Self-trapping of light beams and parametric solitons in diffractive quadratic media

Alexander V. Buryak, Yuri S. Kivshar, and Victoria V. Steblina
Optical Sciences Centre, Australian National University,
Australian Capital Territory 0200, Canberra, Australia
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It is shown that the mutual trapping of the fundamental and second-harmonic beams in a diffractive (or dispersive) medium with quadratic nonlinearity can support a family of two-wave (2+1)-dimensional solitons of circular symmetry. The stability analysis shows that these (2+1)-dimensional solitons are stable in the physically important region of parameters, although unstable solitons are also revealed and their instability dynamics is analyzed numerically. Phase-dependent and, in some cases, nondestructive collisions of these solitons are also considered.

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There is growing interest in the subject of self-guided waves (spatial solitons) in nonlinear materials (see, e.g., [1–5] to cite a few). This interest is not only for fundamental reasons, but also because of the potential use of optical solitons in all-optical switching based on the concept of light guiding light [2,6]. Until recently, it has been usually believed that self-trapping of light beams and stable propagation of spatial optical solitons can be observed only due to the intensity-dependent nonlinear refractive index of a dielectric medium with a third-order (cubic or $\chi^{(3)}$) nonlinear response, whereas the second-order (quadratic or $\chi^{(2)}$) nonlinearities have been discussed in the theory of second-harmonic generation (SHG) (see, e.g., [7]). However, it has been recently shown that effective $\chi^{(3)}$-like nonlinearities of noncentrosymmetric media can be obtained by cascaded second-order effects and they lead to such a phenomenon usually related to $\chi^{(3)}$ materials as a nonlinear phase shift [8] (see also the earlier papers [9,10]). It has also been demonstrated that cascaded nonlinearities can support parametric self-guided waves in the (1+1)-dimensional (or planar geometry) case [5,11–15], which form a family of self-trapped (spatially localized) nonlinear waves [13]. Development of this concept for the three-dimensional [i.e., in fact, (2+1)-dimensional, which means two transverse and one longitudinal directions] case is nontrivial, and it is a problem of fundamental importance which, if solved, could allow one to create reconfigurable guiding structures in bulk $\chi^{(2)}$ materials. Recent numerical simulations [16] and earlier analytical results [5,11] directly indicate that self-trapping of light can be expected due to solely parametric interactions; however, the corresponding family of stationary localized solutions of circular symmetry is not known yet, and its stability properties have not been fully understood. We would like to emphasize that for the self-focusing cubic (the so-called Kerr) nonlinearity this problem meets serious difficulties since any self-trapped beam [1] (spatial soliton of circular symmetry) described by the (2+1)-dimensional nonlinear Schrödinger (NLS) equation is *unstable* and displays collapse (see, e.g., [17]). Several physical mechanisms which can suppress or even eliminate such collapse-type (or blowup) instabilities are known, but these mechanisms usually require one to include either nonlocal [18] or effectively dissipative [4] effects.

The purpose of this paper is to present a family of (2+1)-dimensional [(2+1)D] solitons of circular symmetry which can be supported by solely parametric interactions between the fundamental and second-harmonic waves in a diffractive (or dispersive) quadratic medium. It is shown that these $|(2+1)D|$ solitons are stable in a physically important region of parameters forming a family of self-trapped two-wave beams of circular symmetry. However, we also find the parameter region where the (2+1)D solitons are unstable and show that the development of this instability can have two different scenarios.

We consider interaction of the first ($\omega_1 = \omega$) and second ($\omega_2 = 2\omega$) harmonics in a diffractive dielectric medium with $\chi^{(2)}$ nonlinear susceptibility, and, assuming the harmonic envelopes E_1 and E_2 to be slowly varying, derive from Maxwell's equations the system of two nonlinear equations coupled parametrically through components $\chi^{(2)}_{ijk}$ of the nonlinear susceptibility tensor,

$$\begin{aligned} 2ik_1\frac{\partial E_1}{\partial Z} + \nabla_t^2 E_1 + \chi_1 E_1^* E_2 e^{-i\Delta k Z} &= 0, \\ 4ik_1\frac{\partial E_2}{\partial Z} + \nabla_t^2 E_2 + \chi_2 E_1^2 e^{i\Delta k Z} &= 0, \end{aligned} \tag{1}$$

where $\nabla_t^2 \equiv \partial^2/\partial X^2 + \partial^2/\partial Y^2$, the coefficients χ_1 and χ_2 characterize the nonlinear (quadratic) response of the medium, Z is the propagation distance, and $\Delta k \equiv 2k_1 - k_2 = 2[n(\omega) - n(2\omega)]\omega/c$ is the wave-vector mismatch between the harmonics, where $n(\omega)$ and $n(2\omega)$ are refractive indices of an optical medium for the first and the second harmonics, respectively. A similar model can be derived to include the effect of dispersion on (1+1)-dimensional spatial solitons, i.e., for the so-called light bullets [19] in $\chi^{(2)}$ planar waveguides.

