PHYSICAL REVIEW A VOLUME 52, NUMBER 2 AUGUST 1995

Two-mode quantum systems: Invariant classification of squeezing transformations and squeezed states

Arvind*
Department of Physics, Indian Institute of Science, Bangalore 560 012, India

B. Dutta
Jawaharlal Nehru Centre for Advanced Scientific Research, IISc Campus, Bangalore 560 012, India

N. Mukunda†
Centre for Theoretical Studies and Department of Physics, Indian Institute of Science, Bangalore 560 012, India

R. Simon
Institute of Mathematical Sciences, Central Institutes of Technology Campus, Madras 600 113, India

(Received 31 May 1994)

A general analysis of squeezing transformations for two-mode systems is given based on the four-dimensional real symplectic group $Sp(4,\mathbb{R})$. Within the framework of the unitary (metaplectic) representation of this group, a distinction between compact photon-number-conserving and noncompact photon-number-nonconserving squeezing transformations is made. We exploit the U(2) invariant squeezing criterion to divide the set of all squeezing transformations into a two-parameter family of distinct equivalence classes with representative elements chosen for each class. Familiar two-mode squeezing transformations in the literature are recognized in our framework and seen to form a set of measure zero. Examples of squeezed coherent and thermal states are worked out. The need to extend the heterodyne detection scheme to encompass all of U(2) is emphasized, and known experimental situations where all U(2) elements can be reproduced are briefly described.

PACS number(s): 42.50.Dv,42.50.Ar,03.65.Fd

I. INTRODUCTION

The theoretical analysis [1] and experimental [2–4], realization of squeezed states of radiation continue to receive a great deal of attention. While much of the work so far has concerned itself with single-mode situations [5,6] some analysis of two-mode states has also been presented [7,8]. Other nonclassical effects of radiation beyond second order have also received attention in the literature [9]. More recently, a general invariant squeezing criterion for n-mode systems has been developed by some of us elsewhere [10].

The purpose of the present paper is to study squeezing transformations for two-mode systems and to develop a classification scheme for them motivated by the abovementioned invariant squeezing criterion. Basic to all such discussions is the four-dimensional real symplectic group $Sp(4,\mathbb{R})$, of real, linear, homogeneous canonical transformations and the unitary metaplectic representation of this group acting on the Hilbert space of states for a two-mode quantum system. The structure of the noncompact group $Sp(4,\mathbb{R})$ [and its n-mode counterpart $Sp(2n,\mathbb{R})$] leads to a natural separation of its elements into photon number conserving and nonconserving types. The former give rise to a maximally compact U(2) subgroup and will be referred to as passive; the latter are responsible for the noncompactness of the underlying group and will be referred to as active. This group-theoretical framework gives us an unambiguous way of defining precisely the family of squeezing transformations; they are the active elements of $Sp(4,\mathbb{R})$ and they do not form a subgroup. The action of U(2) on the set of squeezing transformations by conjugation leads to a natural equivalence relation, leading to the emergence of equivalence classes and convenient representative elements as well. In studying the physical properties of a state subjected to squeezing, therefore, we are able to isolate the dependence on intrinsic squeezing parameters and separate them from other passive factors. As might be expected, the single *squeeze factor* encountered in the studies of single-mode states gets enlarged here to *two independent intrinsic* squeeze factors; it turns out that the two-mode squeezing transformations so far studied in the literature form a very small subset of all the independent available possibilities.

The material in this paper is arranged as follows. Section II sets up the basic kinematics for two-mode systems. The variance matrix for a general state and its change under $Sp(4,\mathbb{R})$ are derived. After identifying the maximal compact or passive U(2) subgroup of $Sp(4,\mathbb{R})$, the

*Electronic address: email arvind@physics.iisc.ernet.in
†Also at Jawaharlal Nehru Centre for Advanced Scientific Research, Indian Institute of Science Campus, Bangalore 560 012, India.

U(2) invariant squeezing criterion for two-mode systems is discussed. Section III introduces the polar decomposition theorem for general elements of $Sp(4,\mathbb{R})$, which then leads to a precise definition of squeezing transformations: These are single exponentials of linear combinations of the noncompact generators. We then proceed to break up the set of all squeezing transformations into equivalence classes under U(2) action. We find that these classes form a continuous two-parameter family describable by points in an octant in a two-dimensional plane. The families of Caves-Schumaker transformations and essentially single-mode transformations correspond to one-dimensional lines bounding this octant and so are of measure zero. Section IV applies our formalism to two-mode squeezed coherent and thermal states. In Sec. V we see how the heterodyne detection scheme fits into our analysis. We argue that it is necessary to experimentally realize all elements of the U(2) subgroup of $Sp(4,\mathbb{R})$; the heterodyne scheme handles only a one-parameter subset of U(2). Two examples of two-mode situations, where all elements of U(2) can be experimentally realized, are briefly described. Section VI contains some concluding remarks.

II. SYMPLECTIC GROUP FOR TWO MODES AND THE SQUEEZING CRITERION

We consider two orthogonal modes of the radiation field, with annihilation operators $a_j, j = 1,2$, and corresponding creation operators $a_j^\dagger$. These two modes could, for example, be two different frequencies for the same or different propagation directions and polarizations, two different propagation directions at a common frequency, two different polarization states of plane waves degenerate in frequency and direction of propagation, etc. We arrange these operators in the form of a four-component column vector

$$\xi^{(c)} = (\xi_a^{(c)}) = \begin{pmatrix} a_1 \\ a_2 \\ a_1^\dagger \\ a_2^\dagger \end{pmatrix}, \quad a = 1,2,3,4. \tag{2.1}$$

The superscript (c) indicates that the entries here are complex, i.e., non-Hermitian, operators. For a discussion of quadrature squeezing, however, we need to also deal with the Hermitian quadrature components of these operators. Therefore we define another column vector ξ with four Hermitian entries as

$$\xi = (\xi_a) = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix},$$

$$q_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger) \ , \ p_j = \frac{-i}{\sqrt{2}}(a_j - a_j^\dagger),$$
$$\xi^{(c)} = \Omega\xi \ , \ \xi = \Omega^{-1}\,\xi^{(c)},$$
$$\Omega = (\Omega^{-1})^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \end{pmatrix}. \tag{2.2}$$

The canonical commutation relations can now be written either in terms of ξ or in terms of $\xi^{(c)}$,

$$[\xi_a, \xi_b] = i\beta_{ab},$$
$$[\xi_a^{(c)}, \xi_b^{(c)}] = \beta_{ab},$$
$$(\beta_{ab}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

A general real linear homogeneous transformation on the q's and p's preserving the commutation relations is described by a 4×4 real matrix S acting as

$$\xi \to \xi' = S\xi \ ,$$
$$\xi'_a = \sum_b S_{ab}\xi_b \ ,$$
$$S\,\beta\,S^T = \beta. \tag{2.4}$$

This is the defining property for the elements of the group $Sp(4,\mathbb{R})$.

We denote the Hilbert space on which ξ and $\xi^{(c)}$ act by $\mathcal{H}$. Since the Hermiticity properties and commutation relations of the ξ_a are maintained by the transformation (2.4) for any $S \in Sp(4,\mathbb{R})$, and since the ξ_a act irreducibly on $\mathcal{H}$, it follows from the Stone–von Neumann theorem [11] that it should be possible to construct a unitary operator $\mathcal{U}(S)$ on $\mathcal{H}$ implementing (2.4) via conjugation

$$S \in Sp(4,\mathbb{R}) : \sum_b S_{ab}\xi_b = \mathcal{U}(S)^{-1}\,\xi_a\,\mathcal{U}(S)\,,$$
$$\mathcal{U}(S)^\dagger\,\mathcal{U}(S) = \mathbf{1} \text{ on } \mathcal{H}\,. \tag{2.5}$$

