# Pseudomomentum conservation for one-body and two-body relativistic dynamics in a constant electromagnetic field

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The pseudomomentum is conserved for a single charge in any constant (over space and time) electromagnetic field. It is also conserved for a neutral two-body system, not only in a pure (constant) magnetic field, but also in a pure (constant) electric field, and explicit wave equations can be written. A neutral two-body system in a constant electric field is treated in a way that parallels our previous approach to this system in a constant magnetic field. Relative motion is separated, but the relative time now survives, whereas a spacelike degree of freedom gets eliminated. In other cases, there is, in general, no evidence of a closed form of the wave equations, but conservation of the pseudomomentum can be required as a reasonable condition for implicitly determining the so-called "three-body terms" in the wave equation.

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## I. INTRODUCTION

In the framework of Galilean mechanics, it is well known that the pseudomomentum of a single particle with charge $e$ is conserved in a constant magnetic field. This result was implicitly contained in an early paper by Johnson and Lippmann [1], where the center of the orbit was treated in terms of operators. In fact, the center of the Landau orbit has a simple relationship with the pseudomomentum three-vector $\mathbf{C}=\mathbf{p}+e\,\mathbf{A}$ which is conserved if the vector-potential $\mathbf{A}$ generates a constant magnetic field (constant in space and time). Under very general assumptions, a similar result holds for a system of several charges undergoing mutual interactions in addition to their coupling with a constant (external) magnetic field [2,3].

When the total charge is not zero, the components of the pseudomomentum vector have nonvanishing Poisson brackets (or commutators) among themselves. But these commutators are proportional to the total charge and vanish for globally neutral systems. Thus in the special case of neutral systems (not necessarily two-body systems in the Galilean theory), the three components of $\mathbf{C}$ can be simultaneously diagonalized with the energy, which permits the disentanglement of the relative motion from center-of-mass dynamics. Even in the nonrelativistic theory, this separation is not a trivial issue when external fields are present (but this point is often overlooked because most treatments available in textbooks just neglect recoil effects). So, the conservation of pseudomomentum is a fortunate result, especially when the system under consideration is neutral.

For a single particle, the relativisitic version is straightforward. But many-body systems require more caution. Recently we have investigated a relativistic generalization of these matters for the case of a system made of two scalar particles [4]. Indeed a covariant Hamiltonian theory of interacting particles (treated as a system with a finite number of degrees of freedom) is quite possible [5–7]. We refer to situations where pair creation and annihilation can be neglected, whereas other relativistic effects are significant, as they occur for instance, in atomic physics. Although a general Hamiltonian framework exists for $N$-particle systems, almost all tractable or well-understood relativistic models presented in the literature concern two-body systems. For two-body systems, the relationship between relativistic Hamiltonian dynamics and the standard methods of quantum-field theory [in particular, quantum electrodynamics (QED) and the Bethe-Salpeter equation] has been well established [8,9]. Actually, a lot of technical simplifications are possible only in this case. Therefore we have essentially focused on the two-body problem. We have considered a system of two spinless relativistic particles interacting between themselves, as well as with an external electromagnetic field. Even if we know the interaction terms in closed form when no external field is applied, a preliminary problem is to write down the wave equations explicitly in the presence of an external field. This complication stems from the use of *a pair* of coupled equations [5–7,10] in order to describe a two-particle system. As these equations have to be mutually compatible, it turns out that *relativistic interactions cannot be linearly composed.*

A problem of this kind was first solved by Bijtebier [11] for the case of an external potential that is static in a unique way (the laboratory frame is uniquely defined in space time). But if a constant electromagnetic field is purely magnetic in some frame, this frame cannot be unique. This remark was among the motivations for undertaking a systematic generalization. We undertook exhibiting a general symmetry ensuring the construction of wave equations in closed form for a two-body system submitted to an external field [4]. This work resulted in an ansatz that can be carried out, in particular, when the external field is constant in space and time and purely *magnetic* in some frame.

 

According to this construction, and for *neutral* systems only, the *four-vector* $C^\alpha = \sum(p^\alpha + eA^\alpha)$ is conserved in a magnetic field, and we have proved that its components commute among themselves. These results enable one to separate relative motion and to eliminate the relative time from the wave equations, which seems to promise a covariant relativistic theory of the Zeeman effect, including recoil.

In their formal simplicity the covariant formulas obtained in [4] were strongly suggesting that a similar analysis could also be carried out in the case of a constant *electric* field, which is of interest, for instance, in the hope of a relativistic theory of the Stark effect.

Let us stress that in both the magnetic and electric cases, we are concerned with a theory that is nonlinear in the field strength. In the magnetic case, strong fields are relevant because they exist in the real world: very strong macroscopic magnetic fields are believed to be present on the surface of neutron stars. And the search for a relativistic two-body theory in strong magnetic fields has motivated various investigations [12]. In contrast, it seems that strong macroscopic and permanent electric fields are not present in nature [13].

Nevertheless, quadratic terms in the electric field are significant in perturbative expansions: already in the nonrelativistic theory of the Stark effect, it is necessary to retain these terms because the effect is of second order. And finally in both cases, the conservation of the pseudomomentum of neutral systems is an exact result that takes into account all nonlinear terms.