We are interested now in stationary, spatially localized solutions of Eqs. (1) and apply the following exact transformation: $E_1 = (2k_1\beta/\sqrt{2\chi_1\chi_2})we^{i\beta Z}$ and

$E_2 = (2k_1\beta/\chi_1)ve^{i(2\beta+\Delta k)Z}$. Now the equations for w and v take the form (cf. [14])

$$i\frac{\partial w}{\partial \zeta} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - w + w^* v = 0,$$
$$i\sigma\frac{\partial v}{\partial \zeta} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \alpha v + \frac{1}{2}w^2 = 0, \quad (2)$$

where $\alpha \equiv (2\beta + \Delta k)\sigma/\beta$, $\zeta = \beta Z$, $x = \sqrt{2k_1\beta}\, X$, and $y = \sqrt{2k_1\beta}\, Y$. Equations (2) are the exact reduction of the system (1) and they take into account explicitly the nonlinearity-induced shift β of the propagation constant. We should note that for the spatial solitons discussed in this paper $\sigma = 2$ since for deriving Eqs. (1) we have assumed $\Delta k/k_1 \ll 1$. The system (2) with a different definition of α can also be derived [in a way similar to that shown in [14] for the (1+1)D problem] for a more general case when a small spatial walk-off effect between the harmonics is taken into account. This makes the corresponding stationary solutions and the linear stability analysis of these two problems (i.e., with and without the walk-off effect) mathematically identical, in spite of the fact that physically the beam dynamics and beam generation are expected to be different (see, e.g., [16]).

For the stationary solutions of circular symmetry we omit in Eqs. (2) the derivatives in ζ and treat the real envelopes $w(r)$ and $v(r)$ with $r = \sqrt{x^2 + y^2}$ as two coordinates in the corresponding mechanical problem with the potential $U(w, v) = \frac{1}{2}(w^2 v - \alpha v^2 - w^2)$ and the effective "anisotropic" dissipation $\sim (1/r)(\dot{w}, \dot{v})$ where $\dot{w} \equiv dw/dr$ and $\dot{v} \equiv dv/dr$. Then spatially localized solutions of Eq. (2) correspond to special (separatrix) trajectories in the phase space $(w, v, \dot{w}, \dot{v})$ which start at the point $(w_0, v_0, 0, 0)$ at $r = 0$ and approach asymptotically the point $(0, 0, 0, 0)$ for $r \to \infty$. We have found these separatrix trajectories and therefore soliton profiles numerically by the shooting technique for various values of α, and the corresponding results are summarized in Fig. 1. As can be seen in Fig. 1(a), (2+1)D solitons exist for any value of $\alpha > 0$ and the maximum amplitudes of the fundamental, w_0, and second-harmonic, v_0, waves grow monotonically with increasing α. When $\alpha \ll 1$, the amplitude of the second-harmonic wave becomes larger than that of the fundamental harmonic [see Fig. 1(b)]. In the other limit $\alpha \gg 1$ the amplitude w_0 of the first harmonic is much larger than v_0 [see Fig. 1(c)]. This situation resembles the case of the (1+1)D solitons discussed in [13] and it can also be explained quantitatively. Indeed, for $\alpha \gg 1$ we can neglect the derivatives in the second equation of the system (2) and obtain approximately the relation $v \approx w^2/2\alpha$, reducing the first equation of the system (2) to a standard (scalar) NLS equation which is usually analyzed in the theory of self-focusing in a Kerr medium [1]. However, as is shown below, in the limit $\alpha \gg 1$ the solitons of the model (2) are stable, in contrast to the solitons of the (2+1)D NLS equation which display a blowup instability (see also [20]).

