Dynamical properties of a classical-like entropy in the Jaynes-Cummings model

Arkadiusz Orłowski
Instytut Fizyki, Polska Akademia Nauk, Aleja Lotników 32/46, 02-668 Warszawa, Poland

Harry Paul and Georg Kastelewicz
Arbeitsgruppe "Nichtklassische Strahlung" der Max-Planck-Gesellschaft an der Humboldt-Universität zu Berlin, Rudower Chaussee 5, 12489 Berlin, Germany

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The concept of the classical-like Wehrl entropy is shown to be a very sensitive and informative measure describing the time evolution of a quantum system. Its applications to the Jaynes-Cummings model reveal relevant aspects of the field dynamics. The Wehrl entropy provides a compact one-parameter description giving, among other things, a clear signature for the splitting of the Q function. It gives also a clear indication of the phase randomization of the coherent electromagnetic field states during the interaction with a two-level atom.

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I. INTRODUCTION

Solvable models in quantum theory are so rare that they are worth studying in their own right. Even simplified they provide a clear understanding of the physical phenomena involved and can be useful in controlling various approximations indispensable for the treatment of more realistic cases. In quantum optics, the Jaynes-Cummings model (JCM), where a single two-level atom is resonantly coupled to a single mode of the quantized electromagnetic field of a lossless cavity, is probably the best example of this sort [1–3]. In the dipole and rotating wave approximations, the dynamics of this model is governed by the Hamiltonian

$$\hat{H} = \hbar\frac{\omega}{2}\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+), \tag{1}$$

where the boson operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the electromagnetic field and λ is a coupling constant assumed to be real, without loss of generality. Operators $\hat{\sigma}_i$ $(i = 3, \pm)$ are the Pauli matrices. In the following we set $\hbar = 1$ for convenience.

In spite of its apparent simplicity, the JCM exhibits a quite complicated behavior and fully quantum-mechanical effects such as the vacuum Rabi oscillations, sub-Poissonian statistics, and squeezing of the radiation field (see, e.g., [2,3] and references therein). The JCM is also an important ingredient of the micromaser theory and, because of the recent progress in experiments with single Rydberg atoms in high-Q micromaser cavities, it is now possible to test theoretical predictions with great accuracy [4]. Many details of the JCM dynamics strongly depend on the initial conditions, i.e., on states in which the field and the atom are prepared at the beginning. It is interesting to note that most of the above-mentioned nonclassical effects are particularly transparent when the field is initially prepared in a coherent state [5]

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle \ . \tag{2}$$

Coherent states are known to be as close as possible to classical states: quantum fluctuations are minimal in these states and equal to those of the vacuum. Just in this case we have very striking collapses and revivals of the atomic inversion and the state preparation effect by the quantum apparatus [6]. Also squeezing of quantum fluctuations of the quadrature operators can be observed when the field is initially in a coherent state. Moreover, the squeezing increases with the increasing coherent component of the initial electromagnetic field [7]. In contrast, if the field is prepared in a photon-number state (Fock state), being an eigenstate of the photon-number operator $\hat{N}$, the inversion exhibits a completely periodic behavior. The latter is equivalent to that of the mean photon number of the radiation field, as the operator $\hat{N} + \frac{1}{2}\hat{\sigma}_3$ is the constant of motion. Although the atomic inversion is very illustrative and can be easily handled, it contains only a restricted amount of information about the system dynamics. It was advocated by Risken and collaborators [8–11] that the behavior of this parameter is connected with the phase-space dynamics of the Q representation of the radiation field $\hat{\rho}$, namely,

$$Q(\alpha, \alpha^*; t) = \langle\alpha|\hat{\rho}(t)|\alpha\rangle \ . \tag{3}$$

Let us note that, for simplicity, we use a rescaled Q representation as compared to that defined by Eisett and Risken in Ref. [10]. In the case of an initially inverted atom they observed that the Q representation splits into two parts rotating in opposite directions. They found that revivals occur at the same time when these two parts collide. Matsuo [12] investigated the Q function in even more detail. He found that this function can be divided into clockwise, counterclockwise, and nonrotating terms. It happens that just the last term is closely connected with properties of the atomic inversion: it appears when the inversion oscillates and vanishes during collapse times. The influence of the initial atomic state on the splitting of the Q function has also been investigated: atomic coherences can cause asymmetrical splitting [12]. In fact, it is possible to suppress the splitting completely by the proper choice of the initial atomic states [13].

The Q representation, when known as an analytical function, contains all information about the dynamics of the electromagnetic field in the JCM, being completely equivalent to the density operator $\hat{\rho}(t)$. From the knowledge of the Q function we can recover all matrix elements of the density operator in an orthonormal basis provided, e.g., by the photon-number states [14]. It is, however, rather complicated to use it in practice: for careful monitoring of the time evolution of a quantum system we would need a real function of two variables (or a three-dimensional picture) for any time moment t. There are no doubts that the existence of a reasonably simple parameter that nevertheless still contains enough information to display many of the rich manifold of nontrivial features of the system is desirable.

