\chi=2E\chi\ , \quad (4)$$

where χ, the reduced radial wave function, is defined by

$$\chi(r)=r^{(N-1)/2}R(r) \quad (5)$$

and

$$M=N+2l\ . \quad (6)$$

It should be noted that N and l enter into expression (4) in the form of the combination $N+2l$. Consequently, the solutions for a particular central potential $V(r)$ are the same as long as $N+2l$ remains unaltered. Thus the s-wave eigensolutions (χ) and eigenvalues (E) in four-dimensional space are identical to the p-wave two-dimensional solutions.

We substitute $r=\alpha\rho^2/2$ and $R=F(\rho)/\rho$ and transform Eq. (3) to another Schrödinger-type equation in ($N'=2N-4$)-dimensional space with angular momentum $L=2l+1$,

$$-\frac{1}{2}\left[\frac{d^2F}{d\rho^2}+\frac{N'-1}{\rho}\frac{dF}{d\rho}\right]+\frac{L(L+N'-2)}{2\rho^2}F=[\hat{E}-\hat{V}(\rho)]F\ , \quad (7)$$

where

$$\hat{E}-\hat{V}(\rho)=E\alpha^2\rho^2-\alpha^2\rho^2V(\alpha\rho^2/2)$$

and α is a parameter to be adjusted suitably. Thus, by this transformation, in general, the N-dimensional radial wave Schrödinger equation with angular momentum l can be transformed to a ($2N-4$)-dimensional problem with angular momentum $2l+1$. If we choose $\alpha^2=1/|E|$, the perturbed Coulomb problem (1) with eigenvalue E can be transformed to a sextic anharmonic oscillator problem

$$\hat{V}(\rho)=\rho^2+\frac{b}{2|E|^{3/2}}\rho^4+\frac{c}{4|E|^2}\rho^6\ , \quad (8)$$

with eigenvalue

$$\hat{E}=2a/|E|^{1/2}\ , \quad (9)$$

and the Coulomb problem in N-dimensional space is reduced to a harmonic oscillator problem in ($2N-4$)-dimensional space.

It should be noted that the eigenvalue $E_{\rm CLH}(a,b,c)$ of the Coulomb plus linear plus harmonic (CLH) potential has the following scaling properties in N-dimensional space for any angular-momentum state:

$$E_{\rm CLH}(a,b,c)=a^2E_{\rm CLH}(1,b/a^3,c/a^4)\ , \quad (10a)$$

$$E_{\rm CLH}(0,b,c)=b^{2/3}E_{\rm CLH}(0,1,c/b^{4/3})\ , \quad (10b)$$

$$E_{\rm CLH}(0,0,c)=c^{1/2}E_{\rm CLH}(0,0,1)\ . \quad (10c)$$

The energy eigenvalue $E_{\rm AHO}(\mu,\lambda,\eta)$ of the anharmonic oscillator (AHO) potential (2) has the scaling properties

$$E_{\rm AHO}(\mu,\lambda,\eta)=\mu^{1/2}E_{\rm AHO}(1,\lambda/\mu^{3/2},\eta/\mu^2)\ , \quad (11a)$$

$$E_{\rm AHO}(0,\lambda,\eta)=\lambda^{1/3}E_{\rm AHO}(0,1,\eta/\lambda^{4/3})\ , \quad (11b)$$

$$E_{\rm AHO}(0,0,\eta)=\eta^{1/4}E_{\rm AHO}(0,0,1)\ . \quad (11c)$$

III. SUPERSYMMETRIC POTENTIALS

Since the equation (4) for the reduced radial wave $\chi(r)$ in the N-dimensional space has the structure of the one-dimensional Schrödinger equation for a spherically symmetric potential $V(r)$, we may compute the superpotential [36,37]

$$V_{\pm}(r)=W^2(r)\pm W'(r)\ , \quad (12)$$

which has a zero-energy solution, and the corresponding eigenfunction is given by

$$\chi(r)\sim\exp\left[\pm\int^r W(r)dr\right]\ . \quad (13)$$

In constructing these potentials one should be careful about the behavior of the wave function $\chi(r)$ near $r=0$ and $r\rightarrow\infty$. It may be mentioned that $\chi(r)$ behaves like $r^{(M-1)/2}$ near $r=0$ and it should be normalizable. For the CLH potential (1) we set