In the magnetic case, a detailed nonrelativistic theory is well known [2,3,14] and the physical meaning of the pseudomomentum *three-vector* is clearly understood. For the moment, it seems that the similar theory corresponding to the purely electric case has not received as much attention (see, however, a digression in Ref. [3]). This is not surprising: for a single particle the nonrelativistic analog of $C$ is simply $m\dot{\mathbf{x}} - e\mathbf{E}t$, and its conservation trivially yields the initial value of velocity; for a two-body system, the nonrelativistic pseudomomentum reduces to $m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2 - (e_1+e_2)\mathbf{E}t$ and gives information on the initial value of the center-of-mass velocity. In particular it trivially reduces to the linear momentum $m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2$ in the neutral case (still, the relevance of a charged system should have motivated more developments).

A covariant relativistic treatment is desirable, anyway: First, for the sake of a unified approach. Second, because the nonrelativistic approximation is hardly reasonable in the charged case, as it allows for uniform acceleration of the center of mass and therefore leads to its infinite asymptotic velocity.

Not only shall we consider the electric and magnetic cases in parallel ways, but we shall also intend, as much as possible, to deal with an arbitrary combination of both fields (Sec. III). Section IV is devoted to the case of arbitrary charges in a constant electric field, and Sec. V deals especially with a (globally) neutral system in such a field. The signature is $+---$, and Greek labels $=0, 1, 2, 3$. They will be omitted whenever possible, and the contraction of indices is denoted by a dot: for instance, $\xi\cdot F$ stands for $\xi_\nu F^{\nu\alpha}$, and similarly, $\xi\cdot F\cdot\eta = \xi_\mu F^{\mu\nu}\eta_\nu$.

## II. ONE-PARTICLE DYNAMICS IN THE EXTERNAL ELECTROMAGNETIC POTENTIAL

In this section we collect useful results about the motion of a single scalar particle in an external field. In the quantum theory the wave equation is a Klein-Gordon equation with a squared-mass operator of the form

$$2K = p^2 + 2G \ . \tag{2.1}$$

The *scalar* interaction term $G$ will currently be referred to as the *potential* (not to be confused with the electromagnetic vector potential).

For the moment it is not necessary to assume that the field one applies to the particle is purely electric or magnetic. We consider a constant, but otherwise arbitrary electromagnetic field and review some properties of one-body motion in this field.

We first notice that the *classical* equations of motion can be explicitly integrated [15]. In the *quantum* equation of motion we use the operator $K = \frac{1}{2}(p - eA)^2$ and the Lorentz-covariant gauge

$$A = \tfrac{1}{2} q\cdot F \ . \tag{2.2}$$

Hence we compute

$$[p, A] = -\frac{i}{2} F \ . \tag{2.3}$$

According to (2.1) the interaction term is

$$2G = -eA\cdot p - ep\cdot A + e^2A^2 \ . \tag{2.4}$$

Now the following result is straightforward: The *pseudomomentum defined as the four-vector* $p + eA$ *is conserved.* That is to say,

$$[p^\mu + eA^\mu, K] = 0 \ , \tag{2.5}$$

as we can see by computing the commutator $[p^\mu + eA^\mu, p^\nu - eA^\nu]$, which turns out to be zero, taking the gauge (2.2) and the standard commutation relations into account. In the presence of a constant *magnetic* field this result is already contained in the paper by Johnson and Lippman, who have also considered a relativistic equation of motion [1]. But we wish to emphasize that our covariant derivation is valid *for all kinds of electromagnetic fields, provided it is constant* in space and time.

The four-vector $p + eA$ is a natural generalization of linear momentum when a constant electromagnetic field is applied to the charge we consider. Its physical relevance is well established in the purely magnetic case. In other cases its physical interpretation is clarified if we remark that (2.5) could be alternatively derived from stress-energy tensor considerations.

However, its components do not commute in general, and we find

$$[p^\mu + eA^\mu, p^\nu + eA^\nu] = ieF^{\mu\nu} \ . \tag{2.6}$$

Beside pseudomomentum, what are the other constants of the motion? The answer to this question depends on the form of the field tensor $F$. When $G$ admits some directions of translation invariance *in a strong sense* that we have earlier defined ($s$-translation invariance [4]), other conserved quantities can be found. But a systematic search of all first integrals will be displayed in a separate work; here, we simply focus on pseudomomentum.

We have previously pointed out [4] that a strong translation invariance arises when $F$ is purely magnetic. One of the main goals of this paper is to show that a constant *electric* field enjoys a similar symmetry.

*Remark:* since pseudomomentum is conserved anyway in the one-body sector (assuming a constant field), it may seem at first sight unnecessary to pay so much attention to the special cases of *s*-translation invariance. The reason why we insist on this special symmetry is that it will be of great practical interest when turning, in the next section, to the two-body sector.

Let us recall exactly what we mean by *s*-translation invariance. $G$ is said to be strongly translation invariant along a space-time direction $w$ when

$$[w\cdot q, G]=0 \ , \tag{2.7}$$

$$[w\cdot p, G]=0 \ . \tag{2.8}$$

The directions like $w$ in Eqs. (2.7) and (2.8) will be called *longitudinal.* They span a linear space of four-vectors (the longitudinal space), say, $(E_L)$. If this linear manifold is not tangent to the light cone, there is a unique $c$-number tensor $\tau$ projecting any vector onto $(E_L)$. In view of the second equation above, it is clear that, in the case of strong translation invariance, the longitudinal projection of $p$ is a constant of the motion. Indeed $w\cdot p$ commutes with both pieces of $K$, and this holds true for all longitudinal direction $w$. In few words the $s$-translation invariance of $G$ implies an ordinary translation invariance of $K$ along the direction $w$.