The generators of the operators $\mathcal{U}(S)$ are the ten independent Hermitian quadratic expressions in a_j and $a_j^\dagger$ [12]. We define the four photon-number-conserving generators $Q, J_r, r = 1,2,3$, and six photon-number-nonconserving generators $K_r, L_r, r = 1,2,3$, as

$$Q = \frac{1}{2}(N+1) = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1); \tag{2.6a}$$

$$J_1 = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1),$$
$$J_2 = \frac{i}{2}(a_2^\dagger a_1 - a_1^\dagger a_2),$$
$$J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2); \tag{2.6b}$$

$$K_1 = \frac{1}{4}(a_1^{\dagger 2} + a_1^2 - a_2^{\dagger 2} - a_2^2),$$
$$K_2 = -\frac{i}{4}(a_1^{\dagger 2} - a_1^2 + a_2^{\dagger 2} - a_2^2),$$
$$K_3 = -\frac{1}{2}(a_1^\dagger a_2^\dagger + a_1 a_2); \tag{2.6c}$$

$$L_1 = \frac{i}{4}(a_1^{\dagger 2} - a_1^2 - a_2^{\dagger 2} + a_2^2),$$
$$L_2 = \frac{1}{4}(a_1^{\dagger 2} + a_1^2 + a_2^{\dagger 2} + a_2^2),$$
$$L_3 = -\frac{i}{2}(a_1^\dagger a_2^\dagger - a_1 a_2). \tag{2.6d}$$

They obey the commutation relations

$$[J_r, J_s] = i\epsilon_{rst}J_t,\quad [Q, J_r] = 0; \tag{2.7a}$$

$$[J_r, K_s \text{ or } L_s] = i\epsilon_{rst}(K_t \text{ or } L_t),\quad [Q, K_r \pm iL_r] = \mp(K_r \pm iL_r); \tag{2.7b}$$

$$[K_r, K_s] = [L_r, L_s] = -i\epsilon_{rst}J_t,\quad [K_r, L_s] = i\delta_{rs}Q. \tag{2.7c}$$

The elements of U(2) and their effects on a_j, $a_j^\dagger$ are

$$U = X - iY \in \mathrm{U}(2) \to S(X,Y) = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{R}),$$

$$\mathcal{U}(S(X,Y))a_j\mathcal{U}(S(X,Y))^\dagger = \sum_{k=1}^{2} U_{kj}a_k\ . \tag{2.8}$$

These operators $\mathcal{U}(S(X,Y))$ are generated by Q and J_r. On the photon-number-nonconserving generators K_r,L_r the effect is

$$\begin{aligned} e^{i\theta Q}\left(\vec{K} \pm i\vec{L}\right)e^{-i\theta Q} &= e^{\mp i\theta}\left(\vec{K} \pm i\vec{L}\right),\\ e^{i\vec{\alpha}\cdot\vec{J}}\left(K_r \text{ or } L_r\right)e^{-i\vec{\alpha}\cdot\vec{J}} &= R_{sr}(\vec{\alpha})\left(K_s \text{ or } L_s\right),\\ R_{rs}(\vec{\alpha}) &= \delta_{rs}\cos\alpha + \alpha_r\alpha_s\frac{1-\cos\alpha}{\alpha^2} + \epsilon_{rst}\alpha_t\frac{\sin\alpha}{\alpha},\\ \alpha &= |\vec{\alpha}|. \end{aligned} \tag{2.9}$$

Now we consider physical states of the two-mode system, the action of $\mathrm{Sp}(4,\mathbb{R})$ on them, and the statement of a suitable squeezing criterion. Let ρ be the density operator of any (pure or mixed) state of the two-mode radiation field. With no loss of generality we may assume that the means $\mathrm{Tr}\,(\rho\xi_a)$ of ξ_a vanish in this state. (Any nonzero values for these means can always be reinstated by a suitable phase-space displacement, which has no effect on the squeezing properties.) Squeezing involves the set of all second-order noise moments of the quadrature operators q_j and p_j. To handle them collectively we define the variance or noise matrix V for the state ρ as

$$\begin{aligned} V &= (V_{ab}),\\ V_{ab} = V_{ba} &= \frac{1}{2}\mathrm{Tr}\,(\rho\{\xi_a,\xi_b\}),\\ \{\xi_a,\xi_b\} &= \xi_a\xi_b + \xi_b\xi_a\ . \end{aligned} \tag{2.10}$$

This definition is valid for a system with any number of modes. For a two-mode system it can be written explicitly in terms of q_j and p_j as

$$V = \begin{pmatrix} \langle q_1^2\rangle & \langle q_1q_2\rangle & \frac{1}{2}\langle\{q_1,p_1\}\rangle & \langle q_1p_2\rangle \\ \langle q_1q_2\rangle & \langle q_2^2\rangle & \langle q_2p_1\rangle & \frac{1}{2}\langle\{q_2,p_2\}\rangle \\ \frac{1}{2}\langle\{q_1,p_1\}\rangle & \langle q_2p_1\rangle & \langle p_1^2\rangle & \langle p_1p_2\rangle \\ \langle q_1p_2\rangle & \frac{1}{2}\langle\{q_2,p_2\}\rangle & \langle p_1p_2\rangle & \langle p_2^2\rangle \end{pmatrix}. \tag{2.11}$$

This matrix is real symmetric positive definite and obeys additional inequalities expressing the Heisenberg uncertainty principle of quantum mechanics [10].

When the state ρ is transformed to a new state ρ' by the unitary operator $\mathcal{U}(S)$ for some $S \in \mathrm{Sp}(4,\mathbb{R})$, we see easily from Eqs. (2.5) and (2.10) that the variance matrix V undergoes a symmetric symplectic transformation

$$S \in \mathrm{Sp}(4,\mathbb{R}) : \rho' = \mathcal{U}(S)\,\rho\,\mathcal{U}(S)^{-1} \Rightarrow V' = S\ V\ S^T. \tag{2.12}$$

This transformation law for V preserves all the properties mentioned at Eq. (2.11).

As discussed in detail elsewhere [10,13], for a multimode system it is physically reasonable to set up a definition of squeezing that is invariant under the subgroup of passive transformations of the full symplectic group. For the present case of two-mode systems, we evidently need a U(2)-invariant squeezing criterion. That is, our definition must be such that if a state ρ with variance matrix V is found to be squeezed, then the state $\mathcal{U}(S(X,Y))\,\rho\,\mathcal{U}(S(X,Y))^{-1}$ with variance matrix $V' = S(X,Y)\ V\ S(X,Y)^T$ must also be squeezed, for any $U = X - iY \in \mathrm{U}(2)$.

Conventionally a state is said to be squeezed if any one of the diagonal elements of V is less than 1/2. The diagonal elements correspond, of course, to fluctuations in the "chosen" set of quadrature components of the system. The U(2)-invariant definition is as follows: the state ρ is a quadrature squeezed state if either some diagonal element of V is less than 1/2 (and then we say that the state is manifestly squeezed) or some diagonal element of $V' = S(X,Y)\ V\ S(X,Y)^T$ for some $U = X - iY \in \mathrm{U}(2)$ is less than 1/2:

$$\rho \text{ is a squeezed state } \Leftrightarrow \left(S(X,Y)\ V\ S(X,Y)^T\right)_{aa} < \frac{1}{2} \quad \text{for some } a \text{ and some } X - iY \in \mathrm{U}(2). \tag{2.13}$$

That is, running over all $S(X,Y)$ is the same as running over all possible sets of quadrature components. We may say that since any element of U(2) passively mixes the two modes, the appropriate $S(X,Y)$ that achieves the above inequality for some a (assuming that the given V permits the same) just chooses the right combination of quadratures to make the otherwise possibly hidden squeezing manifest.

To implement this definition in practice, it would appear that even if a state is intrinsically squeezed, we may have to explicitly find a suitable U(2) transformation that when applied to V makes the squeezing manifest. This, however, could be complicated. Here the point to be noticed and appreciated is that diagonalization of a noise matrix V generally requires a real orthogonal transformation belonging to SO(4) that may not lie in $\mathrm{U}(2) = \mathrm{O}(4) \cap \mathrm{Sp}(4,\mathbb{R})$. It is therefore remarkable that, as shown in [10], the U(2)-invariant squeezing criterion (2.13) can be expressed in terms of the spectrum of eigenvalues of V, namely,

ρ is a squeezed state

$$\Leftrightarrow \ \ell(V) = \text{least eigenvalue of } V < \frac{1}{2}. \quad (2.14)$$

That is, while the diagonalization of V is in general not possible within U(2), which is a proper subgroup of O(4), any one particular (and hence the smallest) eigenvalue of V can be made to become one of the diagonal elements of V transformed by an appropriate $S(X,Y)$. In other words, any one quadrature component can be taken to any other quadrature component by a suitable element of U(2). We shall hereafter work with the U(2)-invariant squeezing criterion (2.13) and (2.14).