It is the main objective of this paper to show that *there is* indeed an interesting *single quantity*, namely, the Wehrl entropy [15], which contains in a condensed, one-parameter form essential information about the field dynamics. We show that this entropic parameter, first introduced as a classical entropy of a quantum state, gives additional insight into the field dynamics, as compared to other entropies. It will be clearly seen below that the Wehrl entropy is completely different from the other well-known entropic concepts such as, e.g., the quantum-mechanical von Neumann entropy [16]

$$S_Q(t) = -\mathrm{Tr}[\hat{\rho}(t)\ln\hat{\rho}(t)] \tag{4}$$

and the Shannon information entropy for the photon-number distribution [17]

$$S_I = -\sum_{n=0}^{\infty} p(n;t)\ln p(n;t), \tag{5}$$

where $p(n;t) = \langle n|\hat{\rho}(t)|n\rangle$. Both entropies (4) and (5) were extensively used to study the field dynamics of the JCM [18], but both have some deficiencies as compared to the Wehrl entropy: the von Neumann entropy measures the purity of quantum states, being different from zero only for mixed states. The Shannon entropy for the photon number uses only diagonal elements $p(n;t) = \langle n|\hat{\rho}(t)|n\rangle$ of the density operator in the Fock basis and contains no phase information, although it can provide some useful information about the behavior of the photon-number distribution.

This paper is organized as follows. In Sec. II we introduce the concept of the Wehrl entropy and present its basic properties, including the Wehrl-Lieb inequality and another inequality that gives an interesting relationship between the Wehrl entropy and the von Neumann entropy. In Sec. III the time evolution of the Wehrl entropy in the Jaynes-Cummings model is studied in some detail. Different initial states of the field and the atom are considered. The dynamical behavior of the Wehrl entropy is compared to the time evolution of the atomic inversion, the von Neumann entropy, and the Shannon information entropy (in some cases the last two entropies are equal to each other). The usefulness of the Wehrl entropy as a compact but nevertheless informative one-parameter measure describing the field dynamics is clearly demonstrated. We conclude in Sec. IV with some remarks and comments.

II. WEHRL ENTROPY

In full analogy to the classical entropy in phase space, the Wehrl entropy is defined as

$$S_W(t) = -\frac{1}{\pi}\int Q(\alpha,\alpha^*;t)\ln Q(\alpha,\alpha^*;t)d^2\alpha. \tag{6}$$

There are important reasons that a classical phase-space probability distribution function is replaced in the above expression just by the Q function of the radiation field and not by other quasiprobabilities. It is known that there exist many different quasiprobability distribution functions over quantum-mechanical phase space [19]. They are closely related to the existence of different operator orderings of the noncommuting operators in quantum mechanics. The Wigner function [20], connected with the symmetrical ordering, is known to be very useful in many branches of physics, but it can take on negative values. The Glauber-Sudarshan P representation of the density operator [5,21], connected with the normal ordering, is singular for relevant pure states such as Fock states [24] and can take on negative values too. Having many advantages, e.g., providing a formal equivalence between classical and quantum coherence theories [22,23], it cannot be used to define entropy. In contrast, the Q representation is always a positive and well-behaved function. It is normalizable

$$\frac{1}{\pi}\int Q(\alpha,\alpha^*;t)d^2\alpha = 1\,, \tag{7}$$

but it does not possess correct marginal properties imposed on any distribution pretending to be a true phase-space probability distribution in quantum mechanics (see, e.g., [19] and references therein). Indeed, the Q function gives broader marginal distributions than actual quantum-mechanical expressions [25]. Therefore it is still only a *quasidistribution.* Despite this, it has an appealing physical meaning in the case of the so-called simultaneous measurements of noncommuting quantities: under certain conditions, the Q function can be interpreted as the joint probability distribution for the simultaneous (noisy) measurements of the two field quadratures [26–28]. In this context, the Wehrl entropy can be interpreted as being an information measure for such a joint measurement.

Let us note that the Wehrl entropy cannot be negative. It follows from the properties of the Q representation, namely, from the fact that $0 \leq Q(\alpha) \leq 1$ and from the normalization condition (7). The Q function can never be so concentrated as to make S_W negative. On the contrary, classical distributions can be arbitrarily concentrated in phase space and classical entropies can take on negative values. They may even tend to $-\infty$ if the distributions tend to δ functions. Moreover, Wehrl [15] proved the even stronger relationship

$$S_W(t) \geq S_Q(t), \tag{8}$$

which establishes a connection between the Wehrl and the von Neumann entropies of a given state.

