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Causality and quantization of time-delay systems: A model problem

R. A. Moore
Guelph-Waterloo Program for Graduate Work in Physics, Waterloo Campus University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

T. C. Scott
Mathematical Institute, University of Oxford, 24/29 St. Giles, Oxford, OX1 3LB, England
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In a time-symmetric form, classical relativistic formulations involve not only a reference time but also relatively advanced and retarded times. This work examines a simple model problem having this structure with the equations of motion having closed-form solutions. The complexity of more realistic models is avoided while the general features are retained. The determinism and casuality of the theory are clearly demonstrated. Some problems associated with the quantization of power series representations using conventional procedures are illustrated and discussed.

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I. INTRODUCTION

In recent years, physical modeling involving relativistic delayed interactions and the related problem of higher than first-order Lagrangians have received considerable attention. One or the other or both of these aspects now appear in physical theories ranging from relativistic classical dynamics [1–10] through its canonical form and the structure of relativistic quantum mechanics [2,3,11–16], including the relationship to a field-theoretic description (see, e.g., [17–24], and references therein) to nonlocal field theories [25–28], to gravity [29–37], and on to string theory [38–41].

In this work, considerations are limited to the properties of a classical relativistic action-at-a-distance dynamics theory for point particles based on a Lagrangian formulation, without self-interactions and on a par with the nonrelativistic case, namely, the Fokker-Wheeler-Feynman (FWF) theory of electrodynamics, its generalizations [1,4,6–9], and its quantization. Although the exact classical theory satisfies Lorentz covariance, time-reversal symmetry, and particle interchange symmetry, there are possible ambiguities in the input assumptions and interpretations [1,9,10,41–43] of the theory. In addition, since conventional quantization of classical systems requires a single time formalism, all multitimes theories, such as the one under consideration, must be converted to a manageable form. This is done by making a power series expansion about some suitably chosen time. In this way, the multitimed problem is converted to an infinite-order Lagrangian problem. Next, the Lagrangian is truncated at some order. Thus the quantization of such systems is necessarily approximate. At the present time, two approaches are possible. In one, an order reduction technique is applied to limit the canonical variables to their nonrelativistic numbers [2,3,14,19–21,44,45]. In the other, the higher-order Lagrangian is treated exactly [13,15,16,46–49] with the number of canonical variables increasing with the order of the Lagrangian. We remark that this latter approach has been recently considered for gravity [36] and other problems [50,51]. However, as yet, the effects of any of these approximations are unknown. Unfortunately, the complexities of the complete expressions tend to obscure the results. In order to avoid these problems, herein, we examine a simple model problem, introduced by Feynman and Hibbs [52], consisting of a particle whose acceleration depends upon both past and future "forces." Thus this model problem has features in common with the relativistic theory. However, it has an exact closed-form solution, making the characteristics of these features transparent.

The model problem is defined in Sec. II. Also, the exact classical solutions are given in that section along with illustrations of both their deterministic character and the causal nature of the equations of motion. There we use the traditional definition [53] of causality, namely, an event at a given time is influenced, directly or indirectly, only by past events. In this context, it is seen that the equations of motion can be integrated stepwise forward in time using only information from the past to generate the exact future time evolution of the system. The inclusion of arbitrary initial conditions is discussed briefly. Thus the causal nature [8–10] of this type of theory is clearly displayed.

Quantization of the model problem is considered in Sec. III. The present model problem permits an examination of the two approaches to quantization mentioned above. It is seen that a straightforward application of exact techniques [13,15,16,46–49] can yield either spurious effects or incomplete descriptions. Thus, in general, additional information, through exact solutions or observation of physical systems, is required for a physically meaningful quantization of multitimed systems. Section IV contains further discussion.