$$W(r)=a_{-1}/r+a_0+a_1r \quad (14)$$

and identify the $+$ sector (i.e., W^2+W') with the effective potential so that

$$V_+(r)=-2a/r+2br+2cr^2+(M-1)(M-3)/4r^2-2E=W^2+W'\ . \quad (15)$$

The potential (1) admits the exact solutions (unnormalized)

$$R(r)=r^l\exp\left[-\frac{b(M-1)r^2}{4a}-\frac{2a}{(M-1)}r\right] \tag{16}$$

with the eigenvalues

$$E_{\rm CLH}=\frac{1}{2}\left[\frac{b(M-1)^2}{2a}+\frac{b(M-1)}{2a}-\frac{4a^2}{(M-1)^2}\right] \tag{17}$$

under the constraints

$$b(M-1)=2a\sqrt{2c}\ . \tag{18}$$

For the AHO potential (2) we set

$$W(r)=a_{-1}/r+a_1r+a_3r^3 \tag{19}$$

and identify the $-$ sector (i.e., W^2-W') with the effective potential so that

$$V_-(r)=2\mu r^2+2\lambda r^4+2\eta r^6+\frac{(M-1)(M-3)}{4r^2}-2E$$
$$=W^2-W'\ . \tag{20}$$

In this case the potential (2) admits the exact solutions

$$R(r)=r^l\exp\left[-\frac{\sqrt{2\eta}}{4}r^4-\frac{\lambda}{2\sqrt{2\eta}}r^2\right] \tag{21}$$

with the eigenvalues

$$E_{\rm AHO}=\frac{\lambda M}{2\sqrt{2\eta}}\ , \tag{22}$$

where the parameters satisfy the supersymmetric constraints

$$2\mu=\frac{\lambda^2}{2\eta}-\sqrt{2\eta}(M+2)\ . \tag{23}$$

IV. HILL DETERMINANT APPROACH

A. CLH potential

The radial wave Schrödinger equation for the CLH potential is reduced to

$$g''(r)+f(r)g'(r)+\phi(r)g(r)=0 \tag{24}$$

by substituting

$$R(r)=r^l\exp\left[-\frac{\alpha}{2}r^2-\beta r\right]g(r)\ , \tag{25}$$

where

$$f(r)=(M-1)/r-2\beta-2\alpha r\ , \tag{26}$$

$$\phi(r)=(2a-M\beta+\beta)/r+2E+\beta^2-M\alpha$$
$$+2(\alpha\beta-b)r+(\alpha^2-2c)r^2\ . \tag{27}$$

Next we try a series solution for $g(r)$

$$g(r)=\sum_{k=0}^{\infty}p_kr^{k+\lambda} \tag{28}$$

and substitute (28) into (24). We obtain the following relations for proper normalization of the wave function:

$$\lambda=0\ ,$$
$$P_kp_k+Q_kp_{k-1}+R_kp_{k-2}+S_kp_{k-3}+T_kp_{k-4}=0,\quad k\geq 1 \tag{29}$$
$$p_{-1}=p_{-2}=p_{-3}=0\ ,$$

with

$$P_k=k(M+k-2)\ ,$$
$$Q_k=2a-\beta(2k+M-3)\ ,$$
$$R_k=2E+\beta^2-\alpha(2k+M-4)\ , \tag{30}$$
$$S_k=2(\alpha\beta-b)\ ,$$
$$T_k=\alpha^2-2c\ .$$

The eigenvalue condition of the Hill determinant for large n is

$$\mathrm{Det}D_n=0 \tag{31}$$

with

$$D_n=\begin{vmatrix} Q_1 & P_1 & 0 & 0 & 0 & \cdots \\ R_2 & Q_2 & P_2 & 0 & 0 & \cdots \\ S_3 & R_3 & Q_3 & P_3 & 0 & \cdots \\ T_4 & S_4 & R_4 & Q_4 & P_4 & \cdots \\ 0 & T_5 & S_5 & R_5 & Q_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}\ . \tag{32}$$

The zeros of D_n as a function of the parameter E give the energy eigenvalues of the problem. We can use the following recurrence relation for D_n for evaluation of determinants of any order:

$$D_n=Q_nD_{n-1}-P_{n-1}R_nD_{n-2}+P_{n-1}P_{n-2}S_nD_{n-3}$$
$$-P_{n-1}P_{n-2}P_{n-3}T_nD_{n-4} \tag{33}$$

with $D_0=1$.