### Constant electric field

In the rest of this section we consider the motion of a single charge $e$ in a constant *electric* field. We assume that the electromagnetic tensor $F_{\alpha\beta}$ is purely electric in some frame. So there exists some timelike constant unit vector $u^\alpha$ such that

$$*F_{\alpha\beta}u^\beta=0 \ , \tag{2.9}$$

where the asterisk refers to the dual tensor. *Such a vector is by no means unique.* Indeed, taking a frame adapted to $u$ we can look for a *spacelike* constant unit vector $v^\alpha$ such that

$$*F_{\alpha\beta}v^\beta=0 \ . \tag{2.10}$$

Equation (2.10) reduces to $*F_{ij}v^j=0$, which admits nonvanishing solutions because the rank of $*F_{ij}$ cannot be 3. Any linear combination of $u$ and $v$ is a null eigenvector for $*F_{\alpha\beta}$. We can require that $u\cdot v=0$, but the couple $u,v$ is fixed only up to a Lorentz rotation in its plane $(E)$. In contradistinction, the plane $(E)$ spanned by $u$ and $v$ is intrinsically defined once $F_{\alpha\beta}$ is given. A *frame adapted to the field* is an orthonormal tetrad $\epsilon_{(\alpha)}$, where $\epsilon_{(0)}$ and $\epsilon_{(3)}$ belong to $(E)$. In such a frame the only nonvanishing components of $*F$ are $*F_{12}=-*F_{21}$. Now it is not difficult to check that the only nonvanishing components of $F$ are

$$F^{30}=-F^{03} \ .$$

*Proof.* $F=-**F$, so $F^{10}$ is proportional to $\eta_{1023}(*F)^{23}$, where $*F^{23}$ is zero, etc.

In a frame adapted to the field we have

$$A^\alpha=\tfrac{1}{2}(q^0F_0^\alpha+q^3F_3^\alpha) \ ; \tag{2.11}$$

hence,

$$A^1=\tfrac{1}{2}(q_0F^{01}+q_3F^{31})=0 \ , \tag{2.12}$$

$$A^2=\tfrac{1}{2}(q_0F^{02}+q_3F^{32})=0 \ . \tag{2.13}$$

The vector-potential $A$ is now fully contained within the two-plane $(E)$.

Notice that the application of a constant field does not completely destroy translation invariance. Indeed, it stems from (2.3) that for any constant four-vector $w$,

$$[p\cdot w, A^\lambda]=-\frac{i}{2}w_\sigma F^{\sigma\lambda} \ .$$

If now $w^0=w^3=0$, we find that

$$[w\cdot p, A^\lambda]=0 \ , \tag{2.14}$$

which in turn implies

$$[w\cdot p, A\cdot p]=[w\cdot p, p\cdot A]=[w\cdot p, A^2]=0 \ .$$

Therefore, if $G$ is given by (2.4), Eq. (2.8) is satisfied. Since $w\cdot p$ obviously commutes with the free-particle Hamiltonian $\frac{1}{2}p^2$, we finally get $[w\cdot p, K]=0$. So $w\cdot p$ is a constant of the motion provided $w$ is orthogonal to $(E)$. In other words, the two-dimensional Abelian group generated by the components $p_1$, $p_2$ of $p^\alpha$ keeps being a symmetry.

We claim that this symmetry is a *strong* translation invariance in the sense of Ref. [4]. Indeed we have just proved (2.8), so the only point left to be checked concerns the vanishing of $[w\cdot q, G]$, thus finally the vanishing of $[w\cdot q, p\cdot A]$ and $[w\cdot q, A\cdot p]$. But direct computation yields $iw\cdot A$, which is precisely zero since we assume that $w$ is orthogonal to $(E)$.

Finally $(E)$ is the transverse plane in the terminology of [4] and it must be noted as $(E_T)$. In an adapted frame the directions 1, 2 are longitudinal, whereas 0, 3 are transverse (it was the reverse in the magnetic case). For instance, $A^\mu$ has only transverse nonvanishing components, and these components depend on transverse variables only.

## III. TWO-BODY SYSTEM IN THE EXTERNAL ELECTROMAGNETIC FIELD

When pair creation can be neglected, a system of two interacting relativistic particles is reasonably described by two coupled wave equations [5] of the form

$$2H_a 2H_a\Psi=m_a^2\Psi, \quad a,b=1,2 \ , \tag{3.1}$$

where $2H_1$, $2H_2$ are suitable generalizations of the individual Klein-Gordon operators, including the free term, the external coupling, the mutual interaction, and *an extra term* (often referred to as the *three-body term* because our external field can be felt as created by a third body of very large mass located at infinity), which ensures that the compatibility condition $[H_1,H_2]=0$ remains satisfied [11,4]. Naturally, the three-body term vanishes when either the mutual or the external interaction is turned off, but otherwise it may have the same order of magnitude as the other terms.