II. THE MODEL PROBLEM AND ITS CLASSICAL SOLUTION

The model problem considered in this work is one proposed by Feynman and Hibbs [52], namely, a system obeying the principle of least action with the action being

$$J=\int\{\tfrac{1}{2}m[\dot{x}(t)]^2+\tfrac{1}{4}kx(t)[x(t+\tau)+x(t-\tau)]\}dt\ . \tag{2.1}$$

We have chosen to write J in time symmetric form and the notation is standard with m, k, and τ being constants. The above action is connected to that discussed by Feynman and Hibbs [52] by adding the divergence dF/dt to the integrand, where

$$F(t,\tau)=+\tfrac{1}{4}k\int_t^{t+\tau}x(t')x(t'-\tau)dt'\ . \tag{2.2}$$

The equation of motion is obtained in the usual way, with the additional requirement that variations in $x(t)$ be zero in the range 2τ centered on the end points of the time integral for J, this being the generalization of the fixed end-point condition for no time delays, and is

$$m\ddot{x}(t)=\tfrac{1}{2}k[x(t+\tau)+x(t-\tau)]\ . \tag{2.3}$$

Now Feynman and Hibbs [52] remarked that one has "created the curious situation in which a particle is driven by a force depending on the average value of coordinates that were and that will be." The natural conclusion is that this system is noncausal. However, one can now look to see if this conclusion is correct.

When one fixes $k=-m\omega_0^2$ to be a negative definite, as for a simple harmonic oscillator, (2.3) has exact solutions of the form

$$x(t)=\sum_i A_i\cos(\omega_i t+\delta_i)\ , \tag{2.4}$$

with A_i and δ_i being constants and ω_i being the roots of

$$\omega^2=\omega_0^2\cos(\omega\tau)\ . \tag{2.5}$$

We remark that, for the above choice of k, there are no pure exponentially divergent solutions. It is convenient to rewrite (2.5) by setting $\xi=\omega/\omega_0$ and $\lambda=\omega_0\tau$ to have

$$\xi^2=\cos(\lambda\xi)\ . \tag{2.6}$$

There is always at least one real root and an odd number of real roots, additional roots being added pairwise as λ is increased. A plot of the allowed values of ξ as functions of λ is given in Fig. 1.

Now in this case, the situation is exactly analogous to the case of a second-order differential equation without time delays. Knowing the equation of motion allows one to write the general solution in terms of a set of parameters. These parameters can be determined from a knowledge of a fixed amount of past information about the motion. The future motion of the system is thus completely determined for as long as the equation of motion

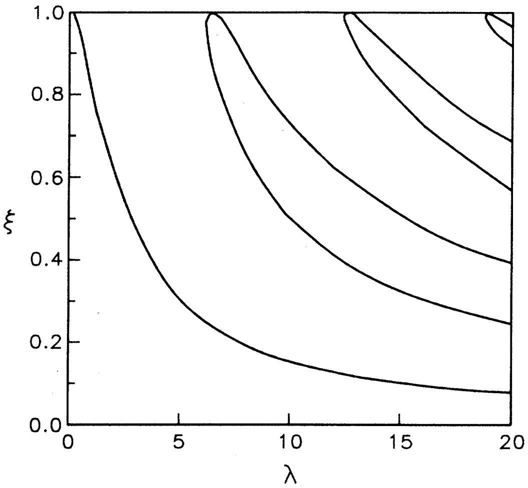

FIG. 1. Roots of Eq. (2.6), with $\xi=\omega/\omega_0$, as functions of $\lambda=\omega_0\tau$.

(2.3) applies. We label this character of (2.3) as deterministic. We also remark that, in both cases, the initial conditions can be over or inconsistently specified. This is related to the incorrect use of the equations and has no bearing on their exact nature. More is said about this point later. Further, since no knowledge of the future is required, causality is implied.

In casual systems, however, one must be able to generate the true time development of the motion by a forward in time integration of the equations of motion, using only past information [10,53,54]. If one assumes that the conventional Newtonian interpretation must apply to this type of double time-delay equation, with the acceleration being the effect and the right-hand side being the cause, and, further, dogmatically insists that any time integration be applied solely to the acceleration, the time integration cannot proceed without knowledge of the complete future motion of the system. This results in a Zeno-like paradox [43,55–57]. Such an insistence is an arbitrary and self-imposed constraint. If, on the other hand, one recognizes that the variational principle only guarantees a prescription for the determination of the motion or if one assumes an alterative physical interpretation of this type of equation [8–10], it can be rewritten as