However, the most important property of this classical-like entropy is perhaps the inequality

$$S_W(t) \geq 1\,, \tag{9}$$

first conjectured by Wehrl [15] and then proved by Lieb [29]. The equality holds if and only if the considered state $\hat{\rho}(t)$ is a coherent state. It is straightforward to check that for coherent states $S_W = 1$. The proof that it corresponds to the global minimum is, however, rather complicated and requires some nontrivial background in functional analysis. The Wehrl entropy clearly distinguishes coherent states and has been recently used by one of the present authors as a measure of the statistical properties of quantum states of light [30]. Indeed it is a good measure of the strength of the coherent component in a given quantum state, i.e., it measures how much "coherence" a given state has. As such it can be used to classify quantum states with respect to their statistical properties. We will see in the following that also the time evolution of the Wehrl entropy gives some insight into important details of the field dynamics governed by the Hamiltonian (1). The Wehrl entropy clearly indicates the times of the revivals of the atomic inversion. Moreover, it also gives indication of other interesting problems such as, e.g., randomization of phase due to the spontaneous emission.

First, we want to elucidate additional basic features of the Wehrl entropy from which its usefulness for the description of the dynamics in question becomes evident. We will study the change the Wehrl entropy undergoes in the processes of spreading (expansion) and splitting of the Q function.

To study the effect of spreading of the Q representation on the Wehrl entropy in its simplest form, we choose the Q function as a Gaussian centered at the origin of the phase space

$$Q(\alpha,\alpha^*) = \frac{1}{a}\exp\left(-\frac{1}{a}|\alpha|^2\right), \tag{10}$$

where the (positive) constant a is bounded according to the inequality $a \geq 1$. The equality sign corresponds to the vacuum state. Since it is a special case of the coherent state, its Wehrl entropy equals 1 (see above). For the Q function (10) the Wehrl entropy is readily calculated to be

$$S_W = 1 + \ln a. \tag{11}$$

From this formula it becomes obvious that spreading of the Q function is reflected by a smooth increase of S_W.

To study the effect of splitting, we start from a Q function that is nicely localized in a certain phase-space region. Let us now assume that this Q function splits, as a result of a certain interaction, into N similar parts Q_i ($i = 1, 2, \ldots, N$) well separated from each other and being all of the same shape as the initial Q representation. Such behavior can be observed, e.g., during Schrödinger cat state formation in the Kerr media [31]. Before splitting, the Wehrl entropy is given by Eq. (6), while after splitting it takes the form

$$\tilde{S}_W = -\frac{1}{\pi}\sum_{i=1}^{N}\int Q_i(\alpha,\alpha^*)\ln Q_i(\alpha,\alpha^*)d^2\alpha, \tag{12}$$

where the assumption of separation is used. It is geometrically evident that the integrals in the above equation are all equal. Their value is readily found, from the requirement of correct normalization of the Q function, to be given by Eq. (6) with Q substituted by $\frac{1}{N}Q$

$$\begin{aligned} S_W^i &= -\frac{1}{N\pi}\int Q_i(\alpha,\alpha^*)[\ln Q_i(\alpha,\alpha^*) - \ln N]d^2\alpha \\ &= \frac{1}{N}S_W + \frac{1}{N}\ln N, \end{aligned} \tag{13}$$

where the normalization condition (7) has been observed. Hence Eq. (12) leads us to the simple result

$$\tilde{S}_W = S_W + \ln N, \tag{14}$$

which indicates that the Wehrl entropy exhibits a jump whose height is simply given by the logarithm of the number of parts into which the Q function splits (without changing its shape).

So we can state that the Wehrl entropy exhibits two basic features: diffusion of the Q function, in the sense of spreading over larger regions in phase space, gives rise to a monotonic, smooth increase of S_W, whereas splitting of the Q function leads to a jumplike increase of S_W. Although the above-mentioned effects will not appear in the pure form in the JCM, we will find some more involved counterparts of both types of these processes reflected in the Wehrl entropy when studying the field evolution in this model.

III. ILLUSTRATIVE EXAMPLES

In this section we investigate the dynamics of the Wehrl entropy of the electromagnetic field in the JCM for different initial conditions. Especially striking and informative behavior is observed when the field is initially in a coherent state, but also in other cases we find some interesting features.

It is well known that the JCM can be solved analytically with arbitrary initial conditions. Let us assume that at $t = 0$ the atom is prepared in an arbitrary pure state, i.e., as a certain coherent superposition of the ground state $|g\rangle$ and the excited state $|e\rangle$

$$|A\rangle = \cos\theta|e\rangle + e^{-i\phi}\sin\theta|g\rangle. \tag{15}$$

We assume that the density operator describing the whole system at time $t = 0$ is decorrelated, i.e., it can be written as a direct product of the initial density operators for the field and the atom, respectively,

$$\hat{\rho}(0) = \hat{\rho}_F(0) \otimes |A\rangle\langle A|. \tag{16}$$

The time evolution of the field density operator $\hat{\rho}_F(t)$ is then described by

$$\hat{\rho}_F(t) = \hat{C}\hat{\rho}_F(0)\hat{C}^\dagger + \hat{S}\hat{\rho}_F(0)\hat{S}^\dagger, \tag{17}$$

