Corrections of order α^6 to S levels of two-body systems

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Corrections to the energy of S levels of positronium of order $m\alpha^6$ that are as large as several hundred kilohertz are obtained. A recoil correction of order $\alpha(Z\alpha)^5(m/M)m$ to the Lamb shift in hydrogen is calculated. This correction turns out to be too small from a phenomenological point of view.

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Recent progress in the spectroscopy of positronium [1–5] triggered theoretical work on the corrections of order $\alpha^6 m$ to the positronium energy levels. All logarithmic corrections of this order to S levels were calculated recently in [6,7]. Complete results for the corrections of order $\alpha^6 m$ to P levels were obtained in [8]. As emphasized in this last work, the large magnitude of the nonlogarithmic corrections to P levels suggests that the calculation of corresponding nonlogarithmic corrections to S levels is also important. Some of these corrections are already known, e.g., contributions induced by the two- and three-photon annihilation kernels [9–11]. We present below results of the calculation of nonlogarithmic contributions of order $\alpha^6 m$ to the S levels of positronium induced by radiative corrections to the Breit potential and by the polarization insertions in the graphs with two-photon exchange.

A radiative-recoil correction of order $\alpha(Z\alpha)^5(m/M)m$ to the Lamb shift in hydrogen induced by a polarization operator insertion in the two-photon exchange graph is also calculated in this Brief Report. Recent experimental achievements in measuring $1S$-$2S$ splitting in hydrogen [12] and the well-known results on the $2S$ Lamb shift [13–15] clearly demonstrate that theoretical calculation of all corrections to the Lamb shift of the order of several Kilohertz for the $1S$ state and about 1 kHz for the $2S$ state is necessary. Several such contributions were obtained quite recently [16–18] and the result presented below is one more such contribution (for more detailed description of the current theoretical status of the Lamb shift calculations see, e.g., [19]).

Let us consider first corrections of order $\alpha^6 m$ to the S levels of positronium connected with radiative insertions in the graph with one-photon exchange in Fig. 1. As is well known, the one-photon exchange produces the Coulomb and Breit potentials. One may easily obtain radiative corrections to the one-photon exchange expression in the form[1]

$$U(\mathbf{p},\mathbf{r})=-\alpha\left\{\frac{1}{r}-\pi\frac{1+8f_1'+2f_2}{m^2c^2}\delta^3(\mathbf{r})+\frac{4\pi p}{m^2c^2}\delta^3(\mathbf{r})+\frac{\mathbf{r}\cdot(\mathbf{r}\cdot\mathbf{p})\mathbf{p}}{2m^2c^2r^3}+\frac{\mathbf{p}^2}{2m^2c^2r}-(3+4f_2)\frac{\mathbf{s}\cdot\boldsymbol{l}}{2m^2c^2r^3}\right.$$
$$\left.+\frac{(1+f_2)^2}{m^2c^2}s_{1i}s_{2j}\left[\frac{\delta_{ij}}{r^3}-3\frac{r_ir_j}{r^5}\right]-\frac{(1+f_2)^2\pi}{m^2c^2}\left[\frac{4}{3}\mathbf{s}^2-2\right]\delta^3(\mathbf{r})\right\}, \tag{1}$$

where m is the electron mass, $\mathbf{p}$ is the relative momentum of the electron and positron, $\mathbf{r}$ is their relative position, f_1' is the slope of the Dirac form factor, f_2 is the Pauli form factor at zero momentum transfer, and p is the polarization operator contribution. With two-loop accuracy we have

$$f_1'=\frac{e_1}{m^2}\frac{\alpha}{\pi}+\frac{e_2}{m^2}\left[\frac{\alpha}{\pi}\right]^2,$$
$$f_2=g_1\frac{\alpha}{\pi}+g_2\left[\frac{\alpha}{\pi}\right]^2, \tag{2}$$
$$p=p_1\frac{\alpha}{\pi}+p_2\left[\frac{\alpha}{\pi}\right]^2.$$

Equation (1) above agrees with a result obtained earlier [20] if the radiative corrections are assumed to arise only from Pauli interactions.