III. CLASSIFICATION OF TWO-MODE SQUEEZING TRANSFORMATIONS

We have shown in the preceding section that the group $\mathrm{Sp}(4,\mathbb{R})$ of linear canonical transformations contains two kinds of elements: passive total photon-number-conserving elements belonging to the maximal compact subgroup U(2) and active *noncompact* elements lying outside this subgroup, which do not conserve total photon number. It is clear from the U(2)-invariant squeezing criterion (2.13) and (2.14) that the former elements cannot produce squeezing. This is because the corresponding changes in the variance matrix $V' = S(X,Y)\ V\ S(X,Y)^T$, being similarity transformations, preserve the eigenvalue spectrum of V; hence $\ell(V) \geq 1/2$ implies $\ell(V') \geq 1/2$ and conversely for every $S(X,Y)$. The *noncompact* elements of $\mathrm{Sp}(4,\mathbb{R})$, on the other hand, have the potential to produce a squeezed state starting from a nonsqueezed state. Thus they may be called squeezing transformations. The following questions then naturally arise: what are the really distinct squeezing transformations that are not related to each other by just passive transformations, and how can they be invariantly labeled or parametrized?

It is a well known fact [14–16] that each matrix $S \in \mathrm{Sp}(4,\mathbb{R})$ can be decomposed, globally and uniquely, into the product of two particular kinds of $\mathrm{Sp}(4,\mathbb{R})$ matrices by a polar decomposition: one factor belongs to the subgroup U(2), the other to a subset Π defined in the following way:

$$\Pi = \left\{ S \in \mathrm{Sp}(4,\mathbb{R}) \ \middle|\ S^T = S = (\text{positive definite}) \right\} \subset \mathrm{Sp}(4,\mathbb{R}). \quad (3.1)$$

We shall hereafter denote elements in Π by $P, P', \ldots$. The decomposition mentioned above is then

$$S \in \mathrm{Sp}(4,\mathbb{R}): \quad S = P\ S(X,Y), \quad P \in \Pi, X - iY \in \mathrm{U}(2), \quad (3.2)$$

with both factors being uniquely determined by S. For the operators $\mathcal{U}(S)$ we have the corresponding statement

$$\mathcal{U}(S) = \mathcal{U}(P)\ \mathcal{U}(S(X,Y)),$$

$$\mathcal{U}(P) = \exp\left[i(\text{real linear combination of } \vec{K} \text{ and } \vec{L})\right],$$

$$\mathcal{U}(S(X,Y)) = \exp\left[i(\text{real linear combination of } Q \text{ and } \vec{J})\right]. \quad (3.3)$$

We may now identify precisely the most general squeezing transformation within the $\mathrm{Sp}(4,\mathbb{R})$ framework, as the operator $\mathcal{U}(P)$ characterized by two numerical three-dimensional vectors $\vec{k}$ and $\vec{l}$ appearing as coefficients of $\vec{K}$ and $\vec{L}$ in the exponent

$$\mathcal{U}(\vec{k},\vec{l}) = \exp\left[i(\vec{k}\cdot\vec{K} + \vec{l}\cdot\vec{L})\right]. \quad (3.4)$$

Thus we reserve the name *squeezing transformations* for these elements of Π within $\mathrm{Sp}(4,\mathbb{R})$, with $\mathcal{U}(S)$ represented by a single exponential factor.

We may relate the decomposition (3.3) to a general quadratic Hamiltonian quite directly. Any such Hamiltonian containing both photon-conserving and -nonconserving terms with possibly time-dependent coefficients would lead via the Schrödinger equation to a unitary finite time evolution operator that can be uniquely decomposed into the product form (3.3). Thus integration of the Schrödinger equation leads in general to a specific passive factor and another specific squeezing transformation. In the case where the Hamiltonian is time independent and a combination only of the generators $\vec{K}$ and $\vec{L}$, this evolution operator is already of the form $\mathcal{U}(P)$.

Since we have a U(2)-invariant squeezing criterion, as we have seen, elements of U(2) have no effect on the squeezed or nonsqueezed status of any given state. This means that the U(2) transform, by conjugation, of a squeezing transformation is another squeezing transformation that should be regarded as equivalent to the first one. It is clear that, in any case, any equivalence relation among squeezing transformations as defined by us above should be based on processes that take one squeezing transformation to another.

Now from the commutation relations (2.7b) we can see that the squeezing transformations $\mathcal{U}(\vec{k},\vec{l})$ defined in Eq. (3.4) behave as follows under conjugation by elements of U(2):

$$e^{i\theta Q}\mathcal{U}(\vec{k},\vec{l})e^{-i\theta Q} = \mathcal{U}(\vec{k}',\vec{l}'),$$
$$\begin{pmatrix} \vec{k}' \\ \vec{l}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{k} \\ \vec{l} \end{pmatrix}; \quad (3.5a)$$
$$e^{i\vec{\alpha}\cdot\vec{J}}\mathcal{U}(\vec{k},\vec{l})e^{-i\vec{\alpha}\cdot\vec{J}} = \mathcal{U}(\vec{k}'',\vec{l}''),$$
$$k''_r \text{ or } l''_r = R_{rs}(\vec{\alpha})\,(k_s \text{ or } l_s)\,. \quad (3.5b)$$

Here we have listed separately the effects of U(1) and SU(2) within U(2) on the squeezing transformations. The questions raised at the start of this section can now be posed more precisely: if the set of squeezing transformations $\mathcal{U}(\vec{k},\vec{l})$ is separated into distinct nonoverlapping equivalence classes based on the U(2) action (3.5), how can we conveniently choose U(2)-invariant parameters to label these classes and then pick a convenient representative element from each class? We answer these questions in this sequence.

It is clear that we need to construct a complete set of independent expressions in $\vec{k}$ and $\vec{l}$, invariant under both U(1) and SU(2) actions (3.5a) and (3.5b). We begin by defining the matrix M of scalar products among $\vec{k}$ and $\vec{l}$, which is then SU(2) invariant:

$$M(\vec{k},\vec{l}) = \begin{pmatrix} \vec{k}\cdot\vec{k} & \vec{k}\cdot\vec{l} \\ \vec{k}\cdot\vec{l} & \vec{l}\cdot\vec{l} \end{pmatrix}. \quad (3.6)$$

This is a real, symmetric positive semidefinite matrix. With respect to U(1) action, we see from Eq. (3.5a) that $M(\vec{k},\vec{l})$ undergoes a similarity transformation by the rotation matrix of angle θ:

$$M(\vec{k}',\vec{l}') = R(\theta)\, M(\vec{k},\vec{l})\, R(\theta)^{-1},$$
$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \quad (3.7)$$

One now sees that there are two independent U(2) invariants that can be formed

$$\Im_1(\vec{k},\vec{l}) = \det M(\vec{k},\vec{l}) = |\vec{k}\wedge\vec{l}|^2,$$
$$\Im_2(\vec{k},\vec{l}) = \operatorname{Tr} M(\vec{k},\vec{l}) = |\vec{k}|^2 + |\vec{l}|^2 \quad (3.8)$$

and it is easily checked that there are no other invariants independent of these.