where the time-dependent operators $\hat{C}$ and $\hat{S}$ are given by

$$\hat{C} = \cos\theta\cos\left(\lambda t\sqrt{\hat{a}\hat{a}^\dagger}\right) - ie^{-i\phi}\sin(\theta)\hat{a}\frac{\sin(\lambda t\sqrt{\hat{a}^\dagger\hat{a}})}{\sqrt{\hat{a}^\dagger\hat{a}}}, \tag{18a}$$

$$\hat{S} = e^{i\phi}\sin\theta\cos\left(\lambda t\sqrt{\hat{a}^\dagger\hat{a}}\right) - i\cos(\theta)\hat{a}^\dagger\frac{\sin(\lambda t\sqrt{\hat{a}\hat{a}^\dagger})}{\sqrt{\hat{a}\hat{a}^\dagger}}. \tag{18b}$$

In the above expressions we corrected a misprint overlooked in the analogous expressions in Ref. [12]. The Q representation of the field is simply given by Eq. (3) with $\hat{\rho}(t)$ replaced by $\hat{\rho}_F(t)$.

A. Field initially in a coherent state

In this subsection we assume that the field is at the beginning in a coherent state $|\alpha_0\rangle$

$$|\alpha_0\rangle = e^{-\frac{1}{2}|\alpha_0|^2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle. \tag{19}$$

The parameter α_0 is assumed to be real. It causes no loss of generality because any phase of the field can simply be added to the atomic phase ϕ. The mean photon number is then given by $\bar{n} = \alpha_0^2$.

First, let us assume that the atom is initially prepared in the excited state $|e\rangle$. This case was extensively studied and there are many interesting results available, especially in the limit of the large mean photon number [8–11,13]. To describe the field evolution it is very useful to investigate the phase-space behavior of the Q representation. In the considered case, the initial well-localized Gaussian Q function splits into two parts rotating clockwise and counterclockwise. Their maximal separation takes place at half the revival time. Then they start to collide and overlap again at the revival time. As the Wehrl entropy (6) is defined using the Q representation, we expect that the rich dynamical properties of the latter should somehow be seen in the behavior of the former. In Fig. 1 we show the time evolution of the Wehrl entropy of the field for $\bar{n} = 25$ under the assumption that the atom is initially fully inverted. The dynamical behavior of the Wehrl entropy is compared to the evolution of the atomic

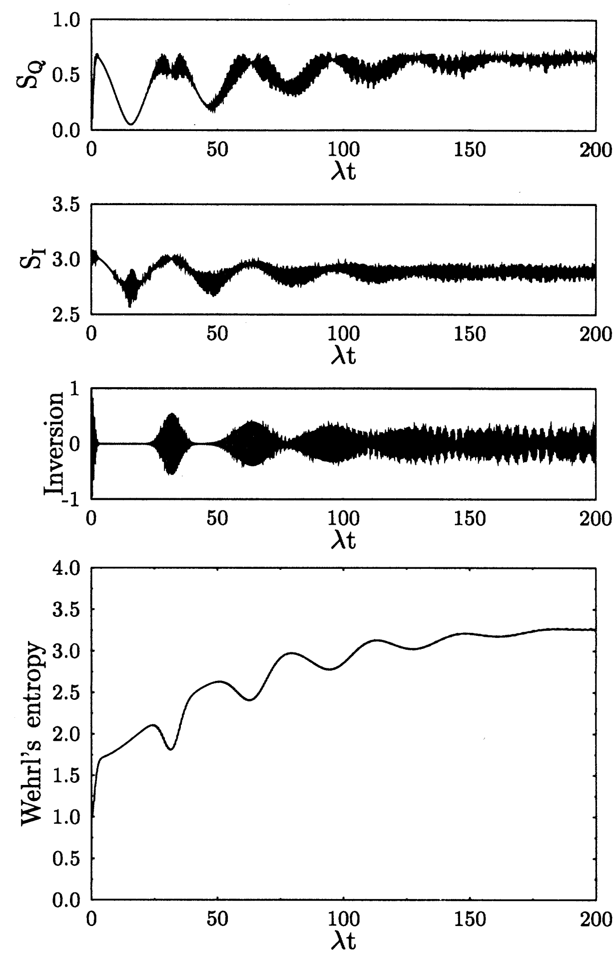

FIG. 1. Time evolution of the Wehrl entropy. The field is initially in the coherent state with $\bar{n} = 25$. The atom is initially in the upper state. The atomic inversion, the Shannon information entropy, and the von Neumann entropy are plotted for comparison.

inversion, the von Neumann entropy, and the Shannon information entropy for the photon-number distribution. The first eye-catching property of the Wehrl entropy is the existence of the pronounced local minima appearing in the middle of the revival times. There also are the well-developed maxima at the times corresponding to the beginning of the revival regions. Thus the essential properties of the atomic inversion are also exhibited by the Wehrl entropy. The above-mentioned modulation imposed on the systematic increase of the Wehrl entropy indicates its sensitivity to the splitting of the Q representation. The von Neumann entropy indicates other aspects of the field dynamics, namely, the degree of the purity of the field state. The first local minimum of the von Neumann entropy, corresponding to the almost pure state, appears approximately in the middle of the collapse region. The Shannon information entropy is in this case quite similar (qualitatively) to the von Neumann entropy.