### Notation, conventions

For a system of two spinless particles with particle labels $a,b,\ldots=1,2$, the wave function is a distribution in the eight independent variables $q_1^\alpha, q_2^\beta$. For free particles the *Hamiltonian generators* $H_a$ reduce to $\frac{1}{2}p_a^2$, where $p_a=-i\partial/\partial dq_a$ in the coordinate representation. With the particle masses being $m_1,m_2$ we set

$$\mu=\tfrac{1}{2}(m_1^2+m_2^2),\quad \nu=\tfrac{1}{2}(m_1^2-m_2^2)\ .$$

We separate the relative variables according to

$$z=q_1-q_2,\quad y=\tfrac{1}{2}(p_1-p_2)\ ,$$

$$P=p_1+p_2,\quad Q=\tfrac{1}{2}(q_1+q_2)\ ,$$

hence the canonical commutation relations

$$[z^\alpha,y_\beta]=[Q^\alpha,P_\beta]=i\delta_\beta^\alpha\ .$$

As in Refs. [4,17] we define

$$Z=z^2P^2-(z\cdot P)^2\ .$$

The most natural picture for relativistic quantum mechanics consists in having the Poincaré group generated by $P$ and

$$M=(q\wedge p)_1+(q\wedge p)_2=Q\wedge P+z\wedge y\ ,$$

where the wedge symbol denotes the tensor formed by the exterior product. In the absence of an external field, $H_1$, $H_2$ reduce to $H_1^{(0)}$, $H_2^0$. More generally, the label (0) refers to the isolated system. Constants of motion are characterized by a vanishing commutator with both $H_1,H_2$.

Let $V^{(0)}$ be the mutual interaction. This term already arises in the absence of an external field; it can be obtained from QED by reduction of the Bethe-Salpeter equation [9]. It may also be motivated by phenomenology. In order to write down coupled wave equations in the presence of an external field, a simple procedure, consisting in adding all the interactions plus the "three-body term," can be considered as a modification of the mutual interaction. The only practical difficulty is the *explicit* determination of the extra term. The compatibility condition alone does not select a unique solution, but the presence of special symmetries in the external potential may result in a reasonable ansatz.

In principle, a more accurate treatment should be warranted: If the mutual interaction present in the absence of an external field is considered as being obtained from QED by reduction of the Bethe-Salpeter equation, one should resume the same treatment in the presence of the external field. But such a procedure is not straightforward. [According to Bethe and Salpeter (1951), an extension to the case where an external field is present is "straightforward, but computational complications can make it prohibitive." According to Grotch and Hegstrom [2], "the BS equation may be minimally coupled provided the external field is time independent."] It has been carried out for some *static* external potentials [16]. In that case the results obtained so far agree (at least at first order) with the ansatz (notice that the ansatz produces no extra term at this order for two opposite charges in a constant field). Moreover, in the simple situation of a constant *electric field* the external *potential* is not any more static; in fact, it is linear in the time coordinate, therefore it seems that no result that originated from reducing the BS equation is available in this case.

In view of these arguments it remains reasonable to describe as follows a two-particle system with scalar constituents, coupled to an external electromagnetic field. The wave function $\Psi(q_1,q_2)$ is submitted to the coupled wave equations (1) with

$$H_1=K_1+V,\quad H_2=K_2+V\ , \tag{3.2}$$

where $K_1$, $K_2$ correspond to the independent-particle approximation; that is,

$$2K_1=(p_1-e_1A_1)^2,\quad 2K_2=(p_2-e_2A_2)^2\ , \tag{3.3}$$

with $A_1=A(q_1)$ and $A_2=A(q_2)$ and the Lorentz gauge.

In (3.2) the extra term has been incorporated into the mutual interaction, in the sense that $V$ is a suitable modification of $V^{(0)}$, which must reduce to $V^{(0)}$ itself in the absence of the field.

Hereafter, we consider a *constant field* $F^{\mu\nu}$ and choose the *covariant gauge* (2.2), hence

$$A_1^\mu=\tfrac{1}{2}q_{1\sigma}F^{\sigma\mu},\quad A_2^\mu=\tfrac{1}{2}q_{2\sigma}F^{\sigma\mu}\ . \tag{3.4}$$

Unless special symmetries are present in the external field, the exact determination of $V$ is impossible. We have explained in detail in previous papers [4,17] how *s*-translation invariance can be defined in the two-body sector and how it may arise in special situations, in particular in the presence of a pure magnetic or a pure electric constant field. This latter case will be discussed in detail in the subsequent section. For the moment we consider an arbitrary constant field. The requirement of compatibility provides, for the determination of $V$, an equation which could be, in principle, solved perturbatively,

$$[K_1-K_2,V]=0\ . \tag{3.5}$$

The arbitrariness of $V$ is first limited by an obvious condition of coupling separability: $V$ must reduce to $V^{(0)}$ in the limit where $F$ identically vanishes (no-field limit). Then the choice of $V$ may be further restricted by imposing reasonable symmetries on the two-body system.

