Nonclassical effects in the gray-body state

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The photon-number distribution for the gray-body radiation given by Bekenstein and Schiffer [Phys. Rev. Lett. **72**, 2512 (1994)] is used to study its nonclassical effects. It is shown that as the absorptivity varies from 0 to 1, the gray-body state changes from an extremely nonclassical state to an extremely classical one. The critical point of this transition is located. It is discovered that the negative binomial state is a special case of the gray-body radiation at the transition point; hence, it is an example of borderline states such as the coherent state. The explicit expressions for the factorial moments, the Q parameter, the P function, and the nonclassical depth of the gray-body state as functions of absorptivity and temperature are derived.

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I. INTRODUCTION

The blackbody is an idealized object with absorptivity $a=1$. The importance of the blackbody radiation is due to its universality. The discovery of its phonon-number distribution by Planck [1] to be

$$p(n)=(1-e^{-x})e^{-nx} \tag{1.1}$$

with $x=\hbar\omega/kT$ marks the beginning of quantum theory. Surprisingly, not much attention has been paid to the more realistic case when $a<1$, which has been referred to as the gray body. This is perhaps because of the widespread feeling that the gray-body radiation is not of universal form. The universality of gray-body radiation has been established very recently by Bekenstein and Schiffer [2]. The conditional photon-number distribution for the gray-body radiation when the incident photon number is known to be m has also been established by these authors to be

$$p_a(n|m)=\frac{(e^x-1)e^{mx}a^{m+n}}{(e^x-1+a)^{m+n+1}}\times\sum_{i=0}^{\min(m,n)}\frac{(-1)^i(m+n-i)!}{i!(m-i)!(n-i)!}\times\left[1-4\frac{1-a}{a^2}\sinh^2(x/2)\right]^i, \tag{1.2}$$

which is identical to the distribution derived by Bekenstein and Meisels [3] for a Schwarzschild black hole. In the limit $a\to 1$, the square bracket in Eq. (1.2) is unity; then the authors of Ref. [2] have demonstrated by numerical method that the summation yields unity and Eq. (1.2) reduces to Eq. (1.1). Since these authors indicated that they were unable to do this analytically, we shall close the gap in the following.

Because the summand is symmetric in m and n, we can arbitrarily assume that $m\geq n$ without losing any generality. Then the summation can be rewritten as

$$S=\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n\\ i\end{bmatrix}\begin{bmatrix}m+n-i\\ m-i\end{bmatrix}, \tag{1.3}$$

which can be identified with the coefficient of x^m in the series expansion of the product

$$(1-x)^n(1-x)^{-n-1}=(1-x)^{-1}=\sum_{k=0}^{\infty}x^k\,; \tag{1.4}$$

therefore we have

$$S=1\,. \tag{1.5}$$

We shall refer to the quantum state of a single-mode radiation with photon-number distribution represented by Eq. (1.2) as the *gray-body state.* We are interested in the nonclassical effects in the gray-body state. Nonclassical effects are those that can never exist or occur if the system is not described quantum mechanically. To be more precise mathematically, whenever the P function (introduced independently by Glauber [4] and by Sudarshan [5]) is not positive definite, the quantum state it represents is defined to be nonclassical.

It will be shown that the gray-body state is the mixture of the thermal (blackbody) state and the number (Fock) state in the proportion of a to $1-a$. It is well known [6] that the number state is as nonclassical as a radiation state can be. On the other hand, we are not aware of any other state that is more classical than the thermal state. Therefore, by varying the value of a from 0 to 1, we can observe the transition of the gray-body state from an extremely nonclassical state to an extremely classical state. This is why we are so interested in studying the gray-body state.

II. FACTORIAL MOMENTS

Factorial moments of photon-number distribution can play a key role in the evaluation of photon statistics of a radiation field. In this section we shall derive the expression for the factorial moments of the gray-body state.

First of all, it will be more convenient to replace the upper limit, $\min(m,n)$, of the summation in Eq. (1.2) by just a fixed m; this is all right because, if $m>n$, then the summand vanishes for $m\geq i>n$ since the factorial of a negative integer is infinity. It will also be more convenient to rewrite the expression inside the square brackets of Eq. (1.2) as

$$1-4\frac{1-a}{a^2}\sinh^2(x/2)=-(e^x-1+a)(1-a-e^{-x})/a^2\ . \tag{2.1}$$

Then Eq. (1.2) can be rewritten as

$$p_a(n|m)=\frac{(e^x-1)e^{mx}a^{m+n}}{(e^x-1+a)^{m+n+1}}\sum_{i=0}^{m}\frac{(m+n-i)!}{i!(m-i)!(n-i)!}[(e^x-1+a)(1-a-e^{-x})/a^2]^i\ . \tag{2.2}$$