$$x(t)=-x(t-2\tau)+2\frac{m}{k}\ddot{x}(t-\tau)\ . \tag{2.7}$$

Now, knowing $x(t)$ for the finite period $-2\tau\le t\le 0$ allows the time integration to proceed. We remark that an observer always has complete freedom to proceed in this way, independently of any other considerations. There are two points that should be emphasized. First, these equations of motion are the generators of the motion of the system and are not merely constraints on this motion, as has been previously claimed [56,57]. Second, since the time evolution of the motion can be generated with no knowledge of or influence of future events, the equations satisfy all of the requirements of causality [53,54]. This is exactly consistent with previous results found for FWF-like equations of motion [10].

To illustrate the above points, we consider the choice of two sets of initial conditions. As a first example, we take $x(t)$ to be given by (2.4) over the required time interval of 2τ with the A_i's and δ_i's being set and with the ω_i's satisfying (2.5). Now one can carry out the forward integration. On stepping through time, the correct $x(t)$ will be exactly generated. The only problem is that of maintaining numerical accuracy. Note that the input consists of an exact solution to (2.3) in this example. As a second example, consider x being held at a constant value, say, x_0, for a time interval of 2τ and then being released instantaneously. In the first time step, (2.7) requires $x(t)$ to change instantly from x_0 to $-x_0$. Thus $\ddot{x}(t)$ is undefined and $x(t)$ does not belong to C^2. As time progresses, $x(t)$ changes by $2x_0$ at each interval of 2τ, and $\ddot{x}(t)$ and hence $x(t)$ become undefined at each interval of τ. Any attempt at a numerical forward integration fails. An immediate interpretation might be that the system lacks causality. Not only does this interpretation lead to a contradiction, but it can be seen to be incorrect as follows. The Newtonian motion of a thrown object consists of the held stage, the impulsive throwing stage, and the free flight stage. In order to get the motion correct in the last state, it is necessary to include the second stage correctly. This is also true of the present problem, that is, a transition period must be included correctly in going from an arbitrary set of initial conditions, which do not satisfy (2.3), to the application of (2.3) alone. In the present model problem, the requirement is that, by the time any external force has been removed, the system must be left in an exact solution to (2.3) extending over a time interval of at least 2τ. There is an infinite number of ways to achieve this result. This situation arises for any input not satisfying (2.3) exactly. Generalizing, one should expect analogous requirements for any time-delay systems. Any difficulties would arise entirely from an incorrect use of these equations. This point is not addressed further here and we go on to consider quantization of the system.

III. APPROXIMATIONS AND QUANTIZATION

Considering the integrand in (2.1) as the Lagrangian, one has a multitimed Lagrangian. At the present time, it is not known how to quantize such Lagrangians exactly and hence one proceeds by approximation. Standard procedures require a single-time formalism and therefore one makes those approximations that reduce the problem to an appropriate form. In such cases [2–4,11–21, 44–49] an expansion is made about a single time, naturally being t in this case. Thus one writes, with $k=-m\omega_0^2$,

$$L'=\tfrac{1}{2}m[\dot{x}(t)]^2-\tfrac{1}{2}m\omega_0^2\sum_{n=0}^{\infty}\frac{\tau^{2n}}{(2n)!}x(t)x^{(2n)}(t)\ , \tag{3.1}$$

where $x^{(2n)}$ is the $2n$th time derivative of x. Thus one converts from a multitimed problem to an infinite-order problem. The quantization of L' is now discussed stepwise as one truncates at various values of n.

The zeroth-order Lagrangian is

$$L_0'=\tfrac{1}{2}m[\dot{x}(t)]^2-\tfrac{1}{2}m\omega_0^2[x(t)]^2\ . \tag{3.2}$$

This is the standard first-order simple harmonic-oscillator Lagrangian with the single root [see (2.6)] $\xi_1=1$. Quantization proceeds in the usual way. However, this approximation is valid only for $\lambda=\omega_0\tau\ll 1$ (see Fig. 1) being a poor approximation over the entire range of λ.