It is an easy task now to obtain corrections of order $\alpha^6 m$ to the positronium energy levels

$$\Delta E_{F_1}=e_2\frac{\alpha^6}{\pi^2n^3}m\delta_{l0}=0.469\,94\frac{\alpha^6}{\pi^2n^3}m\delta_{l0},$$
$$\Delta E_{F_2,|l=0}=g_2\frac{\alpha^4m}{4n^3}=-0.082\frac{\alpha^6m}{\pi^2n^3}, \tag{3}$$
$$\Delta E_{p2}=-p_2\frac{\alpha^6}{2\pi^2n^3}m\delta_{l0}=-\frac{41}{324}\frac{\alpha^6}{\pi^2n^3}m\delta_{l0},$$

[1] The annihilation diagram contribution is missing in this expression since we do not consider annihilation contributions in this paper.

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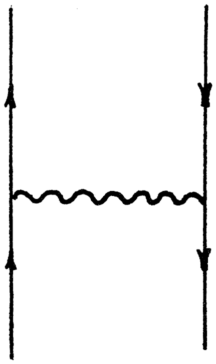

FIG. 1. One-photon exchange skeleton graph.

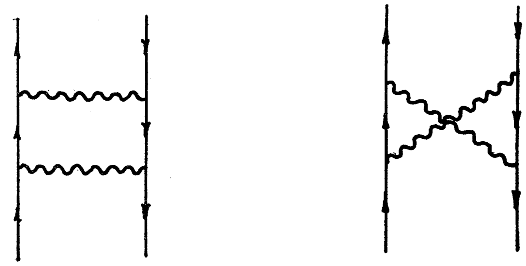

FIG. 2. Two-photon exchange skeleton graphs.

where we used the value of two-loop contribution e_2 to the slope of the Dirac form factor obtained numerically in [21] and analytically in [22], the explicit results for the two-loop electron magnetic moment g_2 [23,24], and the two-loop irreducible vacuum polarization operator [25] obtained a long time ago. With the help of the effective potential in Eq. (1) we may also easily calculate radiative corrections to the levels of positronium that have nonvanishing angular momentum. Our results in this case reproduce and confirm the respective results in [8,18].

Consider now corrections of relative order α^6 to the energy levels of two-body systems that are generated by the diagrams with intermediate momenta, which are high on the scale of the typical atomic momenta. It is well known that all such corrections are generated by the diagrams with two exchanged photons containing also either a polarization operator insertion in one of the exchanged photons or radiative photon insertions in the electron line (see, e.g., [26]). To sufficient accuracy external electron lines in the diagrams under consideration may be safely taken to be on-mass shell. It is not difficult to obtain an explicit expression for the infrared divergent skeleton integral corresponding to the sum of ladder and crossed diagrams in Fig. 2. Direct integration over loop momentum advocated in [27] leads to the expression for the skeleton integral

$$
\begin{aligned}
\Delta E_{\text{skel}} &= -32mM(Z\alpha)^2|\psi(0)|^2\int_0^\infty \frac{dk}{\pi k}\int_0^\pi d\theta \frac{\sin^2\theta(1+2\cos^4\theta)}{(k^2+4m^2\cos^2\theta)(k^2+4M^2\cos^2\theta)} \\
&= -16mM(Z\alpha)^2|\psi(0)|^2\int_0^\infty \frac{dk}{k^3}\frac{1}{(m^2-M^2)}\left\{ m\left[1+\frac{k^2}{4m^2}\right]^{1/2}\left[\frac{1}{k}+\frac{k^3}{8m^4}\right]\right. \\
&\qquad -M\left[1+\frac{k^2}{4M^2}\right]^{1/2}\left[\frac{1}{k}+\frac{k^3}{8M^4}\right] \\
&\qquad \left. -\frac{k^2}{8m^2}\left[1+\frac{k^2}{2m^2}\right]+\frac{k^2}{8M^2}\left[1+\frac{k^2}{2M^2}\right]\right\},
\end{aligned}
\tag{4}
$$

where m and M are the masses of the negatively and positively charged particles, respectively, Z is the charge of the positive particle in terms of the proton charge, and $\psi(0)$ is the value of the reduced mass Schrödinger-Coulomb wave function at the origin. All contributions to hydrogen Lamb shift of order $\alpha(Z\alpha)^5m$, both recoil and nonrecoil, calculated over years by different methods [28–30], may be obtained from the expression for the skeleton integral in Eq. (4) by insertion of radiative corrections.