The sensitivity of the Wehrl entropy to splitting of the

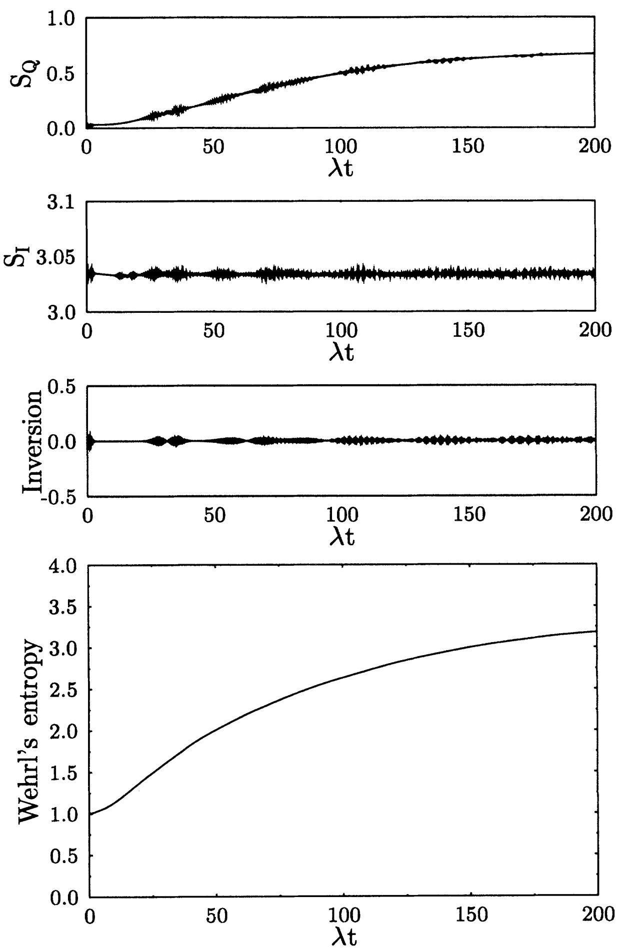

FIG. 2. Time evolution of the Wehrl entropy. The field is initially in the coherent state with $\bar{n} = 25$. The atom is initially in the symmetrical superposition of the upper and the lower levels, corresponding to the eigenstate of the semiclassical Hamiltonian $|+\rangle$. The atomic inversion, the Shannon information entropy, and the von Neumann entropy are plotted for comparison.

Q function is further supported by Fig. 2. As in the previous case, we plot the Wehrl entropy as well as the atomic inversion, the von Neumann entropy, and the Shannon information entropy. Now, however, the atom is initially in a symmetrical coherent superposition

$$|+\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle). \tag{20}$$

Under assumption of real α_0 this state is one of the two eigenstates of the semiclassical JCM Hamiltonian. As shown by Gea-Banacloche in Ref. [13], the Q function of the field does not split in this case. It forms a hump in the phase space, which rotates clockwise. In addition to the rotation it is diffusively deformed and squeezed, but it does not split. Consequently, the behavior of the Wehrl entropy is apparently different from that observed in Fig. 1. It slowly increases without any essential modulation, thus reflecting a diffusionlike evolution of the Q representation. The Q function of the field corresponding to the atomic state

$$|-\rangle = \frac{1}{\sqrt{2}}(|e\rangle - |g\rangle), \tag{21}$$

i.e., to the second eigenstate of the semiclassical Hamiltonian, rotates counterclockwise without splitting and the Wehrl entropy again exhibits the smooth and systematic growth analogous to that presented in Fig. 2. All other quantities plotted in this figure also exhibit rather smooth behavior. The atomic inversion remains close to zero. The von Neumann entropy also rises quite slowly. The reason is again the same: the state (20) is the eigenstate of the semiclassical JCM Hamiltonian and therefore the state of the whole system remains approximately disentangled for a reasonably long time. The Shannon entropy changes only a little and the reason can be traced to another property observed in Ref. [13]. In the limit of the large (real) α_0, i.e., for large values of the mean photon number, the state of the field corresponding to the initial atomic state $|+\rangle$ can be approximated with great accuracy by the (pure) state

$$|\Phi_+\rangle = \exp(-\tfrac{1}{2}\alpha_0^2) \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} \exp(-i\lambda t\sqrt{n})|n\rangle. \tag{22}$$

It is an example of the so-called generalized coherent states [32–34]. It is easily seen that the photon-number distribution of the state (22) does not change in time. Thus it is clear that the Shannon entropy calculated for the photon-number distribution of the state $\hat{\rho}_F(t)$ does not change appreciably if the quality of this approximation is good enough. The considerably flat curve describing the time evolution of the Shannon entropy shows that for $\alpha_0^2 = 25$ it is a very good approximation indeed.