A natural possibility (suggested by the magnetic example) consists in trying to impose the conservation of the total pseudomomentum,

$$C=C_1+C_2\ , \tag{3.6}$$

where

$$C_1=p_1+e_1A_1,\quad C_2=p_2+e_2A_2\ . \tag{3.7}$$

The symmetry associated with this property should not be confused with $s$-translation invariance. As we have seen in Sec. II, pseudomomentum is always conserved in the one-body sector (whereas $s$-translation invariance corresponds to exceptional forms of the tensor $F$). In a continuation of the terminology used in the noncovariant approaches [3] (and noticing that $[C^\mu,C^\nu]$ is not always zero), we refer to the transformations generated by $C$ as *twisted translations.* The vector operator (3.6) is known to be conserved for a neutral system in a magnetic field [4]. (Notice that it reduces to $P$ in the no-field limit.)

For a charged system the components of $C$ do not commute among themselves, but at least their commutator is a $c$ number. In fact, (2.6) implies

$$[C^\alpha,C^\beta]=-2ieF^{\alpha\beta} . \tag{3.8}$$

Therefore we propose to complement (3.5) with the additional conditions

$$[C^\alpha,V]=0 , \tag{3.9}$$

and we claim that (3.9) ensures conservation of $C^\alpha$.

*Proof.* The results of the preceding section entail

$$[C_a^\alpha,K_b]=0 , \tag{3.10}$$

which can be interpreted as the statement that in the absence of mutual interaction each particle can be separately described and has its individual pseudomomentum conserved. Then a glance at (3.2) indicates that (3.9) implies

$$[C_a^\alpha,H_b]=0 , \tag{3.11}$$

which is the conservation of $C^\alpha$. The crucial point is that *Eqs. (3.9) are compatible among themselves and with (3.5).* Indeed $[C,C]$ is a $c$ number, and the subsidiary condition (3.9) finally ensures the conservation of $C^\alpha$ *without knowing $V$ explicitly.*

Condition (3.9) is satisfactory at the level of formal calculations. Indeed the components of $C$ commute among themselves up to $c$ numbers, and they commute with $K_1-K_2$. Therefore applying the Jacobi identity to the system [(3.5) and (3.9)] yields no contradiction. A more rigorous proof that an operator $V$ satisfying (3.5) and (3.9) exists for all constant $F$ is beyond the scope of this paper. [Actually, if we consider the (classical) relativistic limit where commutators are replaced by Poisson brackets, equations like (3.5) and (3.9) become local differential equations. Their compatibility in terms of Lie brackets immediately ensures only the local existence of V.]

A first example where such an operator exists concerns the neutral system in a *magnetic* field. In this case we constructed $V$ explicitly. Formal analogy suggests that a similar result also arises for a neutral system in a purely *electric* field. In both pure cases, the external potential $G$ admits directions of strong translation invariance. This exceptional situation allows for the determination of explicit formulas, at the price of a change of picture for quantum mechanics. Insofar as constant electromagnetic fields are concerned, only the pure magnetic case and the pure electric case are known to provide this symmetry [4]. We have previously worked out the pure magnetic case. But the ansatz used in Ref. [4] is general and can be carried out along the same lines in the pure electric case, as we shall do in the next section.

## IV. TWO-BODY SYSTEM IN A PURE ELECTRIC (CONSTANT) FIELD

When $F$ is pure electric, the study made in Sec. II indicates that the single-particle potential $G$ is $s$-translation invariant along a spacelike two-plane $(E_L)$ (spanned by $\epsilon_{(1)},\epsilon_{(2)}$ in an adapted frame). It is straightforward to define a two-body version of $s$-translation invariance and to check that in the presence of a constant electric field, the external potentials $G_1$, $G_2$ defined through

$$2K_a=p_a^2+2G_a ,$$

enjoy this symmetry along the directions in $(E_L)$.

In an adapted frame, the longitudinal canonical variables are $q_{a1},q_{a2},p_{a1},p_{a2}$ (respectively, transversal, $q_{a0},q_{a3},p_{a0},p_{a3}$). Since $(E_L)$ is not a null plane, we are in a position to construct the wave equations in closed form with the help of a suitable ansatz, provided the mutual interaction can be written as

$$V^{(0)}=f(Z,P^2,y\cdot P) . \tag{4.1}$$

Before we focus on details about the constant electric field, let us recollect general results about this method, given in [4] as a generalization of a result of [11].

Any *four-vector* is uniquely decomposed into a longitudinal and a transverse part. An *operator* is said to be longitudinal (transverse) when it commutes with all the transverse (longitudinal) canonical variables. But in general this terminology must be used with care because it may happen that a *vector-operator* has components only in the longitudinal plane, whereas these components are functions of the transverse canonical variables only. Fortunately, no confusion is possible about $A_1^\mu$, $A_2^\mu$. In the electric case (as well as in the magnetic one), they are transverse vectors and they depend only on transverse variables and therefore commute with any function of the longitudinal canonical variables (for instance, with $L$; see below).

For an electric field, we have seen in Sec. II that $A_L^\alpha$ is zero. When going over to the two-body sector it is clear that $A_{1_L}^\alpha$ and $A_{2_L}^\alpha$ also vanish. Then a glance at (3.6) and (3.7) shows that

$$C_L^\mu=P_L^\mu .$$

Let us emphasize this difference: As in the magnetic case, $C$ and $P$ coincide when projected onto the longitudinal plane, but now the longitudinal plane is purely spacelike.