It may be noted that when $T_4=S_3=R_2=Q_1=0$, the Hill determinant vanishes and we get the supersymmetric solutions (17) with the constraint (18). When $b=c=0$, we set $\alpha=0$ so that $S_k=T_k=0$ for all k. By setting $R_k=0$ or $E=-\beta^2/2$ we get the solutions of the Coulomb problem in N-dimensional space with $Q_k=0$ or $\beta=2a/(2k+M-3)$, $k=1,2,3,\ldots$, so that the eigenenergies are given by

$$E_k=-\frac{\beta^2}{2}=-\frac{2a^2}{(2k+N+2l-3)^2},\quad k=1,2,3,\ldots\ . \tag{34}$$

B. AHO potential

For the AHO potential (2) we substitute

$$R(r)=r^l\exp[-\tfrac{1}{4}\alpha r^4-\tfrac{1}{2}\beta r^2]\sum_{k=0}^{\infty}p_kr^{k+\sigma} \tag{35}$$

into the radial wave Schrödinger equation (3) and obtain the relations

$$\sigma=0\ ,$$
$$U_k p_k+V_k p_{k-2}+W_k p_{k-4}+X_k p_{k-6}+Y_k p_{k-8}\ , \quad k=2,4,6,\ldots \tag{36}$$
$$p_{-2}=p_{-4}=p_{-6}=0\ ,$$

with

$$\begin{aligned} U_k&=k(M+k-2)\ ,\\ V_k&=2E-\beta(M+2k-4)\ ,\\ W_k&=\beta^2-2\mu-\alpha(2k+M-6)\ ,\\ X_k&=2(\alpha\beta-\lambda)\ ,\\ Y_k&=\alpha^2-2\eta\ . \end{aligned} \tag{37}$$

The eigenvalue condition of the Hill determinant for large n is now

$$\mathrm{Det}D_n=0\ ,$$

with

$$D_n=\begin{bmatrix} V_2 & U_2 & 0 & 0 & 0 & \cdots \\ W_4 & V_4 & U_4 & 0 & 0 & \cdots \\ X_6 & W_6 & V_6 & U_6 & 0 & \cdots \\ Y_8 & X_8 & W_8 & V_8 & U_8 & \cdots \\ 0 & Y_{10} & X_{10} & W_{10} & V_{10} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}\ . \tag{38}$$

The energy eigenvalues of the problem are obtained from the zeros of $\mathrm{Det}D_n$ as a function of the parameter E. The determinants now satisfy the recurrence relations

$$D_n=V_{2n}D_{n-1}-U_{2n-2}W_{2n}D_{n-2}+U_{2n-2}U_{2n-4}X_{2n}D_{n-3} - U_{2n-2}U_{2n-4}U_{2n-6}Y_{2n}D_{n-4} \tag{39}$$

with $D_0=1$.

It may be noted now that the Hill determinant vanishes when $V_2=W_4=X_6=Y_8=0$ and we obtain the supersymmetric solutions (22) under the constraint (23). When $\lambda=\eta=0$ we set $\alpha=0$ so that $X_k=Y_k=0$ for all k

TABLE I. The first four eigenvalues of the potential $V(r)=-a/r+br+cr^2$ having supersymmetric character for $l=0,1,2$ in three- and four-dimensional space.

a	b	c	l	N	Hill determinant method with determinant of size 200×200	Results of supersymmetric quantum mechanics [Eq. (41)]
4	1	$\frac{1}{32}$	0	3	−7.625 000	−7.625
					−0.606 517	
					1.773 300	
					3.418 695	
8	1	$\frac{1}{32}$	1	3	−7.375 000	−7.375
					−2.048 308	
					0.567 860	
					2.402 064	
12	1	$\frac{1}{32}$	2	3	−7.125 000	−7.125
					−2.795 083	
					−0.215 883	
					1.682 357	
6	1	$\frac{1}{32}$	0	4	−7.500 000	−7.5
					−1.449 294	
					1.105 228	
					2.865 414	
10	1	$\frac{1}{32}$	1	4	−7.250 000	−7.25
					−2.480 227	
					0.134 448	
					2.011 818	
14	1	$\frac{1}{32}$	2	4	−7.000 000	−7.0
					−3.025 529	
					−0.499 358	
					1.404 025	

TABLE II. The first four eigenvalues of the potential $V(r)=r$ for $l=0,1,2$ in three- and four-dimensional space.