Let us stress that if the amplitude α_0 of the initial coherent state is not real, the atomic states $|\pm\rangle$ are no longer the eigenstates of the semiclassical Hamiltonian. In the general case of the complex $\alpha_0 = |\alpha_0| \exp(i\psi)$ such eigenstates are given by the superpositions $|\psi_\pm\rangle = \exp(i\psi)|e\rangle \pm |g\rangle$. Thus it is clear that if we start evolution from the coherent state of the field with the complex α_0 and from the atomic state $|\pm\rangle$, the Q representation of the field density operator can split during the time evolution. It can be shown that this splitting is asymmetrical [12].

The second essential feature, easily seen from Figs. 1 and 2, is the systematic loss of coherence by the radiation field. It manifests itself by the saturation level of the Wehrl entropy observed for longer times. In all cases corresponding to the field prepared initially in a coherent state, the Wehrl entropy stabilizes at a certain level after a sufficiently long time and then it exhibits only some oscillations with a relatively small amplitude. This amplitude increases in time and for longer times we see some additional quasiperiodic structures. The long-time properties can be better observed in Fig. 3, presenting the behavior of the Wehrl entropy of the field (for parameters corresponding to Fig. 1) at longer times and

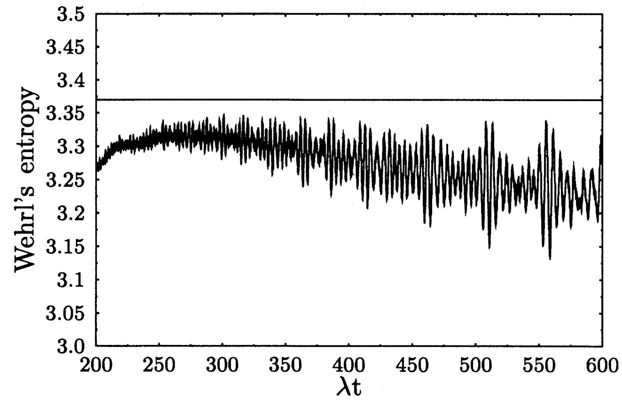

FIG. 3. Long-time behavior of the Wehrl entropy. The parameters are the same as in Fig. 1. The saturation level given by the Wehrl entropy corresponding to the random-phase coherent state with $\bar{n} = 25$ is plotted.

with higher resolution.

Since the coherence properties are very sensitive to phase relations, this lack of coherence should correspond to the randomization or diffusion of the phase. This phase redistribution seems to be irreversible. The saturation limit is a clear limit for chaotic behavior of the field. Using results of Ref. [30] we find that the Wehrl entropy saturates below the level corresponding approximately to that of random-phase coherent states, i.e., the mixed states described by the density operator [30,35,36]

$$\hat{\rho}_F(0) = \exp(-\bar{n}) \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} |n\rangle\langle n|. \tag{23}$$

It is seen that the probability of finding n photons is given by the Poisson distribution, which is exactly the same as for the pure coherent state $|\beta\rangle$ with the mean photon number $|\beta|^2 = \bar{n}$. The Q representation of the random-phase coherent state reads

$$Q(\alpha, \alpha^*) = \exp\big(-|\alpha|^2 - \bar{n}\big) I_0(2|\alpha|\sqrt{\bar{n}}), \tag{24}$$

where I_0 denotes the modified Bessel function. The Wehrl entropy of the random-phase coherent states as a function of the mean photon number was calculated numerically by Orłowski in Ref. [30]. For $\bar{n} = 25$ it is approximately equal to 3.37. The Wehrl entropy of the random-phase coherent state corresponding to $\bar{n} = 25$ is plotted in Fig. 3 to indicate the above-mentioned saturation level.

It is seen from the above figures that the phase randomization is practically irreversible, at least for the periods of time investigated. Let us repeat that the Wehrl entropy enables us to estimate *quantitatively* the degree of this phase randomization. The resulting states of the field cannot be more chaotic than the random-phase coherent states. In particular it proves that the thermalization of the coherent state during its interaction with a two-level atom is not possible: the Wehrl entropies corresponding to the thermal state with the same mean photon numbers lie highly above the saturation levels. For example, the Wehrl entropy of the thermal state with $\bar{n} = 25$ is approximately equal to 4.26.

The Wehrl entropy of the field, calculated under the assumption that the initial atomic state is $|\pm\rangle$, is always smaller than for the atom fully inverted. This is due to the fact that it is possible to introduce additional phase information into the system via atomic coherences. Therefore the saturation limit of the Wehrl entropy is also dependent on the initial atomic state. The maximum is given by the random-phase coherent states.