Consider first recoil contributions of order $\alpha(Z\alpha)^5m$ to the Lamb shift in hydrogen. The contribution induced by the radiative photon insertions in the electron line was obtained in [30]. With the help of the explicit expression in Eq. (4) above, it is easy to confirm the result of [31] that the correction induced by the radiative photon insertions in the heavy line is suppressed by the factor $(m/M)^2$ relative to the contribution induced by the radiative photon insertion in the electron line. It is also easy to see that the recoil correction corresponding to the polarization operator insertion in the exchanged photon is suppressed by the factor m/M relative to the respective nonrecoil correction. Let us calculate this last correction. The general expression in Eq. (4) contains the skeleton integral both for recoil and nonrecoil corrections. The skeleton integral for the recoil corrections may be obtained by subtracting the heavy pole residue in Eq. (4) and has the form

$$\Delta E_{\text{skel-rec}}=\frac{16(Z\alpha)^2|\psi(0)|^2}{m^2(1-\mu^2)}\times\int_0^\infty\frac{kdk}{(k^2+\lambda^2)^2}\left\{\mu\left[1+\frac{k^2}{4}\right]^{1/2}\left[\frac{1}{k}+\frac{k^3}{8}\right]-\left[1+\frac{\mu^2k^2}{4}\right]^{1/2}\left[\frac{1}{k}+\frac{\mu^4k^3}{8}\right]-\frac{\mu k^2}{8}\left[1+\frac{k^2}{2}\right]+\frac{\mu^3k^2}{8}\left[1+\frac{\mu^2k^2}{2}\right]+\frac{1}{k}\right\},\tag{5}$$

where $\mu=m/M$ and we transformed to a dimensionless integration momentum measured in units of the electron mass. In this expression λ provides an infrared cutoff, which is allowed to approach zero later.

For calculation of the radiative-recoil contribution to the Lamb shift induced by the polarization operator insertions one has to make a substitution in the integrand in Eq. (5)

$$\frac{1}{k^2}\to\frac{\alpha}{\pi}I_1(k)\ ,\tag{6}$$

where

$$I_1(k)=\int_0^1 dv\frac{v^2(1-v^2/3)}{4m^2+(1-v^2)k^2}\ .\tag{7}$$

However, the skeleton integrand in Eq. (5) behaves as μ/k^4 at small momenta and naive substitution in Eq. (6) leads to divergence. This divergence dk/k^2 actually diminishes the power of the $Z\alpha$ factor and the respective contribution turns out to be of order $\alpha(Z\alpha)^4$. In order to get the recoil correction of order $\alpha(Z\alpha)^5m$ we have to subtract the leading low-frequency asymptote of the product of the skeleton integrand and the polarization operator. Then we obtain the integral for the radiative-recoil correction (one has to insert an additional factor of 2, which takes into account possible insertions of the polarization in both photon lines)

$$\Delta E_r=\frac{32(Z\alpha)^2|\psi(0)|^2}{m^2(1-\mu^2)}\left[\frac{\alpha}{\pi}\right]\times\int_0^\infty\frac{dk}{k}\left\{I_1(k)\left[\mu\left[1+\frac{k^2}{4}\right]^{1/2}\left[\frac{1}{k}+\frac{k^3}{8}\right]-\left[1+\frac{\mu^2k^2}{4}\right]^{1/2}\left[\frac{1}{k}+\frac{\mu^4k^3}{8}\right]-\frac{\mu k^2}{8}\left[1+\frac{k^2}{2}\right]+\frac{\mu^3k^2}{8}\left[1+\frac{\mu^2k^2}{2}\right]+\frac{1}{k}\right]-\frac{\mu}{15k}\right\}.\tag{8}$$