The determination of a closed form of the wave equations requires that we perform a change of picture [11,4]. In the *external-field picture,* the wave function is $\Psi'$ and the wave equations can be written as

$$H_a'\Psi'=\tfrac{1}{2}m_a^2\Psi' . \tag{4.2}$$

Now (3.2) is replaced by

$$H_a'=K_a'+V' \tag{4.3}$$

and $K'_a$ is formally defined by $K'_a = \exp(iB)K_a\exp(-iB)$, where

$$B = TL \tag{4.4}$$

is a product of suitably chosen factors, namely,

$$T = K_1 - K_2 - y_L \cdot P_L \ , \tag{4.5}$$

and $L$ is required to verify

$$[L, y_L \cdot P_L] = i \ .$$

Following Bijtebier [11] we have proposed the standard choice

$$L = \frac{P_L \cdot z_L}{P_L^2} \tag{4.6}$$

(notice that $T$ and $L$ respectively depend on transverse and longitudinal variables) and $V'$ is obtained by making the ansatz

$$V' = f(\hat{Z}, P^2, y_L \cdot P_L) \ , \tag{4.7}$$

where

$$\hat{Z} = Z + 2(P_L^2 z \cdot P - P^2 z_L \cdot P_L)L + P_T^2 P_L^2 L^2 \ . \tag{4.8}$$

Most operators arising in the external-field picture are more correctly defined through the formula

$$\Omega' = \Omega + i[B, \Omega] + \frac{i^2}{2}[B, [B, \Omega]] + \cdots \ , \tag{4.9}$$

which has only a finite number of terms irrespective of the self-adjointness of $B$, provided some $n$-fold commutator $[B, \ldots [B, \Omega]]$ is zero for a finite order $n$. This is precisely what happens when $\Omega$ is $K_1$ or $K_2$. In the framework of a general treatment elaborated in [4], where the ingredients $T$ and $L$ of $B$ are explicitly written down as in (4.5) and (4.6), we have found in [4]

$$K'_1 = K_1 - T(L \cdot y + \tfrac{1}{2}) + \tfrac{1}{2} T^2 L \cdot L \ , \tag{4.10}$$

$$K'_2 = K_2 - T(L \cdot y - \tfrac{1}{2}) + \tfrac{1}{2} T^2 L \cdot L \ . \tag{4.11}$$

Notice that $K'_1 - K'_2 = y_L \cdot P_L$. (Remark: A more rigorous, but far less intuitive, treatment would *a priori* consider the external-field picture and start from (4.2) and (4.3), then postulate (4.10), (4.11), and (4.7) and check that $[H'_1, H'_2]$ is zero.)

In order to represent pseudomomentum in the external-field picture, we must transform $C$ according to (4.9). But we shall prove that $[C^\alpha, B]$ vanishes, implying the following.

*Lemma.* Pseudomomentum is not altered by transformation (4.9).

*Proof.* We first show that $[T, C^\alpha] = 0$. Indeed, $T$ is given by (4.5), where $K_1$ and $K_2$ commute with $C$ as a consequence of (3.10). Hence $[T, C^\alpha] = -[y_L \cdot P_L, C^\alpha]$. But (3.6) and (3.7) ensure that $[y_L \cdot P_L, C^\alpha]$ reduces to $[y_L \cdot P_L, e_1 A_1^\alpha + e_2 A_2^\alpha]$, where the components of $A_{(1)}$, $A_{(2)}$ are known to commute with all longitudinal variables, hence $[y_L \cdot P_L, C^\alpha] = 0$; therefore $[T, C^\alpha] = 0$. Then we show that $[L, C^\alpha] = 0$. Indeed, $L$ obviously commutes with $P$ in Eq. (4.6). It remains to be checked whether $[L, e_1 A_{(1)} + e_2 A_{(2)}]$ actually vanishes. But this is true because $A_{(1)}$, $A_{(2)}$ only depend on the transverse variables. Finally, it stems from (4.4) that $[B, C^\alpha] = 0$ as we claimed.

As a result, applying formula (4.9) to $C$ leaves it unchanged. So pseudomomentum is represented, also in the external picture, by

$$C' = C = P + e_1 A_{(1)} + e_2 A_{(2)} \ . \tag{4.12}$$

Here we arrive at the main point of this section, i.e., the statement that, under very general conditions, the commutators $[C^\alpha, H'_a]$ vanish. More precisely we have the following.

*Theorem.* In a constant electric field, the pseudomomentum $C^\alpha$ of a neutral system with mutual interaction of the form (4.1) is conserved provided the wave equations are determined through the ansatz (4.7) and (4.8), where $[L, z]$ and $[L, P]$ vanish [e.g., $L$ as in (4.6)].