	Hill determinant method with determinant of size 150×150	
l	$N=3$	$N=4$
0	1.855 757	2.279 586
	3.244 608	3.566 111
	4.381 671	4.656 734
	5.386 614	5.633 500
1	2.667 830	3.029 958
	3.876 793	4.177 230
	4.926 994	5.192 033
	5.877 881	6.119 223
2	3.371 785	3.697 190
	4.468 303	4.750 893
	5.451 836	5.706 563
	6.357 305	6.592 069

and the Hill determinant reduces to a tridiagonal form. Now we make $W_k=0$ by setting $\beta^2=2\mu$ and we obtain the solutions of the harmonic oscillator problem in N-dimensional space with $V_k=0$ or

$$E_k=\tfrac{1}{2}\sqrt{2\mu}(N+2l+2k-4),\quad k=2,4,6,\ldots . \tag{40}$$

V. RESULTS AND DISCUSSION

We consider the following supersymmetric potentials having exact eigenvalues and eigenfunctions for the CLH interactions in three- and four-dimensional space:

$$\left.\begin{aligned} &V_1^{(1)}(r)=-\frac{4}{r}+r+\frac{1}{32}r^2\,,\\ &N=3,\quad l=0\,,\\ &R_{10}(r)=\exp\left[-\frac{r^2}{8}-4r\right],\\ &E_{10}=-\tfrac{61}{8}\,;\end{aligned}\right. \tag{41a}$$

$$\left.\begin{aligned} &V_1^{(2)}(r)=-\frac{8}{r}+r+\frac{1}{32}r^2\,,\\ &N=3,\quad l=1\,,\\ &R_{21}(r)=r\exp\left[-\frac{r^2}{8}-4r\right],\\ &E_{21}=-\tfrac{59}{8}\,;\end{aligned}\right. \tag{41b}$$

$$\left.\begin{aligned} &V_1^{(3)}(r)=-\frac{12}{r}+r+\frac{1}{32}r^2\,,\\ &N=3,\quad l=2\,,\\ &R_{32}(r)=r^2\exp\left[-\frac{r^2}{8}-4r\right],\\ &E_{32}=-\tfrac{57}{8}\,;\end{aligned}\right. \tag{41c}$$

$$V_1^{(4)}(r)=-\frac{6}{r}+r+\frac{1}{32}r^2\,,$$

TABLE III. The first four eigenvalues of the linear plus Coulomb potential $V(r)=-1/r+br$ for $l=0,1,2$ in three- and four-dimensional space.

Space dimension (N)	l	$b=10^{-4}$	$b=10^{-3}$	$b=10^{-2}$	$b=0.1$	$b=1$	$b=10$	$b=100$	$b=1000$
3	0	−0.499 850	−0.498 501	−0.485 144	−0.360 900	0.577 930	6.143 45	34.904 45	174.867 14
		−0.124 401	−0.119 063	−0.069 671	0.299 259	2.450 169	13.418 40	66.436 91	317.064 44
		−0.054 212	−0.042 629	0.051 428	0.641 155	3.756 911	19.024 80	91.606 54	432.184 54
		−0.028 885	−0.009 692	0.127 259	0.907 033	4.855 676	23.878 09	113.650 38	533.511 73
	1	−0.124 501	−0.120 057	−0.079 193	0.222 076	1.974 219	10.946 47	54.435 97	260.284 58
		−0.054 312	−0.043 617	0.042 232	0.570 525	3.335 505	16.855 80	81.096 53	382.478 03
		−0.028 985	−0.010 673	0.118 323	0.840 710	4.468 125	21.897 04	104.071 09	488.239 30
		−0.016 472	0.009 877	0.1781 47	1.073 065	5.472 600	26.420 83	124.789 53	583.823 12
	2	−0.054 511	−0.045 525	0.026 915	0.478 196	2.863 086	14.578 48	70.356 51	332.275 22
		−0.029 183	−0.012 532	0.104 051	0.756 605	4.039 438	19.830 42	94.321 00	442.652 32
		−0.016 670	0.008 061	0.164 626	0.994 601	5.074 146	24.501 20	115.733 72	541.482 07
		−0.009 110	0.023 553	0.216 557	1.207 858	6.015 910	28.779 78	135.402 04	632.367 99
4	0	−0.221 922	−0.219 235	−0.193 304	0.021 697	1.400 452	8.797 03	45.369 45	219.991 29
		−0.079 102	−0.071 214	−0.002 928	0.452 842	2.930 040	15.224 04	74.008 68	350.574 32
		−0.039 033	−0.024 085	0.088 658	0.749 997	4.133 481	20.510 35	97.970 33	460.620 82
		−0.021 760	0.001 107	0.154 878	0.996 221	5.178 865	25.183 73	119.306 98	558.920 52
	1	−0.079 252	−0.072 685	−0.015 987	0.363 768	2.446 730	12.842 79	62.681 65	297.434 18
		−0.039 182	−0.025 533	0.076 339	0.669 269	3.700 636	18.381 54	87.844 09	413.106 14
		−0.021 909	−0.000 320	0.143 104	0.921 012	4.779 150	23.222 23	109.981 73	515.171 17
		−0.012 383	0.017 230	0.198 405	1.142 792	5.749 901	27.616 08	130.148 28	608.297 88
	2	−0.039 429	−0.027 817	0.059 793	0.557 036	3.242 828	16.199 24	77.598 61	365.308 27
		−0.022 154	−0.002 523	0.127 732	0.836 302	4.359 057	21.218 59	100.570 87	471.256 50
		−0.012 627	0.015 095	0.183 881	1.063 443	5.356 976	25.742 18	121.346 09	567.220 97
		−0.006 413	0.029 107	0.233 187	1.269 568	6.273 484	29.917 18	140.561 56	656.066 63