Next let us tackle the problems of finding convenient parameters and representative squeezing transformations for the U(2) equivalence classes, one for each class. We see from Eq. (3.8) that if $\Im_1 > 0$ (i.e., $\Im_1 \neq 0$), then the two vectors $\vec{k}$ and $\vec{l}$ are both nonzero and nonparallel, while if $\Im_1 = 0$ they are parallel (and one of them could vanish). These are therefore clearly different geometrical situations. Starting with the matrix $M(\vec{k},\vec{l})$, we see from its U(1) transformation law (3.7) that by a suitable choice of the angle θ we can arrange the transformed matrix $M(\vec{k}',\vec{l}')$ to be diagonal and in the case of unequal eigenvalues to place the larger eigenvalue in the first position. This means that in each equivalence class of squeezing transformations there certainly are elements $\mathcal{U}(\vec{k},\vec{l})$ for which $\vec{k}\cdot\vec{l} = 0$ and $|\vec{k}| \geq |\vec{l}|$. This still leaves us the freedom of action by SU(2). We may now exploit this freedom to put the (mutually perpendicular) vectors $\vec{k}$ and $\vec{l}$ into a convenient geometrical configuration. A look at the forms of the noncompact generators $\vec{K}$ and $\vec{L}$ in Eq. (2.6) suggests that we choose $\vec{k}$ and $\vec{l}$ as follows:

$$\vec{k} = (0,a,0), \quad \vec{l} = (b,0,0), \quad a \geq b. \quad (3.9)$$

(A further reason for this choice will emerge shortly). $\Im_1$ and $\Im_2$ can now be evaluated in terms of a,b to obtain the relations

$$\Im_1(\vec{k},\vec{l}) = a^2b^2,$$
$$\Im_2(\vec{k},\vec{l}) = a^2 + b^2,$$
$$a \geq b \geq 0, (a,b) \neq (0,0). \quad (3.10)$$

With this parametrization we can now say that there is a twofold infinity of distinct equivalence classes of squeezing transformations for two-mode systems, each class corresponding uniquely and unambiguously to a point (a,b) in the octant $a \geq b \geq 0$ in the real a-b plane, excluding the origin. Different points in the octant correspond to intrinsically distinct equivalence classes. Within an equivalence class determined by a point (a,b), of course, one can connect different squeezing transformations $\mathcal{U}(\vec{k},\vec{l})$ by conjugation with suitable U(2) elements. Given a squeezing transformation $P \in \Pi \subset \mathrm{Sp}(4,\mathbb{R})$ its class (a,b) is determined by solving the equations

$$\operatorname{Tr}(P) = 2\left[\cosh\frac{(a-b)}{2} + \cosh\frac{(a+b)}{2}\right],$$
$$\operatorname{Tr}(P^2) = 2\left[\cosh(a-b) + \cosh(a+b)\right] \quad (3.11)$$

subject to the conditions on a and b appearing in Eq. (3.10).

Then we have the following convenient two-mode squeezing transformation representing the equivalence class (a,b):

$$\mathcal{U}^{(0)}(a,b) = \mathcal{U}^{(0)}(a,0)\,\mathcal{U}^{(0)}(0,b),$$
$$\mathcal{U}^{(0)}(a,0) = \exp(iaK_2) = \exp\left[\frac{-ia}{2}(q_1p_1 + p_2q_2)\right],$$
$$\mathcal{U}^{(0)}(0,b) = \exp(ibL_1) = \exp\left[\frac{ib}{2}(q_1p_1 - q_2p_2)\right]. \quad (3.12)$$

The two factors $\mathcal{U}^{(0)}(a,0)$ and $\mathcal{U}^{(0)}(0,b)$ commute and may be written in either order, since according to Eq. (2.7c) the noncompact generators K_2 and L_1 commute.

Finally, one can easily calculate the symplectic matrix $S^{(0)}(a,b) \in \mathrm{Sp}(4,\mathbb{R})$, corresponding to the operator $\mathcal{U}^{(0)}(a,b)$, by using Eq. (2.5). The result is

$$\mathcal{U}^{(0)}(a,b)^{-1}\,\xi\,\mathcal{U}^{(0)}(a,b) = S^{(0)}(a,b)\,\xi,$$

$$S^{(0)}(a,b) = \mathrm{diag}\left(e^{(a-b)/2}, e^{(a+b)/2}, e^{-(a-b)/2}, e^{-(a+b)/2}\right). \tag{3.13}$$

Now we can clarify that the particular choice (3.9) was dictated by the desire to have $S^{(0)}(a,b)$ diagonal. This element of $\mathrm{Sp}(4,\mathbb{R})$ describes independent reciprocal scalings of the standard quadrature components of each mode. This amounts to showing geometrically that it is possible to diagonalize every $P \in \Pi$ using conjugation by U(2).

We illustrate our classification scheme of two-mode squeezing transformations by giving two examples. The extensively studied Caves-Schumaker (CS) [6] transformation uses the operator

$$\mathcal{U}^{(\mathrm{CS})}(z) = \exp\left(z\,a_1^\dagger\,a_2^\dagger - z^*\,a_1\,a_2\right). \tag{3.14}$$

By appearance, this attempts to involve or entangle the two modes maximally. In our notation this squeezing transformation corresponds to the generator combination

$$z\,a_1^\dagger\,a_2^\dagger - z^*\,a_1\,a_2 = i(\vec{k}\cdot\vec{K} + \vec{l}\cdot\vec{L}),$$
$$\vec{k} = -2(0,0,\mathrm{Im}\,z),$$
$$\vec{l} = 2(0,0,\mathrm{Re}\,z). \tag{3.15}$$

Thus the invariant parameters a and b have values

$$a = 2|z|, \quad b = 0. \tag{3.16}$$

The Caves-Schumaker squeezing transformations and their U(2) conjugates, all taken together, form a one-parameter family or a one-dimensional line in the a-b octant. In that sense they are a set of measure zero.

Another interesting case is a squeezing transformation that refers essentially to a single mode but masquerades as a two-mode transformation

$$\mathcal{U}(z;\alpha,\beta) = \exp\left[z(\alpha^* a_1^\dagger + \beta^* a_2^\dagger)^2 - z^*(\alpha a_1 + \beta a_2)^2\right],$$
$$|\alpha|^2 + |\beta|^2 = 1. \tag{3.17}$$

After some simple algebra we find

$$\mathcal{U}(z;\alpha,\beta) = \exp\left[i(\vec{k}\cdot\vec{K} + \vec{l}\cdot\vec{L})\right],$$
$$\vec{k} + i\vec{l} = 2z\left[-i(\alpha^{*2} - \beta^{*2}), (\alpha^{*2} + \beta^{*2}), 2i\alpha^*\beta^*\right]. \tag{3.18}$$

The associated invariants and parameters are

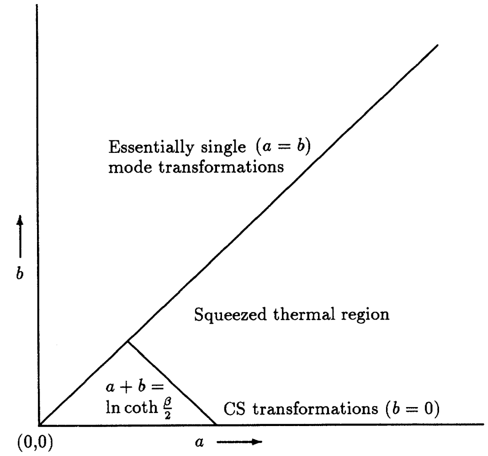

FIG. 1. Equivalence classes of two-mode squeezing transformations, Caves-Schumaker (CS) and single-mode limits, and the squeezed thermal region.

$$\Im_1 = 16|z|^4, \quad \Im_2 = 8|z|^2,$$
$$a = b = 2|z|. \tag{3.19}$$

These equivalence classes thus lie along the line $a = b$ in the octant, again a one-parameter family of zero measure. Our results are depicted in Fig. 1.

We note here that the size of an equivalence class depends sensitively upon the point (a,b). Since for $a \neq 0, b \neq 0$, and $a \neq b$ none of the generators of U(2) or a linear combination of them commute with $aK_2 + bL_1$, we have a full four-parameter equivalence class. For the cases $a \neq 0, b = 0$, and $a = b$, respectively, the vanishing of the commutators $[J_2, K_2]$ and $[Q + J_3, K_2 + L_1]$ leads to reduction of the dimensionality of the equivalence class to 3.

We conclude this section with a few comments. Each point (a,b) in the octant denotes an equivalence class of squeezing transformations, whose dependences on a and b would be of physical significance and would show up in a variety of U(2)-invariant properties. The two-mode transformations discussed so far in the literature lie basically along the two lines shown in Fig. 1. In this sense, *most* of the intrinsically distinct two-mode transformations, their effects on various states, etc., remain to be explored. Those equivalence classes (a,b) for which $a > b$ involve the two modes in an essential way. We may say purely qualitatively that the *distance* of the point (a,b) from the line $a = b$, or perhaps better the expression $(1 - b/a)$, is a measure of the extent to which two independent modes are involved in the transformation. In this sense, as remarked earlier, the Caves-Schumaker transformations involve the two modes maximally.