This integral contains also some contributions of higher order in the electron-proton mass ratio and may be easily calculated numerically. However, these higher-order contributions are clearly negligible and we omit them. Then we obtain the analytic result

$$\Delta E_r=\left[\frac{2\pi^2}{9}-\frac{70}{27}\right]\mu\frac{\alpha(Z\alpha)^5m}{\pi^2n^3}\left[\frac{m_r}{m}\right]^3.\tag{9}$$

Consider now the contribution of order α^6 induced by insertion of the one-loop polarization operator for the positronium case. The calculation is similar to the one for hydrogen. The analog of the skeleton integral in Eq. (4), for the case of equal masses, has the form

TABLE I. Summary of $2S$ and $1S$ energy corrections.

ΔE		$2S$ (kHz)	$1S$ (kHz)
positronium, ΔE_{F_1}	$0.469\ \frac{\alpha^6}{\pi^2n^3}m$	111.05	888.40
positronium, ΔE_{F_2}	$-0.082\frac{\alpha^6}{\pi^2n^3}m$	−19.38	−155.02
positronium, ΔE_{p2}	$-\frac{41}{324}\frac{\alpha^6}{\pi^2n^3}m$	−29.90	−239.22
positronium, ΔE_{p1}	$\left[\frac{\pi^2}{36}-\frac{5}{27}\right]\frac{\alpha^6}{\pi^2n^3}m$	21.02	168.19
hydrogen, ΔE_r	$\left[\frac{2\pi^2}{9}-\frac{70}{27}\right]\frac{\alpha(Z\alpha)^5}{\pi^2n^3}\frac{m}{M}m$	−0.05	−0.41

$$\Delta E = -16m^2(Z\alpha)^2|\psi(0)|^2\int_0^\infty \frac{dk}{k^4}\left\{\frac{1}{\sqrt{k^2+4m^2}}+\frac{k^3}{8m^4}-\frac{k^4(k^2+3m^2)}{8m^6\sqrt{k^2+4m^2}}+\frac{k^5}{8m^6}\right\}$$
$$= -\frac{2\alpha^5m^5}{\pi n^3}\int_0^\infty \frac{dk}{k^4}\left\{\frac{1}{\sqrt{k^2+4m^2}}+\frac{k^3}{8m^4}-\frac{k^4(k^2+3m^2)}{8m^6\sqrt{k^2+4m^2}}+\frac{k^5}{8m^6}\right\}. \quad (10)$$

The consideration of the hydrogen case above teaches us that the integrand for the radiative correction is given by the subtracted product of the vacuum polarization operator and the skeleton integrand. Hence the contribution to the energy levels of positronium of order $m\alpha^6$ is given by the expression (remember combinatorial factor 2)

$$\Delta E_{p1} = -\frac{4\alpha^6m^5}{\pi^2n^3}\int_0^\infty \frac{dk}{k^2}\left\{I_1(k)\left[\frac{1}{\sqrt{k^2+4m^2}}+\frac{k^3}{8m^4}-\frac{k^4(k^2+3m^2)}{8m^6\sqrt{k^2+4m^2}}+\frac{k^5}{8m^6}\right]-\frac{1}{30}\right\}$$
$$= \left[\frac{\pi^2}{36}-\frac{5}{27}\right]\frac{m\alpha^6}{\pi^2n^3}. \quad (11)$$

Numerical values of the corrections obtained above are presented in Table I. In the case of positronium these corrections turn out to be of the same order of magnitude as other corrections to the energy levels calculated recently [6–11] and are significant for comparison of the theory with the current experimental results. In the case of hydrogen the corrections for the S levels obtained above are about an order of magnitude smaller than the corrections of order $\alpha^6(m/M)m$ for the P levels obtained recently [18] and are too small to be interesting from the phenomenological point of view. Detailed derivation of the results of this paper will be presented elsewhere. In the case of positronium there remain some other yet unknown contributions of order $\alpha^6 m$ to the energy shift of S levels. Work on their calculation is in progress now.

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