The first-order Lagrangian is

$$\begin{aligned}L_1'&=\tfrac{1}{2}m[\dot{x}(t)]^2-\tfrac{1}{2}m\omega_0^2[x(t)]^2-\tfrac{1}{4}m\omega_0^2\tau^2x(t)\ddot{x}(t)\\&=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega_0^2x^2+\tfrac{1}{4}m\omega_0^2\tau^2\dot{x}^2-\frac{d}{dt}\left[\frac{1}{4}m\omega_0^2\tau^2x\dot{x}\right]\ .\end{aligned} \tag{3.3}$$

Notice that, in such problems, it is convenient [15,16] to put the Lagrangian in a minimal form, that is, reduce the time derivatives to lowest order by adding divergences, before proceeding with the quantization. Although methods have been developed recently [36,50,51] to deal with the original form of (3.3), the final results are the same. Note also that the divergence satisfies the equation of motion [2,3,15,50] exactly. For

$$\frac{dF}{dt}=\frac{1}{4}\omega_0^2\tau^2(x\ddot{x}+\dot{x}^2) \tag{3.4}$$

and

$$p^{(1)}=\frac{\partial}{\partial\dot{x}}\left[\frac{dF}{dt}\right]-\frac{d}{dt}\left[\frac{\partial}{\partial\ddot{x}}\left[\frac{dF}{dt}\right]\right]\ , \tag{3.5}$$

one has

$$\frac{dp^{(1)}}{dt}=\frac{\partial}{\partial x}\left[\frac{dF}{dt}\right]\ . \tag{3.6}$$

Thus the equation of motion is given by the first three terms in the second form of (3.3) and they may be taken as the Lagrangian of the problem. Again this is a simple harmonic-oscillator problem with the single root $\xi_1=1/(1+\lambda^2/2)^{1/2}$, which is a reasonable approximation to ξ_1 over the entire range of λ. Quantization proceeds in the standard way with the quantum of energy being $\hbar\omega_0/(1+\lambda^2/2)^{1/2}$.

The second-order Lagrangian is

$$\begin{aligned}L_2'&=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega_0^2x^2-\tfrac{1}{4}m\omega_0^2\tau^2x\ddot{x}-\tfrac{1}{48}m\omega_0^2\tau^4xx^{(4)}\\&=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega_0^2x^2+\tfrac{1}{4}m\omega_0^2\tau^2\dot{x}^2-\tfrac{1}{48}m\omega_0^2\tau^4\ddot{x}^2\\&\quad-\frac{d}{dt}\left[\frac{1}{4}m\omega_0^2\tau^2\left[x\dot{x}+\frac{\tau^2}{24}(xx^{(3)}-\dot{x}\ddot{x})\right]\right]\ .\end{aligned} \tag{3.7}$$

Again the total time derivative in (3.7) satisfies the corresponding equation of motion exactly. The remaining second-order Lagrangian appearing in (3.7) is similar to a model problem considered earlier [16]. The equation of motion is

$$m(1+\tfrac{1}{2}\lambda^2)\ddot{x}+\frac{m\lambda^4}{24\omega_0^2}x^{(4)}=-m\omega_0^2x\ . \tag{3.8}$$

The solutions are of the form $A\cos(\omega t+\delta)$, i.e., (2.4), with

$$\tfrac{1}{24}\lambda^4\xi^4-(1+\tfrac{1}{2}\lambda^2)\xi^2+1=0\ , \tag{3.9}$$

where $\xi=\omega/\omega_0$, as before. The solutions are

$$\xi_1^2=\frac{12}{\lambda^4}\left[\left[1+\frac{1}{2}\lambda^2\right]-\left[1+\lambda^2+\frac{\lambda^4}{12}\right]^{1/2}\right] \tag{3.10}$$

and

$$\xi_2^2=\frac{12}{\lambda^4}\left[\left[1+\frac{1}{2}\lambda^2\right]+\left[1+\lambda^2+\frac{\lambda^4}{12}\right]^{1/2}\right]. \tag{3.11}$$

For this approximation, ξ_1 is very close to the first root found for (2.6) while ξ_2 is a poor approximation to the second and third roots. This should not be surprising as this was also the case for the lowest-order approximation to the first root, that is, from (3.2). Note also that one gets a spurious root for λ small. Now, exact quantization using the technique of the problem of Lagrange [48], the Jacobi-Ostragadski procedure [49] or Dirac's method of constraint dynamics [16,36,46,50,51] retains precisely the features of the approximation and therefore, in this case, will not always represent correctly the original problem.