IV. SQUEEZED COHERENT AND THERMAL STATES FOR TWO MODES

The general two-mode coherent state with complex two-component displacement $\tilde{\alpha} = (\alpha_1, \alpha_2)$ is defined by

$$|\tilde{\alpha}\rangle = \exp\left(\tilde{\alpha}\cdot\tilde{a}^\dagger - \tilde{\alpha}^*\cdot\tilde{a}\right)\,|0,0\rangle$$
$$= \exp\left(-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2\right)$$
$$\times\exp\left(\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger\right)\,|0,0\rangle. \qquad (4.1)$$

For this state the means of the quadrature components ξ_a do not vanish in general:

$$\langle\tilde{\alpha}|\xi|\tilde{\alpha}\rangle = \sqrt{2}\,(\mathrm{Re}\,\alpha_1, \mathrm{Re}\,\alpha_2, \mathrm{Im}\,\alpha_1, \mathrm{Im}\,\alpha_2)^T. \qquad (4.2)$$

The variance matrix is, however, independent of $\tilde{\alpha}$:

$$V(|\tilde{\alpha}\rangle) = V(|\tilde{0}\rangle) = \frac{1}{2}\,\mathbf{1}_{4\times 4}. \qquad (4.3)$$

The most general squeezed coherent state is obtained by applying $\mathcal{U}(P)$ for some $P \in \Pi \subset \mathrm{Sp}(4,\mathbb{R})$ to $|\tilde{\alpha}\rangle$ for some $\tilde{\alpha}$. This $\mathcal{U}(P)$ is conjugate, via some U(2) element, to $\mathcal{U}^{(0)}(a,b)$ for some a,b. Now the effect of a U(2) transformation on $|\tilde{\alpha}\rangle$ is to give us another coherent state $|\tilde{\alpha}'\rangle$, $\tilde{\alpha}'$ being the U(2) transform of $\tilde{\alpha}$. But the variance matrix is in any case $\tilde{\alpha}$ independent. To examine the U(2)-invariant squeezing condition, therefore, it suffices to examine the particular class of squeezed coherent states

$$|\tilde{\alpha};a,b\rangle = \mathcal{U}^{(0)}(a,b)\,|\tilde{\alpha}\rangle. \qquad (4.4)$$

From Eqs. (2.12) and (3.13), the calculation of the variance matrix for this state is trivial and it is in fact diagonal:

$$V(|\tilde{\alpha};a,b\rangle) = S^{(0)}(a,b)\,V(|\tilde{\alpha}\rangle)\,S^{(0)}(a,b)$$
$$= \frac{1}{2}\,S^{(0)}(2a,2b)$$
$$= \frac{1}{2}\,\mathrm{diag}\left(e^{(a-b)}, e^{(a+b)}, e^{(b-a)}, e^{-(a+b)}\right). \qquad (4.5)$$

Since a and b are non-negative and in addition $a+b>0$, we see that the least eigenvalue of this variance matrix is

$$\ell(V(|\tilde{\alpha};a,b\rangle)) = \frac{1}{2}\,e^{-(a+b)} < \frac{1}{2}. \qquad (4.6)$$

These states are thus always squeezed.

If we apply any passive U(2) transformation $S(X,Y)$ to any one of the states $|\tilde{\alpha};a,b\rangle$ defined above, the variance matrix will in general change as $V \to V' = S(X,Y)\;V\;S(X,Y)^T$; but its eigenvalue spectrum, and in particular $\ell(V)$, remains unaltered. Thus all the states symbolically written as $\mathcal{U}(\mathrm{U}(2))\,|\tilde{\alpha};a,b\rangle$, for various U(2) elements, are squeezed to the same extent as $|\tilde{\alpha};a,b\rangle$ and have $\ell(V)$ given by Eq. (4.6).

The Schrödinger wave functions for the subfamily of squeezed coherent states (4.4) are particularly simple since they are products of single-mode squeezed coherent state wave functions

$$\langle q_1', q_2'|\tilde{\alpha};a,b\rangle = \psi^{(0)}(q_1';\alpha_1, a-b)\;\psi^{(0)}(q_2';\alpha_2, a+b),$$

$$\psi^{(0)}(q';\alpha,a)$$
$$= \frac{e^{-a/4}}{\pi^{1/4}}\,\exp\left[i\alpha\,\mathrm{Im}\,\alpha - \frac{1}{2}\left(q'\,e^{-a/2} - \sqrt{2}\alpha\right)^2\right]. \qquad (4.7)$$

[For a general state $\mathcal{U}(\mathrm{U}(2))\,|\tilde{\alpha};a,b\rangle$, we do not expect such a product form.] When we set $b=0$ (the Caves-Schumaker limit), both factors show the same amount of squeezing, while when we set $a=b$ (essentially the single-mode situation) we see squeezing only in the factor referring to the second mode. These features are as we would have expected.

The next example we look at is the case of a two-mode thermal state subjected to squeezing. The motivation in making this choice is that the starting density operator is explicitly U(2) invariant. The normalized density operator corresponding to inverse temperature $\beta = \hbar\omega/kT$ is described in the Fock and Schrödinger representations by

$$\rho_0(\beta) = (1-e^{-\beta})^2\,\exp\left[-\beta(a_1^\dagger\,a_1 + a_2^\dagger\,a_2)\right]$$
$$= (1-e^{-\beta})^2\sum_{n_1,n_2=0}^{\infty} e^{-\beta(n_1+n_2)}|n_1,n_2\rangle\langle n_1,n_2|, \qquad (4.8a)$$
$$\rho_0(q_1',q_2',q_1'',q_2'';\beta) = \frac{2}{\pi}\,\tanh^2\frac{\beta}{2}\,\exp\left[-\frac{1}{2}\left(\tanh\frac{\beta}{2} + \coth\frac{\beta}{2}\right)\right.$$
$$\left.\times\left(q_1'^2 + q_2'^2 + q_1''^2 + q_2''^2\right) - \left(\tanh\frac{\beta}{2} - \coth\frac{\beta}{2}\right)\left(q_1'q_1'' + q_2'q_2''\right)\right], \qquad (4.8b)$$

with U(2) invariance expressed by

$$e^{i\theta Q}\,\rho_0(\beta)\,e^{-i\theta Q} = e^{i\vec{\alpha}\cdot\vec{J}}\,\rho_0(\beta)\,e^{-i\vec{\alpha}\cdot\vec{J}} = \rho_0(\beta). \qquad (4.9)$$

Therefore it suffices to examine the properties of the density operator obtained by conjugating $\rho_0(\beta)$ with $\mathcal{U}^{(0)}(a,b)$,

$$\rho(\beta;a,b) = \mathcal{U}^{(0)}(a,b)\;\rho_0(\beta)\;\mathcal{U}^{(0)}(a,b)^{-1}. \qquad (4.10)$$

The most general squeezed thermal state is evidently

$$\mathcal{U}(U(2))\;\rho(\beta;a,b)\;\mathcal{U}(U(2))^{-1}, \qquad (4.11)$$

but this has the same squeezing properties as $\rho(\beta;a,b)$.

For the thermal state $\rho_0(\beta)$ the variance matrix is well known [10]

$$V(\rho_0(\beta)) = \frac{1}{2} \coth\frac{\beta}{2} \mathbf{1}_{4\times 4} . \quad (4.12)$$

Therefore, for the particular set of squeezed thermal states (4.10), we have diagonal variance matrices

$$\begin{aligned} V(\rho(\beta; a, b)) \\ &= S^{(0)}(a,b)\, V(\rho_0(\beta))\, S^{(0)}(a,b)^T \\ &= \frac{1}{2} \coth\frac{\beta}{2}\, S^{(0)}(2a, 2b) \\ &= \frac{1}{2} \coth\frac{\beta}{2} \operatorname{diag}\left(e^{(a-b)}, e^{(a+b)}, e^{(b-a)}, e^{-a-b}\right) . \end{aligned} \quad (4.13)$$

The least eigenvalue is evidently

$$\ell(V) = \frac{1}{2} \coth\frac{\beta}{2}\, e^{-(a+b)}, \quad (4.14)$$

so for a given temperature, squeezing sets in when

$$a + b > \ln\coth\frac{\beta}{2}. \quad (4.15)$$

In Fig. 1, this region consists of all points in the a-b octant *to the right of* the line $a + b = \ln\coth\beta/2$, which is a line perpendicular to the line $a = b$ and at a distance $\ln\coth(\beta/2)/\sqrt{2}$ from the origin.