To analyze the stability of the parametric (2+1)D solitons presented in Fig. 1 we can use the well known criterion of soliton stability theory [21]. In the original form

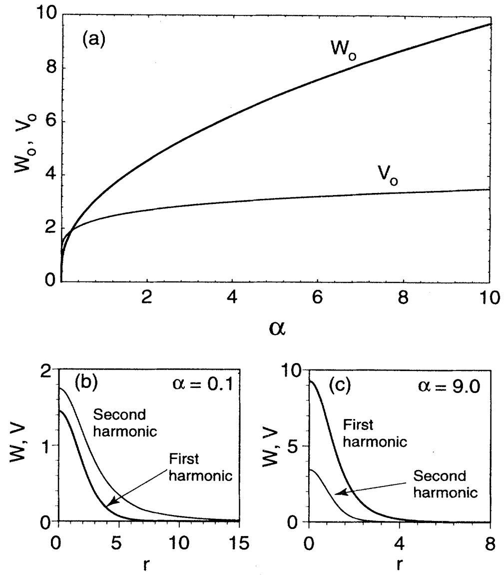

FIG. 1. (a) Family of two-wave (2+1)-dimensional solitons of circular symmetry characterized by the harmonics amplitudes w_0 and v_0 vs the effective parameter α. (b), (c) Two particular examples of the soliton profiles at $\alpha = 0.1$ and $\alpha = 9.0$, respectively.

this approach has been proposed for NLS-type models, and it is based on the analysis of the behavior of a system Hamiltonian H as a function of the system energy, which is $Q = \int_{-\infty}^{\infty}(|w|^2 + 2\sigma|v|^2)dxdy$ for the model (2). If a system of equations has fixed parameters, then for a family of stationary soliton solutions the stability properties change at a critical point in the dependence $H(Q)$, so that this point determines the instability threshold. However, for the model (2) we have obtained the family of soliton solutions presented in Fig. 1 by *varying* the parameter α. Thus, we do not have a family of soliton solutions for a system of equations with fixed parameters. To be able to use the criterion of soliton stability theory [21], we should renormalize the system (2) and its stationary soliton solutions to make α be an internal solution parameter, but not the parameter of the system (2) itself. It can be done by using the following scaling transformations:

$$\tilde{w} = a^2 w(ax, ay, a^2\zeta)e^{ia^2\zeta},$$
$$\tilde{v} = a^2 v(ax, ay, a^2\zeta)e^{2ia^2\zeta \pm i\zeta}, \quad (3)$$

where we have "$-$" and $a \equiv \sqrt{\sigma/(2\sigma - \alpha)}$ for $\alpha < 2\sigma$ or "$+$" and $a \equiv \sqrt{\sigma/(\alpha - 2\sigma)}$ for $\alpha > 2\sigma$. [The critical case of exact phase matching (or $\alpha = 2\sigma$, in our notation) has already been investigated in [20], and the stability of the corresponding solitons has been proven there.] The transformations (3) change the energy invariant Q and Hamiltonian H of the system (2). For the family of stationary soliton solutions of circular symmetry shown in Fig. 1 the values of the renormalized invariants $\tilde{Q}$ and $\tilde{H}$ can be calculated using the following expressions:

$$\tilde{Q} = 2\pi a^2 \int_0^\infty (w^2 + 2\sigma v^2) r dr,$$
$$\tilde{H} = \pi a^4 \int_0^\infty \left[\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 - w^2 v \right] r dr. \quad (4)$$

The dependence $\tilde{H}$ vs $\tilde{Q}$ for the family of (2+1)D solitons discussed in this paper is shown in Fig. 2. Using the standard reasoning of soliton stability theory [21] or an exact stability analysis analogous to that presented for the (1+1)-dimensional problem in [22] we obtain the result that the upper (dashed) branch of the curve $\tilde{H}(\tilde{Q})$ corresponds to unstable solitons whereas the lower (solid) branch corresponds to stable solitons. We have carried out direct numerical simulations which confirmed that this statement is correct. The same result follows from the analysis of the growing modes in the equations linearized around the corresponding soliton solution. Thus we reveal that the parametric (2+1)D solitons of the family shown in Fig. 1 are unstable provided $\alpha < \alpha_{cr} \approx 0.38$ [at $\sigma = 2.0$ in Eqs. (2)], but stable otherwise. It is important to note that there is a threshold value $\tilde{Q}_{cr} = \tilde{Q}(\alpha_{cr})$ of the normalized energy $\tilde{Q}$ which should be exceeded to generate such (2+1)D solitons (see Fig. 2), i.e., in fact, to observe self-trapping of light beams due to solely parametric interactions.