The third-order Lagrangian is

$$\begin{aligned}L_3'&=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega_0^2x^2-\tfrac{1}{4}m\omega_0^2\tau^2x\ddot{x}-\tfrac{1}{48}m\omega_0^2\tau^4xx^{(4)}-\tfrac{1}{1440}m\omega_0^2\tau^6xx^{(6)}\\&=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}m\omega_0^2x^2+\tfrac{1}{4}m\omega_0^2\tau^2\dot{x}^2-\tfrac{1}{48}m\omega_0^2\tau^4\ddot{x}^2+\tfrac{1}{1440}m\omega_0^2\tau^6[x^{(3)}]^2\\&\quad-\frac{d}{dt}\left[\frac{1}{4}m\omega_0^2\tau^2\left[x\dot{x}+\frac{\tau^2}{24}(xx^{(3)}-\dot{x}\ddot{x})+\frac{\tau^4}{720}(xx^{(5)}-\dot{x}x^{(4)}+\ddot{x}x^{(3)})\right]\right].\end{aligned} \tag{3.12}$$

The total time derivative again exactly satisfies the generalized equation of motion and the physical equation of motion is

$$\frac{\lambda^6}{720\omega_0^6}x^{(6)}+\frac{\lambda^4}{24\omega_0^4}x^{(4)}+\frac{(2+\lambda^2)}{2\omega_0^2}x^{(2)}+x=0\ . \tag{3.13}$$

Looking for solutions of the form (2.4) gives

$$\tfrac{1}{720}\lambda^6\xi^6-\tfrac{1}{24}\lambda^4\xi^4+(1+\lambda^2/2)\xi^2-1=0\ . \tag{3.14}$$

This is a cubic equation in ξ^2 and the roots are known. It is straightforward to show that (3.14) has one real root, very close to the first root of (2.6), and two complex roots over the entire range of λ. Thus, at this point, we do not have a valid expression for meaningful quantization.

The situation improves only marginally as one increases the order of approximation. For example, the fourth-order approximation gives two real roots and two complex roots. The one real root is again very close to the first root of (2.6). The other real root is spurious for λ small but approximates the second and third roots of (2.6). In fifth order, only one real root appears. Thus, for this model problem, since the standard quantization procedures maintain the features of the approximation precisely, the quantum versions will, in general, be invalid.

IV. DISCUSSION

A simple model problem, having "forces" that depend upon both "past" and "future" times, has been examined. This model problem has a number of characteristics similar to FWF-like models but is simple enough to have a closed-form solution and to be relatively transparent. Two aspects have been considered: the causal nature of the model and the quantization of this multitimed problem.

It is natural to assume that a Newtonian interpretation applies to (2.3), namely, the acceleration, the effect, is caused by forces generated at both past and future times. In this case, one concludes that the future affects the present and hence the system is noncausal. However, it is seen from the exact solution that this model problem is precisely as deterministic as the usual Newtonian case. Thus one must conclude that the above assumption is invalid and that an alternative interpretation applies [10]. Further, it was shown that the equations of motion can be integrated forward in time using only the past history to generate the time evolution of the motion. Thus the system satisfies the usual definition of causality [53,54]. However, it was also noted that considerably more care is required with the initial conditions than for Newtonian-like problems. If one starts with initial conditions that do not satisfy (2.3), they can be considered to satisfy (2.3) with additional "forces." It is inconsistent with the concept of time delays to remove these additional forces instantaneously. The corresponding transition period must be incorporated appropriately in order to obtain meaningful results.