The factorial moments are defined as

$$\langle n^{(k)}\rangle=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}p_a(n|m)\ . \tag{2.3}$$

Using Eq. (2.2) in Eq. (2.3) and changing the order of summations, we obtain

$$\langle n^{(k)}\rangle=\frac{(e^x-1)e^{mx}a^m}{(e^x-1+a)^{m+1}}\sum_{i=0}^{m}\begin{bmatrix}m\\ i\end{bmatrix}\left[\frac{1-a-e^{-x}}{a}\right]^i S_i^{(k)}\ , \tag{2.4}$$

where

$$S_i^{(k)}=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}\begin{bmatrix}m+n-i\\ n-i\end{bmatrix}\left[\frac{a}{e^x-1+a}\right]^{n-i}\ . \tag{2.5}$$

Using the following identity

$$\frac{n!}{(n-k)!}=\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\frac{(n-i)!i!}{(n-i-k+j)!(i-j)!} \tag{2.6}$$

in Eq. (2.5) and changing the order of summations, we have

$$\begin{aligned}S_i^{(k)}&=k!\sum_{j=0}^{k}\begin{bmatrix}i\\ j\end{bmatrix}\begin{bmatrix}m+k-j\\ k-j\end{bmatrix}\sum_{n=i+k-j}^{\infty}\begin{bmatrix}m+n-i\\ m+k-j\end{bmatrix}\left[\frac{a}{e^x-1+a}\right]^{n-i}\\ &=k!\left[\frac{e^x-1+a}{e^x-1}\right]^{m+1}\sum_{j=0}^{k}\begin{bmatrix}i\\ j\end{bmatrix}\begin{bmatrix}m+k-j\\ k-j\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{k-j}\ .\end{aligned} \tag{2.7}$$

Using Eq. (2.7) in Eq. (2.4) and changing the order of summations, we obtain

$$\begin{aligned}\langle n^{(k)}\rangle&=k!\left[\frac{e^xa}{e^x-1}\right]^m\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\begin{bmatrix}m+k-j\\ k\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{k-j}\sum_{i=j}^{m}\begin{bmatrix}m-j\\ i-j\end{bmatrix}\left[\frac{1-a-e^{-x}}{a}\right]^i\\ &=k!\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\begin{bmatrix}m+k-j\\ k\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{k-j}\left[\frac{1-a-e^{-x}}{1-e^{-x}}\right]^j\ .\end{aligned} \tag{2.8}$$

But we can put the last expression in a more convenient form by using the following binomial expansion:

$$\left[\frac{1-a-e^{-x}}{1-e^{-x}}\right]^j=(-1)^j\sum_{i=0}^{j}(-1)^i\begin{bmatrix}j\\ i\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{j-i}(1-a)^i\ . \tag{2.9}$$

Substitution of Eq. (2.9) into Eq. (2.8) followed by a change in the order of summations gives

$$\langle n^{(k)}\rangle=k!\sum_{i=0}^{k}(-1)^i\begin{bmatrix}k\\ i\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{k-i}(1-a)^i\sum_{j=i}^{k}(-1)^j\begin{bmatrix}k-i\\ j-i\end{bmatrix}\begin{bmatrix}m+k-j\\ k\end{bmatrix}\ . \tag{2.10}$$

Using the identity

$$\sum_{j=i}^{k}(-1)^j\begin{bmatrix}k-i\\j-i\end{bmatrix}\begin{bmatrix}m+k-j\\k\end{bmatrix}=(-1)^i\begin{bmatrix}m\\i\end{bmatrix} \tag{2.11}$$

in Eq. (2.10), we finally obtain a better expression for the factorial moments as

$$\langle n^{(k)}\rangle=k!\sum_{i=0}^{\min(k,m)}\begin{bmatrix}k\\i\end{bmatrix}\begin{bmatrix}m\\i\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{k-i}(1-a)^i . \tag{2.12}$$

We are now in the position to evaluate the nonclassical effects of the gray-body state by using Mandel's parameter [7]

$$Q\equiv(\langle n^{(2)}\rangle-\langle n\rangle^2)/\langle n\rangle . \tag{2.13}$$

Whenever $Q<0$, we have a sub-Poissonian photon-number distribution, which is possible only for a nonclassical state.