TABLE IV. The first four eigenvalues of the potential $V(r)=-a/r+br+cr^2$ for large values of the parameters b and c.

a	b	c	l	$N=3$	$N=4$
0	10^6	0	0	18 557.57	22 795.86
				32 446.08	35 661.11
				43 816.71	46 567.34
				53 866.14	56 335.00
			1	26 678.30	30 299.58
				38 767.93	41 772.30
				49 269.94	51 920.33
				58 778.81	61 192.23
			2	33 717.85	36 971.90
				44 683.03	47 508.93
				54 518.36	57 065.63
				63 573.05	65 920.69
0	10^6	10^6	0	18 739.41	23 058.09
				32 998.40	36 318.50
				44 818.59	47 690.14
				55 373.17	57 975.03
1	10^6	10^6	0	18 633.09	22 978.30
				32 923.73	36 257.36
				44 757.63	47 638.57
				55 320.33	57 929.54

$$N=4,\quad l=0\ ,$$
$$R_{10}(r)=\exp\left[-\frac{r^2}{8}-4r\right]\ , \tag{41d}$$
$$E_{10}=-7.5\ ;$$
$$V_1^{(5)}(r)=-\frac{10}{r}+r+\frac{1}{32}r^2\ ,$$
$$N=4,\quad l=1\ ,$$
$$R_{21}(r)=r\exp\left[-\frac{r^2}{8}-4r\right]\ , \tag{41e}$$
$$E_{21}=-7.25\ ;$$
$$V_1^{(6)}(r)=-\frac{14}{r}+r+\frac{1}{32}r^2\ ,$$
$$N=4,\quad l=2\ ,$$
$$R_{32}(r)=r^2\exp\left[-\frac{r^2}{8}-4r\right]\ , \tag{41f}$$
$$E_{32}=-7.0\ .$$

We apply the Hill determinant method to these potentials. Our calculations (see Table I) agree very well with the exact eigenvalues of the supersymmetric potentials. The first four eigenvalues of the linear potential $V(r)=r$ are presented in Table II for $l=0,1,2$ in three- and four-dimensional space. Next we consider the elementary quarkonium potential $V(r)=-1/r+br$ problem and tabulate the first four eigenvalues in Table III for a wide range of values of b. The eigenvalues for different values of a and b for the potential $V(r)=-a/r+br$ may be obtained by using the scaling properties (10). The eigenvalues of the potential $V(r)=-1/r+br$ for large values of b are approximately given by

$$E(1,b)=b^{2/3}E(0,1)\ ,$$

where the eigenvalues $E(0,1)$ are presented in Table II. The error involved in applying this formula is about 6% for $b=1000$ and the error decreases with increasing b. In Table IV we present the first four eigenvalues of the $-a/r+br+cr^2$ CLH potential as obtained by the Hill determinant method for high values of the parameters in three- and four-dimensional space. We consider determinants of size 200×200, which yields very accurate results.