The above features are identical to those found in FWF-like models [1,6–10] and this work gives a further confirmation of the causal nature and the revised interpretation of their equations of motion. The present work examines a single-particle problem. Previous work examined a two-particle problem. Although the extension to many-particle systems was alluded to, no detailed discussion was given. It is straightforward to see that the arguments applicable to the FWF-like two-body problem can be extended to the FWF-like three-body problem. The only noteworthy comment is that care must be taken to ensure that the initial conditions are consistent with the time-delay constraints. This means that although an observer can specify a latest time and the past history of one of the particles up to that time, those of the other two particles may be specified only up to some later times, which must be consistent with the time-delay constraints.

At this point, the procedure follows that of the two-particle case, in principle. Once it is recognized how to deal with the three-particle problem, the extension to four and to N particles is, again, in principle, established. One sees that the requirements are that one has a finite, although possibly large, number of particles in a finite region of space, so that an observer can specify a meaningful latest time, with no external interactions. This establishes the criteria for Havas's [54] closed system. Such a modeling can be expected to apply on a terrestrial scale and even possibly on a galactic scale. Remembering that the intent of the previous work was to establish a relativistic action-at-a-distance dynamics theory on a par with the nonrelativistic case, which is all that is required. One should be under no illusion that such a theory will be suitable for a complete cosmological model. Thus we conclude that the original goal has been accomplished. One might point out that, since the system can be arbitrarily large, statistical methods can be applied, contrary to a recent statement [41].

Although the original model problem has exact classical solutions, rarely do any of the single-time approximations accurately reproduce these solutions. For example, if λ is such that only a single mode exists in the original model problem, except in zeroth and first order, spurious modes are generated in the approximate problems. Further, if three modes exist, none of the low-order approximations considered gives back three real modes. Now,in order to obtain a canonical form and proceed to quantization, the technique of the problem of Lagrange [48], or the Jacobi-Ostragadski procedure [49] or Dirac's method of constraint dynamics [46], must be applied to one of the approximations. However, exact application of each of these methods retains the features of that approximation precisely [16,36,50,51]. Thus the subsequent quantum version contains, identically, the inaccuracies and errors of the starting approximation. Now, in general, unlike this model problem, one does not have a complete set of solutions for the more complex models of physical systems. Quantization always proceeds from a low-order single-time approximation without the benefit of exact solutions. It follows that, at this point, the true physical content of the quantum version will be uncertain. Some criteria must be established in order to determine its validity. This can only come from some additional information [50], such as exact solutions or observation on a relevant physical system. For example, no low-order approximation correctly reproduces the exact three-mode case. If these three modes were observed in a physical system, the above approach would not be useful and they would normally be represented by three independent oscillators. The single-particle source nature would be lost and the physics partially obscured. Clearly, further consideration must be given to and care taken with the quantization procedure for systems with time delays. The applicability of these methods can be expected to be system dependent.

There may be some concern that the model considered herein is too simple and strictly applies to a one-particle problem. Thus any extrapolation to a two- (or more) particle case might be unjustified. A second model problem for two particles can be taken as

$$J=\int \{\tfrac{1}{2}m_1[\dot{x}_1(t)]^2+\tfrac{1}{2}m_2[\dot{x}_2(t)]^2+\tfrac{1}{16}k[x_1(t)-x_2(t+\tau)]^2+\tfrac{1}{16}k[x_1(t)-x_2(t-\tau)]^2 + \tfrac{1}{16}k[x_2(t)-x_1(t+\tau)]^2+\tfrac{1}{16}k[x_2(t)-x_1(t-\tau)]^2\}dt \,, \tag{4.1}$$

with an obvious interpretation. The variational procedure gives

$$m_1\ddot{x}_1(t)=\frac{k}{4}[2x_1(t)-x_2(t+\tau)-x_2(t-\tau)] \tag{4.2}$$

for m_1 and a similar equation for m_2. For equal masses $m_1=m_2=m$, and in the center of momentum frame $x_1(t)=-x_2(t)=x(t)$, (4.2) reduces to

$$m\ddot{x}(t)=\frac{k}{4}[2x(t)+x(t+\tau)+x(t-\tau)] \,. \tag{4.3}$$

This equation has characteristics similar to (2.3).

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