For unstable solitons which exist for $\alpha < \alpha_{cr} \approx 0.38$, small perturbations initially grow exponentially fast, but later nonlinear effects become important. To analyze the nonlinear regime of the instability development we have carried out direct numerical modeling analyzing the evolution of slightly perturbed solitons with the values of α which belong to the unstable (dashed) branch shown in Fig. 2. We have found that there are generally two very different scenarios of unstable soliton evolution. "Negative" small initial perturbations (which slightly decrease the energy Q of an unperturbed soliton) lead to the fast decay and disintegration of two-wave solitons [see Fig. 3(a)]. For "positive" small initial perturbations (which

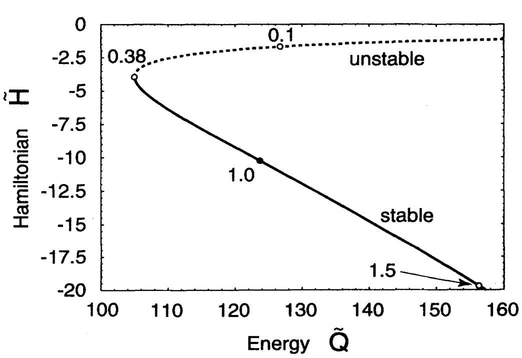

FIG. 2. Renormalized Hamiltonian $\tilde{H}$ vs energy $\tilde{Q}$ calculated at $\sigma = 2$ for the family of (2+1)D solitons presented in Fig. 1. Solid branch corresponds to stable solitons ($\alpha > \alpha_{cr} \approx 0.38$) and dashed branch to unstable solitons. Numbers and open circles show the corresponding values of α. Filled circle at $\alpha = 1$ indicates the particular case considered in [5].

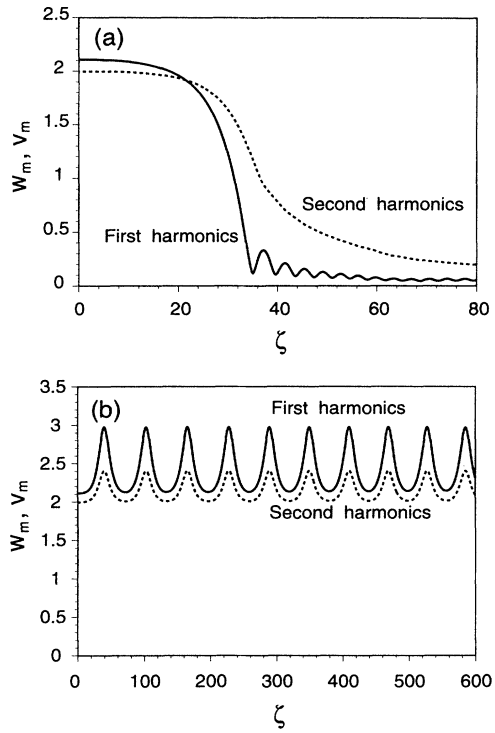

FIG. 3. Two scenarios of the evolution of slightly perturbed unstable solitons at $\alpha = 0.3$. The maximum of the first harmonic amplitude is shown by the solid curve and the maximum of the second harmonic amplitude is shown by the dashed curve. (a) The initial perturbation slightly decreases the energy Q of the unperturbed soliton. (b) The initial perturbation slightly increases the energy Q of the unperturbed soliton.

slightly increase the energy Q of an unperturbed soliton) the initial exponential growth of the amplitudes of both harmonics saturates due to nonlinearity and finally large in-phase oscillations of the amplitudes of the two components are observed. These oscillations are accompanied by periodic beam compressions with a very small amount of radiation emitted [for the case shown in Fig. 2(b) $Q(\zeta = 600) > 0.999Q(\zeta = 0)$]. The large-amplitude beats of two harmonics are observed around certain averaged amplitudes $\bar{w}_m$ and $\bar{v}_m$ which, as can be easily checked with the help of the dependencies presented in Fig. 1, correspond to a stable (2+1)D soliton. Indeed, for the case shown in Fig. 3(b) we have $\bar{w}_m \approx 2.53$ and $\bar{v}_m \approx 2.19$ which correspond to the effective $\alpha \approx 0.49$ (> 0.38). The detailed analysis of such "oscillating optical solitons" will be presented elsewhere for both (1+1)- and (2+1)-dimensional problems.