B. Field initially in a photon-number state

If at the beginning the field is in a pure Fock state so that $\hat{\rho}_F(0) = |n\rangle\langle n|$ and the atom is in the excited state, we get [18]

$$\begin{aligned}\hat{\rho}_F(t) &= \cos^2(\lambda t\sqrt{n+1})|n\rangle\langle n| \\ &\quad + \sin^2(\lambda t\sqrt{n+1})|n+1\rangle\langle n+1|.\end{aligned} \tag{25}$$

Consequently, the Q function reads

$$\begin{aligned}Q(\alpha, \alpha^*; t) = \Bigg(&\frac{|\alpha|^{2n}}{n!} \cos^2(\lambda t\sqrt{n+1}) \\ &+ \frac{|\alpha|^{2(n+1)}}{(n+1)!} \sin^2(\lambda t\sqrt{n+1})\Bigg) e^{-|\alpha|^2}.\end{aligned} \tag{26}$$

The Wehrl entropies plotted in Fig. 4 show the expected behavior: oscillations between the Wehrl entropies of the two Fock states $|n\rangle$ and $|n+1\rangle$. As shown by Orłowski [30], the Wehrl entropy for the photon-number state $|n\rangle$ can by calculated analytically giving

$$S_W = 1 + n + \ln n! + \gamma n - n \sum_{k=1}^{n} \frac{1}{k}, \tag{27}$$

where $\gamma \approx 0.577\,215\,664\,9$ is the well-known Euler constant. There is an interesting aspect of the Wehrl entropy behavior in this case, namely, the flatness at times when the atom tends to be in the ground state. It would be in-

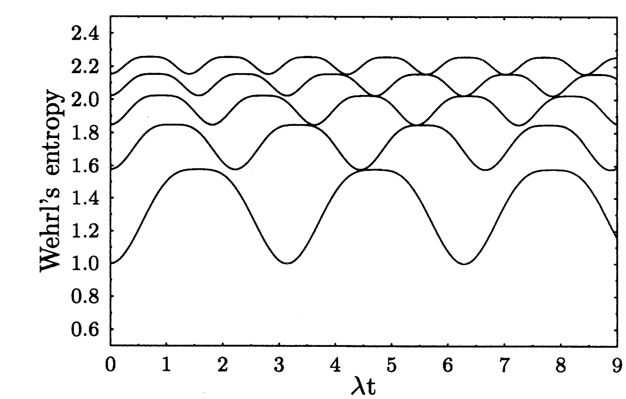

FIG. 4. Time evolution of the Wehrl entropy. The field is initially in different Fock states $|n\rangle$: $n = 0, 1, 2, 3,$ and 4 (from the bottom). The atom is initially fully inverted.

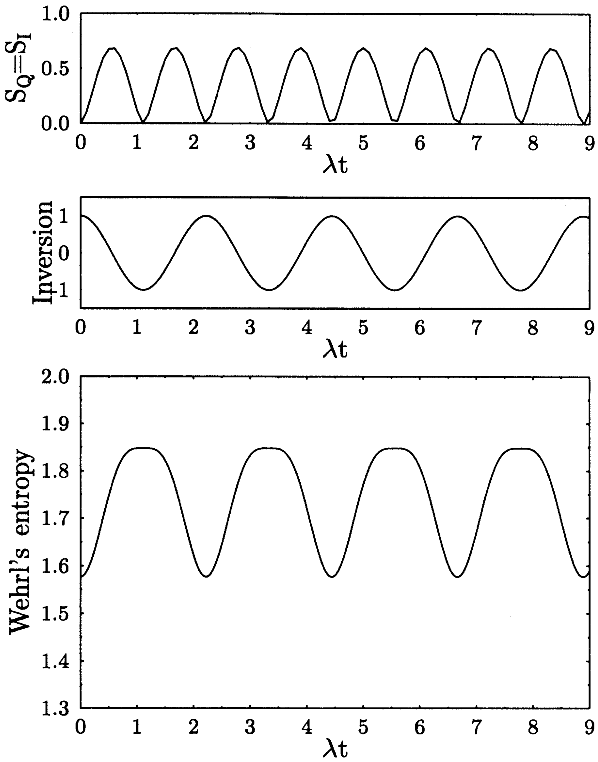

FIG. 5. Time evolution of the Wehrl entropy. The field is initially in the pure one-photon state $|1\rangle$. The atom is initially fully inverted. The atomic inversion and the von Neumann entropy (equal in this case to the Shannon information entropy) are also plotted.

teresting to elaborate in further investigations whether it is only a mathematical artifact or there is some physical explanation.

If the field is initially in a pure photon-number state, the von Neumann entropy and the Shannon information entropy are always equal. Together with the atomic inversion they show the expected periodic behavior. In Fig. 5 we plot the Wehrl entropy, the atomic inversion, and the von Neumann entropy (equal to the Shannon information entropy) in the case of the field prepared initially in the one-photon state $\hat{\rho}_F(0) = |1\rangle\langle 1|$. The Wehrl entropy and the atomic inversion oscillate with the same frequency but with different locations of the minima and the maxima. The von Neumann entropy oscillates with the double frequency as compared to S_W.