*Proof.* For a more rigorous exposition it is better to work directly in the external picture [17]. Look at (4.3) and proceed in two steps. We first realize that $[C^\alpha, K'_a] = 0$. Indeed, this commutator is the transformed version of $[C^\alpha, K_a]$ by formula (4.9), and we know that $[C_b^\alpha, K_a]$ all vanish. Now it remains to be proved that $[C^\alpha, V']$ also vanishes. According to (4.7), all we need to prove is that $C^\alpha$ commutes with $y_L \cdot P_L$, $P^2$, and $\hat{Z}$. Commutation with $y_L \cdot P_L$ has already been checked when proving that $C$ is invariant in the change of picture. In order to discuss more easily commutation with $P^2$, it is convenient to separate external from relative coordinates in the pseudomomentum. In the *neutral case* $e_1 + e_2 = 0$ we simply have

$$C^\alpha = P^\alpha + \frac{e_1 - e_2}{4}(z \cdot F)^\alpha \ . \tag{4.13}$$

(Remark: only in the neutral case does $P^2$ commute with $C^\alpha$.) Only $P$ and $z$ contribute in $C$, and both commute with $Z$. Inspection of (4.8) shows that they also commute with the other terms in $\hat{Z}$, provided $[P, L]$ and $[z, L]$ vanish, which is the case for the standard choice (4.6). Finally all the prerequisites ensuring that $[C^\alpha, V] = 0$ are satisfied, which achieves our proof.

Let us stress that under the present assumption of neutrality, the components of pseudomomentum commute among themselves. Direct computation is carried out using (4.13), where only the two first terms survive, and one finds

$$[C^\alpha, C^\beta] = 0 \ , \tag{4.14}$$

which allows for simultaneous diagonalization of $H_1$, $H_2$, $C^\alpha$. But, similar to the covariant approach in a magnetic field, and in contrast to the nonrelativistic (or at least noncovariant) theory, there is no evidence that pseudomomentum be exactly conserved for globally charged systems ($e_1 + e_2 \neq 0$) when $V$ is constructed through the ansatz (4.7) and (4.8).

## V. PSEUDOSTATIONARY STATES OF THE NEUTRAL SYSTEM

Since the Hamiltonians and all the components of $C$ commute among themselves, we can define *pseudostationary states* as those permitting these operators to diagonalize simultaneously. Such states satisfy (4.2) and

$$C^{\alpha}\Psi' = k^{\alpha}\Psi' , \tag{5.1}$$

where the components of $k^{\alpha}$ are constant $c$ numbers.

In order to explicitly separate the relative motion, we shall proceed exactly as we did in the magnetic case. Let $e_1 = -e_2 = e$ be the constituent charges. We combine the external-field picture and a canonical transformation inspired from Grotch and Hegstrom [2], which maps the twisted translations onto the ordinary ones. This ultimate change of picture is

$$\begin{aligned} \Psi'' &= \exp(i\Gamma)\Psi' , \\ a'' &= \exp(i\Gamma)a'\exp(-i\Gamma) \end{aligned} \tag{5.2}$$

for all operators $a'$ representing a physical quantity in the external-field picture, and $\Gamma$ is given by

$$\Gamma = \frac{e}{2}(z\cdot F\cdot Q) . \tag{5.3}$$

Notice that the transformation (5.2) can always be carried out in closed form. But the main virtue of $\Gamma$ is that $P^{\alpha} = \exp(i\Gamma)C^{\alpha}\exp(-i\Gamma)$; that is, $C'' = P$. This relation permits us to eliminate $Q$ from our formulas; Eq. (5.1) gets transformed into

$$P^{\alpha}\Psi'' = k^{\alpha}\Psi'' . \tag{5.4}$$

(Remark: it would be erroneous to conclude that we have restored translation invariance. In the present picture, space-time translations are now represented by a new operator $P''$, which does not commute with $H''$. In contrast, $P^{\alpha} = C''$ now represents the pseudomomentum, generator of the so-called "twisted translations" associated with the presence of a constant electric field. Nevertheless, invariance under "twisted translations" simplifies the calculations just as well as invariance under the true ones would do.)

Let us emphasize that $H_1''$ and $H_2''$ can be written in closed form because we already know $H_a'$ in closed form by addition of $K_a'$ and $V'$; see Eqs. (4.10), (4.11), (4.7), and (4.8). All we need is to apply (5.2) to $H_a'$. But (5.2) can always be explicitly carried out; see formulas (5.9) and (5.10) of Ref. [4].

Looking for pseudostationary states we have transformed Eq. (5.1) and obtained (5.4). Hereafter, we strictly assume $k^{\alpha}$ to be *timelike.* Taking the sum and difference, the wave equations can be written as

$$(H_1'' + H_2'')\Psi'' = \mu\Psi'' , \tag{5.5}$$

$$(H_1'' - H_2'')\Psi'' = \nu\Psi'' . \tag{5.6}$$

Here, remember that according to (4.3), $H_1'' - H_2'' = K_1'' - K_2''$. This expression is calculated by the application of transformation (5.2) to $(K_1' - K_2')$. But, according to (4.10) and (4.11), $(K_1' - K_2') = y_L\cdot P_L$.