We would like to note that recently multidimensional parametric solitons have been discussed by Hayata and Koshiba [5] where an approximate soliton solution has been found with the help of a Hartree-type ansatz for certain relations between the system and solution parameters. Our simple analysis indicates that the case discussed in [5] corresponds to the particular point $\alpha = 1$ (see Fig. 2) of the soliton family found in this work. As can be seen from the stability analysis, this value of α

does correspond to stable solitons of the model (2) at $\sigma = 2.0$ as was correctly pointed out in [5].

Experimental verification of (2+1)-dimensional self-trapping of light beams in $\chi^{(2)}$ materials and parametric spatial solitons can be based on the conventional method of phase-matched SHG, similar to the recent experimental observation of (1+1)D solitons in $LiNbO_3$ waveguides [23]. Because the (2+1)D parametric spatial solitons of circular symmetry have been shown to exist, being stable in a wide region of the parameters of the system (2), they can be generated by launching a monochromatic Gaussian beam in the direction of the phase matching to generate a second-harmonic beam. In the low-power regime, this leads to standard SHG in a bulk medium; however, above a certain power threshold, one can expect to observe mutual self-trapping of the fundamental and second-harmonic beams, similar to that recently reported in numerical simulations [16]. The results of the stability analysis presented above also give a simple sufficient condition for the existence of stable (2+1)D solitons: $\Delta k \geq 0$, i.e., $n(\omega) \geq n(2\omega)$.

In general, the system (2) is not integrable and it does not possess translational (Galilean) invariance. However, we note that Eqs. (1) [and Eqs. (2) at $\sigma = 2$] have the property of Galilean invariance and moving solitons can be obtained by means of the simple gauge transformation

$$\begin{aligned} E_1(\mathbf{R}, Z) &\rightarrow E_1(\mathbf{R} - \mathbf{V}Z, Z)\, e^{i[k_1(\mathbf{V},\mathbf{R}) - \frac{1}{2}k_1\mathbf{V}^2 Z]}, \\ E_2(\mathbf{R}, Z) &\rightarrow E_2(\mathbf{R} - \mathbf{V}Z, Z)\, e^{i[2k_1(\mathbf{V},\mathbf{R}) - k_1\mathbf{V}^2 Z]}, \end{aligned} \tag{5}$$

where $\mathbf{R} = (X, Y)$ is the transverse coordinate and the "velocity" parameter $\mathbf{V} = (V_X, V_Y)$ characterizes the beam displacement in the transverse plane. Using this property we investigate numerically collisions between two (2+1)D solitons of circular symmetry in the framework of Eqs. (2). In particular, we reveal that these collisions depend strongly on the initial relative phase between the solitons, and two examples of such collisions are presented in Figs. 4(a) and 4(b) in the form of contour plots. The corresponding evolution of maxima of