V. DETECTION SCHEMES AND THE ROLE OF U(2) TRANSFORMATIONS

We have so far not specified in any detail the two orthogonal modes of radiation being subjected to squeezing. Let us at this point consider a situation well studied experimentally by the heterodyne detection scheme [17]. Here the two modes differ only slightly in frequency, but are otherwise similar. In this kind of experimental arrangement what is actually measured is the fluctuation of a certain photocurrent and this in turn gives the fluctuation or variance of the q-quadrature component of a particular (passive) combination of the original modes. The combinations of a_1 and a_2 that are involved form the one-parameter family

$$a(\psi) = \frac{1}{\sqrt{2}} (a_1 + a_2)\, e^{-i\psi/2} , \quad 0 \leq \psi < 4\pi. \quad (5.1)$$

This can be regarded as the first component of a U(2)-transformed pair a'_1, a'_2

$$\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\psi/2} & e^{-i\psi/2} \\ -e^{i\psi/2} & e^{i\psi/2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad a'_1 \equiv a(\psi). \quad (5.2)$$

The Hermitian quadrature component whose fluctuation is measured is

$$\begin{aligned} q(\psi) &= \frac{1}{\sqrt{2}} \left[a(\psi) + a(\psi)^\dagger\right] \\ &= \frac{1}{\sqrt{2}} (q_1 + q_2) \cos\frac{\psi}{2} + \frac{1}{\sqrt{2}} (p_1 + p_2) \sin\frac{\psi}{2}. \end{aligned} \quad (5.3)$$

The only experimentally adjustable parameter here is the angle ψ. The family of U(2) elements realized in the heterodyne scheme is thus only the one-parameter set given in Eq. (5.2) parametrized by ψ and belonging to SU(2):

$$U_H(\psi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\psi/2} & e^{-i\psi/2} \\ -e^{i\psi/2} & e^{i\psi/2} \end{pmatrix} \in \mathrm{SU}(2). \quad (5.4)$$

We notice that this is *not* a one-parameter subgroup of SU(2); in particular, even the identity element of the group is not contained here.

With this description of the heterodyne setup in our framework, let us see to what extent it can be used to detect U(2)-invariant squeezing. Now a general two-mode state ρ with variance matrix V, even if it is squeezed in the intrinsic sense of Eq. (2.14), may not be manifestly squeezed. That is, it may happen that $V_{aa} \geq 1/2$ for all $a = 1, \ldots, 4$. As our discussion in Sec. II shows, we need to be able to *experimentally* realize a general U(2) transformation applied to the state ρ and change its variance matrix to a form where one of its diagonal entries (say the leading one) becomes less than 1/2. However, the heterodyne method is generally unable to do this job for us, as it can only realize the one-parameter subset of SU(2) transformations $U_H(\psi)$ for $0 \leq \psi < 4\pi$.

In the two examples of squeezed coherent states and squeezed thermal states studied in Sec. IV, we have a family of states related to each other by conjugation with U(2) for each point in the a-b plane. Each equivalence class has appropriate dimensionality depending upon the point (a, b) as explained in Sec. III. It turns out that for each (a, b) the heterodyning scheme can detect squeezing in only a one-parameter subset of the family of states. Although heterodyning detection covers the whole a-b plane, it does not reach all the states corresponding to each point in the a-b plane. For example, in the representative chosen in Eqs. (4.4) and (4.10), for which the variance matrix is already diagonal, the squeezing cannot be detected by this scheme because of the absence of the identity in U_H. It should be possible to detect squeezing in these states by a suitably modified scheme. We wish to emphasize that there is a definite need to be able to experimentally implement the most general element of U(2). This would allow the experimenter to detect the degree of squeezing unambiguously, if the state is squeezed, without any prior knowledge of the elements of the initial variance matrix.

Having stressed the need to implement arbitrary U(2) transformations on the two modes of radiation in order to reach the proper quadrature to exhibit squeezing, we now describe how it can be achieved in some situations. We discuss two particular cases of the two modes involved, the first when the two modes have the same frequency but different directions of propagation and the second when the modes have the same frequency and direction

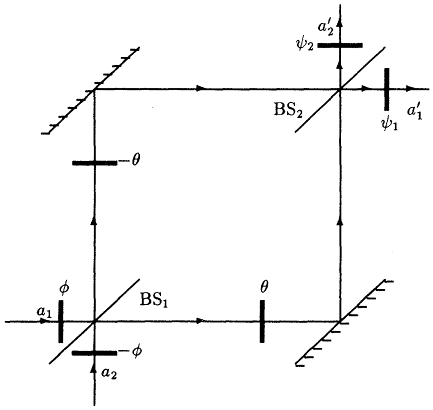

FIG. 2. Mach-Zehnder interferometer implementing arbitrary U(2) transformation on two modes at the same frequency but differing in their direction of propagation. BS$_1$ and BS$_2$ are 50:50 beam splitters and thick lines are phase shifters by angles indicated. a_1, a_2 are the annihilation operators at the input port and a_1', a_2' are the annihilation operators at the output port.

of propagation but different polarizations.

The experimental setup for the first case is shown in Fig. 2. We achieve an arbitrary U(2) transformation on the two modes by using a Mach-Zehnder interferometer with two 50:50 beam splitters (BS$_1$ and BS$_2$) and appropriate phase shifters [18]. The input modes with annihilation operators a_1 and a_2 are subjected to equal and opposite phase shifts by angles ϕ and $-\phi$, then the modes are mixed in the beam splitter BS$_1$, and the mixed modes again undergo equal and opposite phase shifts by angles θ and $-\theta$ and further mixing through the beam splitter BS$_2$. Finally, they undergo unequal phase shifts by angles ψ_1 and ψ_2. If the annihilation operators at the output are a_1' and a_2' then all the above operations when combined are implemented through the transformation

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} e^{i(\phi+\psi_1)}\cos\theta & -ie^{-i(\phi-\psi_1)}\sin\theta \\ -ie^{i(\phi+\psi_2)}\sin\theta & e^{-i(\phi-\psi_2)}\cos\theta \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{5.5}$$

relating the two sets of annihilation operators. The above matrix is the most general U(2) transformation matrix. We note here that if $\psi_2 = -\psi_1$ then the transformation matrix is the most general SU(2) transformation. Going from SU(2) to U(2) is just a matter of overall phase and can also be achieved by free propagation.

For the second case when the two modes differ only in polarization we achieve the arbitrary U(2) transformation by using two quarter wave plates Q_1, Q_2 and a half wave plate H as shown in Fig. 3. The detailed discussion of this setup is given in [19]. It turns out that the configuration Q-H-Q is not the only one, but Q-Q-H and H-Q-Q also accomplish the same result, as shown in [19]. We basically have three elements: a quarter wave plate Q_1, a half wave plate H and a quarter wave plate

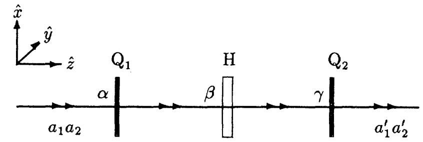

FIG. 3. Implementation of an arbitrary SU(2) element on two modes with the same frequency and directions of propagation, but different polarizations. Here Q_1, Q_2 are quarter wave plates and H is the half wave plate. α, β and γ are the angles that the slow axes of Q_1, H, and Q_2 make with the $\hat{x}$ axis, respectively. a_1, a_2 are the annihilation operators at the input port and a_1', a_2' are the annihilation operators at the output port.