Fortunately, it is straightforward to check that (like in the magnetic case) $\Gamma$ is purely transverse; thus $[\Gamma, y_L\cdot P_L]$ vanishes and $(y_L\cdot P_L)$ is not affected by transformation (5.2). Finally, Eq. (5.6) can be written as

$$y_L\cdot P_L\Psi'' = \nu\Psi'' . \tag{5.7}$$

Now solve the pseudostationarity condition (5.4) by

$$\Psi'' = \exp(ik\cdot Q)\chi(z^{\alpha}) . \tag{5.8}$$

Let $k_L$ be the longitudinal part of $k$. So (5.7) becomes

$$y_L\cdot k_L\chi = \nu\chi . \tag{5.9}$$

A further reduction is possible, but we hit a qualitative difference with respect to the magnetic case. Here *the longitudinal plane is purely spacelike.* Therefore we cannot solve (5.7) in terms of a time derivative of $\chi$. Instead, we determine the dependence upon a spacelike variable, as follows. It is convenient to take, *in the longitudinal plane* $(E_L)$, a frame adapted to $k_L$; we mean a choice of the dyad $\epsilon_1$, $\epsilon_2$ such that $\epsilon_1 = |k_L|^{-1}k_L$. [Beware that, in general, a frame adapted to $(E)$ cannot be additionally adapted to $k^{\alpha}$, since there is no reason why $k_L^{\alpha}$ should vanish.] In such a frame we can write $y_L\cdot k_L = y^1k_1$, where $k_1 = -|k_L|$. Equation (5.9) can be written as

$$-y^1|k_L|\chi = \nu\chi , \tag{5.10}$$

and (5.4) and (5.10) are solved easily by

$$\Psi'' = \exp(ik\cdot Q)\exp i\left[\frac{\nu z^1}{|k_L|}\right]\phi(z^0,z^2,z^3) , \tag{5.11}$$

and we end up with a reduced wave function depending on three degrees of freedom. But, in contradistinction to the magnetic case, the variables in the ultimate reduced wave function belong to a hyperbolic (2+1)-dimensional space. It is clear that $Q$ does not appear in $H_1'' + H_2''$ because of twisted translation invariance. Moreover, after writing (5.10) we may, in $H_1'' + H_2''$, replace $P$ by its eigenvalue $k$. We also can replace $y^1$ by $\nu/k_1$. So (5.5) reduces to a Klein-Gordon equation in a fictitious (2+1)-dimensional space-time.

## VI. CONCLUSION

The requirement that pseudomomentum be conserved seems to be a good prescription for reducing the ambiguities in the determination of the three-body terms in the wave equation. There are at least two interesting cases where this requirement has solutions in closed form: the magnetic case studied in [4] and the electric case displayed herein, provided we consider *neutral* systems. The ansatz (4.7), together with (4.3) and (4.2), where $K_1'$, $K_2'$ are given by (4.10) and (4.11), is a reasonable solution of the compatibility problem for a relativistic two-body system in a constant electric field, in the sense that it yields a pair of compatible equations that reduce to the correct limits if either the mutual interaction or the external field vanish. The no-field limit arises in disguise by the effect of a canonical transformation generated by

$\exp(ib)$, $b=(B)_{G=0}$. The ansatz respects invariance under rotations in the longitudinal plane.

When the system we consider is *charged,* there is no evidence, to our present knowledge, that the ansatz (4.7) fulfills conservation of $C$ (we have already met this situation in the magnetic case). Still, we cannot completely exclude the possibility that conservation is satisfied after all in charged systems, but hidden under complicated calculations. If this hope were to be frustrated one might prefer a perturbative, or implicit, solution to the compatibility problem insofar as it yields exact conservation of $C$, rather than a solution in closed form, which would violate this property. Then it might be interesting to ask whether a suitable modification of (4.7) may provide alternative explicit formulas together with conservation of $C$. A way of investigating these matters consists in looking for a convenient canonical transformation mapping $C$ onto $P$; that is, a covariant version of the Grotch and Hegstrom transformation (which would work also in the charged case). Naturally, even if we were to succeed, pseudomomentum of a charged system would never have commuting components.

The results concerning *neutral systems in a constant electric field* are much more satisfactory, since they allow one to define generalized stationary states where the relative motion gets separated. However, it is worthwhile to mention that, in contrast to the nonrelativistic theory of neutral systems in an electric field, pseudomomentum generally does not reduce to the customary linear momentum (though they may coincide for particular initial data).

After a further reduction one is left with a problem where the unknown wave function depends only on *three variables.* The peculiarity of this situation is that, in contrast to the magnetic case, the "relative time" now survives at the expense of a spacelike degree of freedom (there is some analogy here to a recent result of Bijtebier and Broekaert [18]).

One should not be too puzzled about it. After all, the concept of relative time as the timelike piece of the variable in the wave function has no absolute meaning, except perhaps if it is implicitly assumed that one considers the wave function of the primitive (Schrödinger) picture in the position representation. In this "primitive picture" the Poincaré generators have the conventional form, but this is already *not* the case in the external-field picture, and using $\Psi''$ instead of $\Psi'$ makes it more problematic. Finally, identifying "relative time" in $\Psi''$ is a rather superficial interpretation (the remark is valid also for the magnetic case where the "relative time" is actually eliminated). In our opinion, the important thing here is that the final problem has exactly the same number of degrees of freedom as its Galilean analog.

Needless to say, having only two space variables in the reduced relative wave function $\phi$ is by no means an indication of a planar motion. Rather, it is an artifact of the peculiar picture we are using for computational simplicity, exactly in the same way that, after the first reduction, the wave equations look like translation invariants without being so. Remember that the constant of the motion that allows for the last reduction is always the same in all pictures, namely, the difference of squared masses. In view of possible applications, we plan to incorporate spin in the present formalism.


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