C. Field initially in a thermal state

The electromagnetic field prepared at time $t = 0$ in a thermal state is described by the density operator

$$\hat{\rho}_F(0) = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|, \tag{28}$$

where the initial photon-number distribution is given by the Bose-Einstein distribution

$$p_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}. \tag{29}$$

Using (25) we get

$$\hat{\rho}(t) = \sum_{n=0}^{\infty} p_n [\cos^2(\lambda t\sqrt{n+1})|n\rangle\langle n| + \sin^2(\lambda t\sqrt{n+1})|n+1\rangle\langle n+1|]. \tag{30}$$

The Q representation of the above state has a quite simple form

$$Q(\alpha,\alpha^*;t) = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} p_n \left(\cos^2(\lambda t\sqrt{n+1}) + \frac{|\alpha|^2}{n+1} \sin^2(\lambda t\sqrt{n+1}) \right). \tag{31}$$

The corresponding Wehrl entropy is plotted in Fig. 6. The numerical value of the mean photon number chosen for this figure is $\bar{n} = 5$ and the atom is initially inverted. According to [30] the evolution of the Wehrl entropy starts from

$$S_W = 1 + \ln(1+\bar{n}) \tag{32}$$

(i.e., from $S_W \approx 2.79$) and shows a complicated behavior

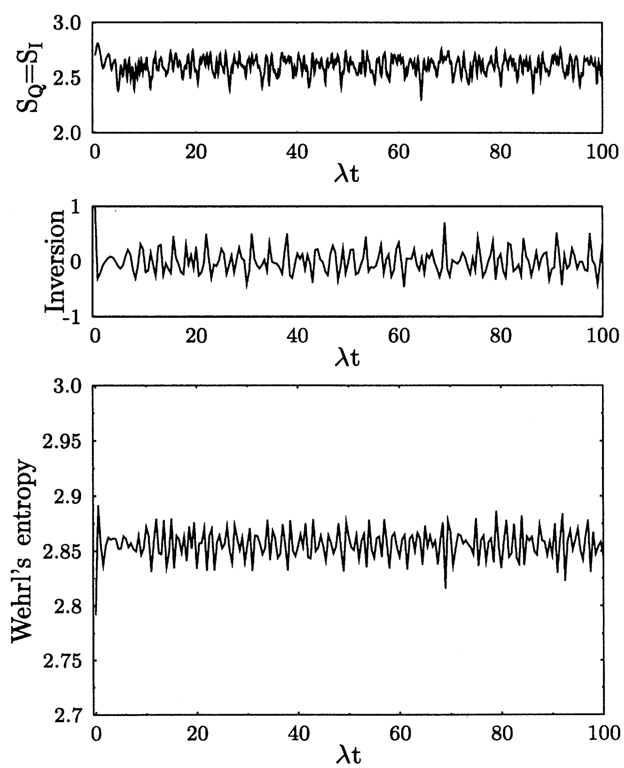

FIG. 6. Time evolution of the Wehrl entropy. The field is initially in the thermal state with $\bar{n} = 5$. The atom is initially fully inverted. The atomic inversion and the von Neumann entropy (equal in this case to the Shannon information entropy) are also plotted.

even after a short time. Also in this case the von Neumann entropy is equal to the Shannon entropy all the time. All quantities mentioned show rather complicated and irregular behavior.

IV. CONCLUSIONS AND REMARKS

We have shown that the Wehrl entropy is a useful one-parameter quantity, providing a compact and informative description of the time evolution of the field in the JCM, especially for the field being initially in a coherent state. It is shown to be very sensitive to the phase-space dynamics (such as, e.g., spreading and splitting) of the Q representation. It extracts from the Q function essential information about the investigated system. Other quantities usually considered in the investigation of the JCM, such as the atomic inversion, the Shannon information entropy, and the von Neumann entropy [18], usually show the rapid oscillations of the Rabi frequency with rather large amplitude already for the relatively short times. It makes a deeper insight into the field dynamics less straightforward. Although the Wehrl entropy also exhibits some oscillations of the Rabi frequency, their amplitude is very small, even for reasonably long times. In Fig. 1 these oscillations of the Wehrl entropy are practically invisible, in contrast to those easily seen for the other quantities considered. The Wehrl entropy illustrates very clearly that during the time evolution the field loses its coherence with the upper limit for the phase randomization corresponding to the random-phase coherent states. The observed phase redistribution seems to be irreversible and we see that the Wehrl entropy saturates up to some oscillations. We see that complete thermalization is not possible: the limit for the chaotic behavior is given by the random-phase coherent states. It should be noted that this level is never reached for the time scales investigated.

As the Wehrl entropy has been proven to be a very sensitive and useful parameter to describe the phase-space dynamics of the field in the JCM, we are convinced that it can also be successfully applied to study other problems. Some of them are currently under consideration and the results should be presently available.

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