Q_2, all three of them being coaxially mounted and with their slow axes in the x-y plane making angles of α, β, and γ, respectively, with the $\hat{x}$ axis. The two modes having annihilation operators a_1 and a_2 moving along the $\hat{z}$ direction pass through this arrangement. If the annihilation operators at the output are a_1' and a_2' then they are related to the operators at the input by an SU(2) transformation given in terms of α, β, and γ. By changing these parameters one can reproduce any desired SU(2) element. As pointed out earlier, going to U(2) now is just a matter of free propagation.

In both the above cases, by going to the proper U(2) element we can make the squeezing (if present) manifest and bring it to the leading diagonal element of the variance matrix, i.e., in the quadrature $q_1' = \frac{1}{\sqrt{2}}(a_1' + a_1^{\dagger\prime})$. The squeezing in this quadrature can now be measured by any standard one-mode detection method.

These remarks show, on the one hand, the way the heterodyning scheme fits into our general analysis and, on the other hand, the need to devise new schemes capable of realizing all elements of U(2), tailored to the definition of the two modes involved.

VI. CONCLUDING REMARKS

We have presented a classification scheme for two-mode squeezing transformations, based on the structure of the real four-dimensional symplectic group Sp(4, $\mathbb{R}$), and the separation of its elements into passive (compact) and active (noncompact) types. The structure and action of the maximal compact subgroup U(2) in Sp(4, $\mathbb{R}$), and the U(n)-invariant squeezing criterion formulated elsewhere for a general n-mode system, have guided our considerations.

We mention at this point two useful algebraic properties: one relevant for any number of modes and the other specifically for two-mode systems. We have seen in Sec. II that the real symplectic transformations acting on irreducible and Hermitian canonical variables are implemented by unitary operators $\mathcal{U}(S)$. However, these operators are determined only up to S-dependent phase factors by this requirement. The question arises whether this freedom can be used to make the composition law for these unitary operators as simple as possible. It turns out that the maximum simplification [20] that can be achieved leads to the composition law

$$S_1, S_2 \in \mathrm{Sp}(4,\mathbb{R}) : \mathcal{U}(S_1)\, \mathcal{U}(S_2) = \pm \mathcal{U}(S_1\, S_2). \quad (6.1)$$

Thus these unitary operators provide a double-valued representation of the symplectic group. It is, however, more proper to recognize that $\mathcal{U}(S)$ is really not entirely determined by S alone; one is in fact dealing with a faithful unitary representation of the metaplectic group, which is a twofold cover of the symplectic group.

For two-mode systems based on $\mathrm{Sp}(4,R)$ we gain geometrical insight by recognizing that this group has the same Lie algebra as the group SO(3,2) [21]. Thus the commutation relations (2.7) can be presented in the alternate form

$$\begin{aligned} [M_{AB}, M_{CD}] &= i(g_{AC}M_{BD} - g_{BC}M_{AD} \\ &\quad + g_{AD}M_{CB} - g_{BD}M_{CA}), \\ g_{AB} &= \mathrm{diag}\,(+1,+1,+1,-1,-1), \\ M_{AB} &= -M_{BA}, \end{aligned} \quad (6.2)$$

where the $\mathrm{Sp}(4,R)$ generators are given in terms of M_{AB} by

$$Q = M_{45}, \qquad J_r = \frac{1}{2}\epsilon_{rst}M_{st},$$
$$K_r = M_{r4}, \qquad L_r = M_{r5}. \quad (6.3)$$

In this form the photon-number-conserving transformations correspond to rotations in the compact subgroup SO(3)×SO(2) while the photon-nonconserving transformations generated by K_r, L_r correspond to pure noncompact boosts in $(r,4)$, $(r,5)$ hyperplanes, respectively.

The squeezing transformations studied in Sec. III, $\mathcal{U}(P), P \in \Pi \subset \mathrm{Sp}(4,\mathbb{R})$, do not form a subgroup of $\mathrm{Sp}(4,\mathbb{R})$. The breakup of these transformations into equivalence classes, based on the effect of conjugation by elements of U(2), is the only natural available classification procedure. This is because the definition of equivalence classes for any set of objects has to be based on an equivalence relation defined on that set. Thus we have treated two elements $P, P' \in \Pi$ as intrinsically equivalent if

$$P' = S(X,Y)\; P\; S(X,Y)^T \quad \text{for some} \quad X - iY \in \mathrm{U}(2). \quad (6.4)$$

It should, however, be realized that the detailed effects of action by $\mathcal{U}(P)$ and $\mathcal{U}(P')$ on a *general* initial two-mode state ρ_0, as seen in the changes caused in the variance matrix $V(\rho_0)$, need not be identical. Since this is a subtle and important point we spell it out in detail. Starting from a general state ρ_0, action by a squeezing transformation leads to a new state

$$\rho = \mathcal{U}(P)\; \rho_0\; \mathcal{U}(P)^{-1}. \quad (6.5)$$

As seen in Sec. III, any $\mathcal{U}(P)$ is expressible in terms of a representative element $\mathcal{U}^{(0)}(a,b)$ as

$$\mathcal{U}(P) = \mathcal{U}(S(X,Y))\; \mathcal{U}^{(0)}(a,b)\; \mathcal{U}(S(X,Y))^{-1} \quad (6.6)$$

for suitable $X - iY \in \mathrm{U}(2)$. Therefore we have

$$\begin{aligned} \rho &= \mathcal{U}(S(X,Y))\; \mathcal{U}^{(0)}(a,b)\; \mathcal{U}(S(X,Y))^{-1}\; \rho_0 \\ &\quad \times \mathcal{U}(S(X,Y))\; \mathcal{U}^{(0)}(a,b)^{-1}\; \mathcal{U}(S(X,Y))^{-1}. \end{aligned} \quad (6.7)$$

This leads by Eq. (2.12) to the relation

$$\begin{aligned} V(\rho) &= S(X,Y)\; S^{(0)}(a,b)\; S(X,Y)^T\; V(\rho_0) \\ &\quad \times S(X,Y)\; S^{(0)}(a,b)\; S(X,Y)^T \end{aligned} \quad (6.8)$$

between the two variance matrices. Now the first and last factors on the right-hand side here have no influence on the spectrum, and so on the least eigenvalue, of $V(\rho)$. Therefore the squeezed or nonsqueezed nature of ρ is actually determined by the least eigenvalue of the matrix

$$\begin{aligned} &S(X,Y)^T\; V(\rho)\; S(X,Y) \\ &\quad = S^{(0)}(a,b)\; S(X,Y)^T\; V(\rho_0)\; S(X,Y)\; S^{(0)}(a,b). \end{aligned} \quad (6.9)$$

But now the right-hand side is in *general* dependent not only on the invariant parameters a, b but also on X, Y. In the examples studied in Sec. IV, namely, where ρ_0 is a coherent state or an isotropic thermal state, $V(\rho_0)$ happens to be a multiple of the identity matrix, so that on the right-hand side of Eq. (6.9) the dependence on X, Y cancels. But this need not happen in general. Thus, for instance, if we take for ρ_0 an anisotropic thermal state with unequal temperatures for the two modes, we have only U(1) × U(1), rather than U(2), invariance for this ρ_0; so the least eigenvalue $\ell(V(\rho))$ of $V(\rho)$ will depend on a, b and on two out of the four U(2) parameters present in $X - iY$. One can easily convince oneself that the only situation where $S(X,Y)^T\; V(\rho_0)\; S(X,Y) = V(\rho_0)$ independent of X and Y is when $V(\rho_0)$ is a multiple of the unit matrix; the isotropic thermal states do reproduce all such cases. Therefore a more detailed study of the effect of a general squeezing transformation on initial states ρ_0 with nontrivial $V(\rho_0)$ is of considerable interest.