We know from (8) and (9) that the perturbed Coulomb problem with supersymmetric potentials $V_1^{(4),(5),(6)}(r)$ in four-dimensional space having exact solutions can be transformed in the sextic anharmonic oscillators (8) in four-dimensional space with eigenvalues (9). We compute the eigenvalues of these conjugate anharmonic oscillators $\hat{V}_1^{(4),(5),(6)}(r)$ in four-dimensional space by the Hill determinant method using Eqs. (38) and compare our results in Table V with the exact values given by (9). A number of supersymmetric anharmonic oscillators may be constructed from (23) that admit exact solutions. These eigenvalues are checked by the Hill determinant method. In Table VI we present the first two eigenvalues of the anharmonic oscillators $V_2(r)=\frac{1}{2}(r^2+r^6)$ in two, three, and four-dimensional space for $l=0,1$, and compare our results with those available in the literature. The agreement is in general very good.

The supersymmetric quantum mechanics yields exact solutions for a single state only for a potential of type (1) or (2) with some constraints on the coupling constants. These correspond to the vanishing of all the elements of the first column of the Hill determinants (32) or (38). Our method is applicable to any general perturbed Coulomb potential or anharmonic oscillator and produces excellent results for the low-lying states. It gives the exact solu-

TABLE V. The eigenvalues of the conjugate sextic anharmonic oscillators $\hat{V}_1^{(4)}(r)=r^2+[1/2(7.5)^{3/2}]r^4+\frac{1}{7200}r^6$, $\hat{V}_1^{(5)}(r)=r^2+[1/2(7.25)^{3/2}]r^4+\frac{1}{6728}r^6$, and $\hat{V}_1^{(6)}(r)=r^2+[1/2(7)^{3/2}]r^4+\frac{1}{6272}r^6$ are compared with the exact values [Eq. (9)].

Conjugate sextic anharmonic oscillator $\hat{V}_1(r)$	l	Hill determinant method with determinant of size 150×150	Exact value [Eq. (9)]
$\hat{V}_1^{(4)}(r)$	1	4.381 780 461	4.381 780 459
$\hat{V}_1^{(5)}(r)$	3	7.427 813 527	7.427 813 526
$\hat{V}_1^{(6)}(r)$	5	10.583 005 244	10.583 005 240

TABLE VI. The first two eigenvalues of the potential $V(r)=\frac{1}{2}(r^2+r^6)$ as obtained by the Hill determinant method are compared with the JWKB results [27] and the exact values [27].

N	l	Hill determinant method with determinant of size 150×150	JWKB results	Exact value
2	0	1.560 968	1.55	1.560 968
		6.457 470		
	1	3.574 964	3.572	3.574 964
		9.593 364	9.593 37	9.593 359
3	0	2.516 698	2.507	2.516 698
		7.994 722		
	1	4.727 768	4.729	4.727 768
		11.251 982	11.252 1	11.251 968
4	0	3.574 964		
		9.593 364		
	1	5.968 601		
		12.969 261		

tions of the Coulomb and the harmonic oscillators in N-dimensional space. The perturbed Coulomb problem and the anharmonic oscillator in N-dimensional space are related through Eq. (8) and are verified in Table V by the Hill determinant method. A class of conjugate anharmonic oscillators having exact eigenvalues may be constructed from the supersymmetric perturbed Coulomb potential. The scaling properties (10) and (11) of the eigenvalues may be used to find the eigenvalues for different values of the parameters. In Table VI we notice that the eigenvalues of central potential $V(r)$ are identical for $N=2$, $l=1$ and $N=4$, $l=0$ states. This is because $M=N+2l$ remains unaltered for these states. The recurrence relations (33) and (39) may be used to find the zeros of the higher-order determinants.

Finally, it should be mentioned that the expectation values $\langle r^k \rangle$ can also be calculated approximately by this method for potentials (1) and (2) by using first-order perturbation result [18]. The eigenvalues of the potential (1) for large values of the parameters are given in Table IV. These results may be used to test the efficacy of any method proposed to tackle quantum-mechanical problems.

ACKNOWLEDGMENT

R.N.C. thanks the International Centre for Theoretical Physics, Trieste, Italy for their kind hospitality.

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