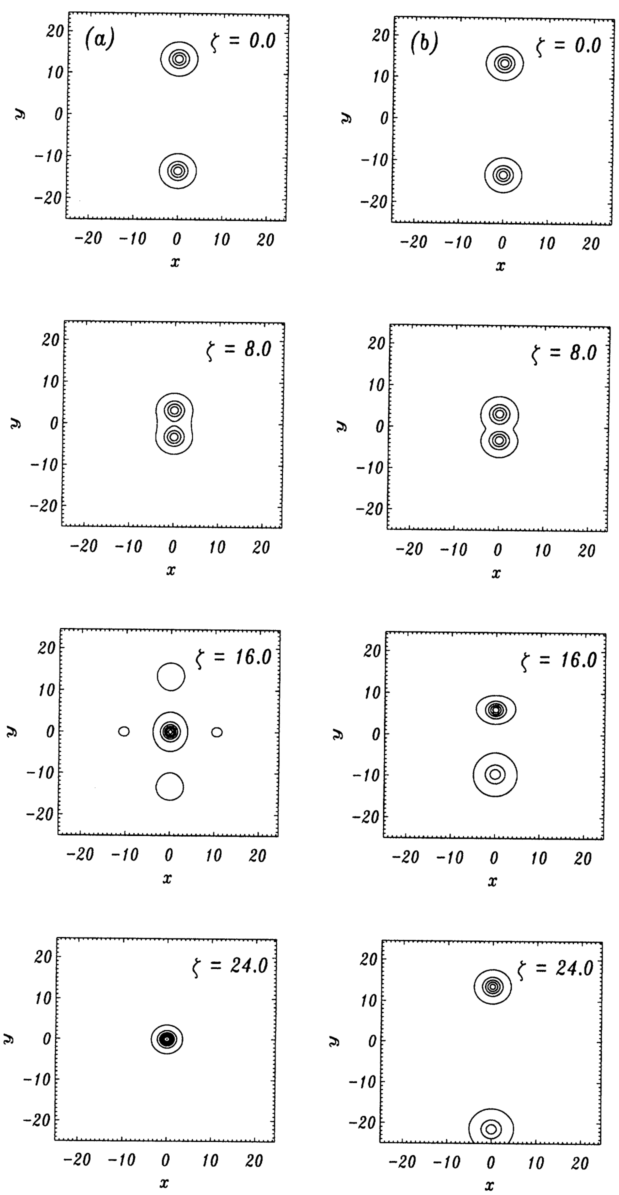

FIG. 4. Collisions of (2+1)D solitons of equal amplitudes (at $\alpha = 4$) for the initial relative phase (a) $\phi = 0$ and (b) $\phi = \pi/2$. Only the first harmonic is shown.

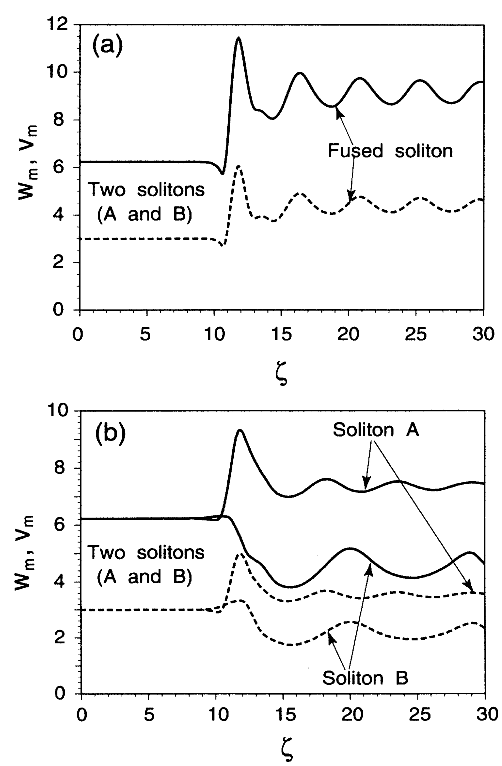

FIG. 5. The same as Fig. 4, but for the dynamics of the soliton amplitude maxima of the first (solid curves) and second (dashed curves) harmonics.

both soliton amplitudes for the first and second harmonics is shown in Figs. 5(a) and 5(b), respectively. When the relative phase of two colliding solitons is zero, the solitons attract each other and finally fuse into a single (2+1)D soliton of larger amplitude [see Figs. 4(a) and 5(a)]. The amplitude of this "fused soliton" oscillates similarly to that shown in Fig. 3(b). However, when the relative phase of two colliding solitons is $\pi/2$, the interaction between them is repulsive, and both solitons, after exchanging some energy, still survive in the collision as shown in Figs. 4(b) and 5(b). However, the soliton amplitudes after the collisions also oscillate. Such phase-dependent collisions of two (2+1)D beams can be useful for soliton-based switching in a bulk $\chi^{(2)}$ medium.

In conclusion, we have shown that the self-trapping of light beams in a diffractive $\chi^{(2)}$ medium leads to the existence of a family of two-wave (2+1)D solitons of circular symmetry which have been shown to be stable in a physically important region of parameters: $\sigma = 2$, $0.38 < \alpha < \infty$ (i.e., near $\alpha = 4$ which corresponds to exact phase matching between the harmonics). The stability of these solitons has been determined by analyzing the diagram $\tilde{H}(\tilde{Q})$, and has been verified by direct numerical simulations and by investigating the instability mode evolution of the corresponding linearized problem. We have also analyzed collisions between the (2+1)D solitons, which have been found to be strongly phase dependent and, in some cases, nondestructive. The approach and results obtained can be readily applied to other models of different physical context where resonant parametric interactions between waves are generated by quadratic nonlinearities.

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