A related important point is the following: let us take two squeezing transformations $P_1, P_2 \in \Pi$ belonging to equivalence classes (a_1,b_1), (a_2,b_2), respectively, which could coincide. The product P_1P_2 will in general be of the form $S(X,Y)P$ with $S(X,Y) \in \mathrm{U}(2)$ and $P \in \Pi$. If P belongs to the equivalence class (a,b) we wish to determine this class in terms of P_1 and P_2. Using that $\mathrm{Tr}(P^2) = \mathrm{Tr}((P_1P_2)(P_1P_2)^T)$ and Eq. (3.11) we arrive at the relations

$$\begin{aligned} &2[\cosh(a-b) + \cosh(a+b)] = \mathrm{Tr}((P_1P_2)(P_1P_2)^T), \\ &2[\cosh 2(a-b) + \cosh 2(a+b)] \\ &\quad = \mathrm{Tr}(((P_1P_2)(P_1P_2)^T)^2), \end{aligned} \quad (6.10)$$

which can be solved to find (a,b). We note here that (a,b) depend not only upon (a_1,b_1) and (a_2,b_2) but also on the actual elements chosen from each of these classes. So we do not have a notion of class multiplication among these equivalence classes.

Finally, we call attention to our considerations in Sec. V and to the need for being able to experimentally implement or realize general passive elements of the subgroup U(2) of Sp(4, ℝ) for each given choice of the independent modes in a two-mode system. Once this is achieved, for any given state we can bring out in an explicit or manifest fashion its squeezing nature (provided it is squeezed) by altering its variance matrix and making the least eigenvalue appear in the leading position on the diagonal. This also means that we would be able to experimentally measure the fluctuation in the quadrature variable isolating the least eigenvalue. As extensively discussed elsewhere, these considerations, which exploit the richness of the geometry underlying the symplectic group, do not require complete diagonalization of the variance matrix at all [10]. We shall elsewhere examine U(2)-invariant properties of two mode squeezed states that go beyond the level of second-order moments of the quadrature operators.

[1] D. Stoler, Phys. Rev. D **1**, 3217 (1970); H. P. Yuen, Phys. Rev. A **13**, 2226 (1976); D. F. Walls, Nature **306**, 141 (1983); J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).

[2] R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985); R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. DeVoe, and D. F. Walls, *ibid.* **57**, 691 (1986); Ling-An Wu, J. H. Kimble, J. L. Hall, and H. Wu, *ibid.* **57**, 2520 (1986); M. W. Maeda, P. Kumar, and J. H. Shapiro, Opt. Lett. **12**, 161 (1987).

[3] B. L. Schumaker, S. H. Perlmutter, R. M. Shelby, and M. D. Levenson, Phys. Rev. Lett. **58**, 357 (1987); M. G. Raizen, L. A. Orosco, M. Xiao, T. L. Boyd, and H. J. Kimble, *ibid.* **59**, 198 (1987); A. Heidmann, R. J. Horowicz, S. Reynaud, E. Giacobino, C. Fabre, and G. Camy, *ibid.* **59**, 2555 (1987); Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, *ibid.* **68**, 3663 (1992).

[4] P. Kumar, O. Aytur, and J. Huang, Phys. Rev. Lett. **64**, 1015 (1990); A. Sizmann, R. J. Horowicz, E. Wagner, and G. Leuchs, Opt. Commun. **80**, 138 (1990); M. Rosenbluh and R. M. Shelby, Phys. Rev. Lett. **66**, 153 (1991); K. Bergman and H. A. Haus, Opt. Lett. **16**, 663 (1991); E. S. Polzik, R. C. Carri, and H. J. Kimble, Phys. Rev. Lett. **68**, 3020 (1992).

[5] For recent reviews see R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987); M. C. Teich and B. E. A. Saleh, Quantum Opt. **1**, 153 (1990); S. Reynaud, A. Heidmann, E. Giacobino, and C. Fabre, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1992), Vol. 30; C. Fabre, Phys. Rep. **219**, 215 (1992); H. J. Kimble, *ibid.* **219**, 227 (1992).

[6] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); B. L. Schumaker and C. M. Caves, *ibid.* **31**, 3093 (1985); B. L. Schumaker, Phys. Rep. **135**, 317 (1986); C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[7] G. S. Agarwal, Phys. Rev. Lett. **57**, 827 (1986); J. Opt. Soc. Am. B **5**, 1940 (1988).

[8] A. K. Eckert and P. L. Knight, Amer. J. Phys. **57**, 692 (1989); C. M. Caves, C. Zhu, G. J. Milburn, and W. Schleich, Phys. Rev. A **43**, 3854 (1991); M. Selvadoray, M. Sanjay Kumar, and R. Simon (unpublished).

[9] C. K. Hong and L. Mandel, Phys. Rev. Lett. **54**, 323 (1985); Phys. Rev. A **32**, 974 (1985); C. C. Gerry and P. J. Moyer, *ibid.* **38**, 5665 (1988); M. S. Kim, F. A. M. de Oliviera, and P. L. Knight, *ibid.* **40**, 2494 (1989); J. J. Gong and P. K. Aravind, *ibid.* **46**, 1586 (1992); M. Hillery, *ibid.* **36**, 3796 (1987).

[10] R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994).

[11] T. F. Jordan, *Linear Operators in Quantum Mechanics* (Wiley, New York, 1974); G. Lion and M. Vergne, *The Weil Representation, Maslov Index, and Theta Series* (Birkhäuser, Basel, 1980).

[12] H. J. Lipkin, *Lie Groups for Pedestrians* (North-Holland, Amsterdam, 1966).

[13] N. Mukunda, Current Sci. **59**, 1135 (1990); B. Dutta, N. Mukunda, R. Simon, and A. Subramaniam, J. Opt. Soc. Am. B **10**, 253 (1993).

[14] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).

[15] C. L. Siegel, Am. J. Math. **65**, 1 (1943) [reprinted as C. L. Siegel, *Symplectic Geometry* (Academic, New York 1964)].

[16] Arvind, B. Dutta, N. Mukunda, and R. Simon (unpublished).

[17] H. P. Yuen and J. H. Shapiro, IEEE Trans. Info. Theory **24**, 657 (1978); M. D. Levenson, R. M. Shelby, A. Aspect, M. Reid, and D. F. Walls, Phys. Rev. A **32**, 1550 (1985); M. J. Collett, R. Loudon, and C. W. Gardiner, J. Mod. Opt. **34**, 881 (1987).

[18] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986); R. A. Campos, B. E. A. Saleh, and M. C. Teich, *ibid.* **40**, 1371 (1989); R. F. Bishop and A. Vourdas, Z. Phys. **71**, 527 (1988); C. T. Lee, Phys. Rev. A **42**, 4193 (1990); D. Han, Y. S. Kim, and M. E. Noz, *ibid.* **41**, 6233 (1990).

[19] R. Simon and N. Mukunda, Phys. Lett. A **143**, 165 (1990).

[20] A. Weil, Acta Mathematica **111**, 143 (1964); V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge University Press, Cambridge, 1984); R. Simon and N. Mukunda, in *Symmetries in Science VI: From the Rotation Group to Quantum Algebras*, edited by B. Gruber (Plenum, New York, 1993), p. 659.

[21] The connection between the groups Sp(4, ℝ) and SO(3,2) has been noticed and exploited in many physical contexts for a very long time. These include the infinite component relativistic wave equations constructed by Majorana, the Dirac equation for a relativistic point particle with two internal harmonic-oscillator degrees of freedom, the theory of relativistic objects with internal structure, and the classification of anisotropic Schell model beams in optics. See E. Majorana, Nuovo Cimento **9**, 335 (1932); P. A. M. Dirac, J. Math. Phys. **4**, 901 (1963); N. Mukunda, Phys. Rev. **183**, 1486 (1969); E. C. G. Sudarshan and N. Mukunda, Phys. Rev. D

1, 576 (1970); P. A. M. Dirac, Proc. R. Soc. London Ser. A **322**, 435 (1971); **328**, 1 (1972); N. Mukunda, H. van Dam, and L. C. Biedenharn, in *Relativistic Models of Extended Hadrons Obeying a Mass Spin Trajectory*, edited by A. Böhm and J. D. Dollard, Lecture Notes in Physics Vol. 165 (Springer-Verlag, Berlin, 1982); R. Simon, E. C. G. Sudarshan, and N. Mukunda, Phys. Rev. A **31**, 2419 (1985); **36**, 3868 (1987). A recent paper exploiting these ideas for squeezing problems is by D. Han, Y. S. Kim, M. E. Noz, and L. Yeh, J. Math. Phys. **34**, 5493 (1993).