From Eq. (2.12) we can obtain the explicit expressions for the first two factorial moments to be

$$\langle n\rangle=(1-a)m+\frac{a}{e^x-1} \tag{2.14}$$

and

$$\langle n^{(2)}\rangle=m(m-1)(1-a)^2+\frac{4ma(1-a)}{e^x-1}+\frac{2a^2}{(e^x-1)^2} . \tag{2.15}$$

Substitution of Eqs. (2.14) and (2.15) into Eq. (2.13) gives

$$Q=\frac{-m(1-a)^2(e^x-1)^2+2ma(1-a)(e^x-1)+a^2}{(e^x-1)[m(e^x-1)(1-a)+a]} . \tag{2.16}$$

Examination of Eq. (2.16) indicates that when the absorptivity $a\to0$, or when the temperature $T\to0$ ($x\to\infty$), we have $Q\to-1$, and we definitely have a nonclassical state. On the other hand, when $n\to\infty$, we will have a nonclassical state if

$$a<\tanh(x/2) . \tag{2.17}$$

III. *P* FUNCTION

To determine the probability distribution function from the knowledge of its moments is generally referred to as the problem of moments [8]. Historically, only normalizable positive distribution functions are considered in such a problem. However, in 1963 Glauber [4] and Sudarshan [5] independently introduced the so-called P representation of quantum states as quasi distributions in the complex domain which are known as P functions. The P functions not only can assume negative values, but also are typically highly singular in the sense that it consists of high-order derivatives of Dirac's delta function in the complex domain. Therefore, the moment problem presents new difficulties.

The moment problem of P functions was recently solved in practical sense by Lee [9], who derived an expression for the P function in terms of Dirac's delta function in the complex domain and its derivatives as follows:

$$P(z,\bar z)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\mu_{k,l}}{k!l!}\delta^{(k,l)}(z,\bar z) , \tag{3.1}$$

where $\bar z$ is the complex conjugate of z and

$$\delta^{(k,l)}(z,\bar z)\equiv\frac{\partial^{k+l}}{\partial z^k\partial\bar z^l}\delta(z,\bar z) \tag{3.2}$$

are the derivatives of the delta function in the complex domain. The moments $\mu_{k,l}$, appearing as the coefficients in this expression, can be expressed as

$$\mu_{k,l}\equiv\langle(\hat a^\dagger)^l(\hat a)^k\rangle=\mathrm{Tr}\{(\hat a^\dagger)^l(\hat a)^k\hat\rho\}=\frac{1}{\pi}\int d^2z\, z^k\bar z^lP(z,\bar z) , \tag{3.3}$$

where $\hat a$ and $\hat a^\dagger$ are the annihilation and creation operators, respectively, $\hat\rho$ is the density operator for the quantum state, and d^2z/π is the measure of the complex domain.

The density matrix for the gray-body state can be written as

$$\hat\rho(a,m)=\sum_{n=0}^{\infty}p_a(n|m)|n\rangle\langle n| . \tag{3.4}$$

Its moments can be calculated as follows:

$$\begin{aligned}\mu_{k,l}&=\sum_{n=0}^{\infty}p_a(n|m)\langle n|(\hat a^\dagger)^l(\hat a)^k|n\rangle\\&=\sum_{n=0}^{\infty}p_a(n|m)\frac{n!}{(n-k)!}\delta_{k,l}\\&=\langle n^{(k)}\rangle\delta_{k,l} .\end{aligned} \tag{3.5}$$

So the moments of the P function are identical to the factorial moments of the photon-number distribution, which have been calculated in the preceding section.

Substitution of Eq. (2.12) into Eq. (3.1) yields

$$P(z,\bar z)=\sum_{k=0}^{\infty}\frac{1}{k!}\delta^{(k,k)}(z,\bar z)\times\sum_{i=0}^{\min(k,m)}\begin{bmatrix}k\\i\end{bmatrix}\begin{bmatrix}m\\i\end{bmatrix}\left[\frac{a}{e^x-1}\right]^{k-i}(1-a)^i . \tag{3.6}$$

Unfortunately, it is not easy to make much sense out of this expression. Sometimes it is possible to convert an apparently singular expression into a regular expression. One way to achieve such a goal, as suggested in Ref. [9], is through a Fourier transformation. We adopt the definition of Cahill and Glauber [10] to calculate the Fourier transform of a function $\varphi(z,\bar z)$ in the complex domain according to the formula

$$\tilde\varphi(w,\bar w)\equiv\frac{1}{\pi}\int\exp(w\bar z-\bar wz)\varphi(z,\bar z)d^2z . \tag{3.7}$$

Such a definition is in conformity with the displacement operator, which generates the coherent states from the vacuum, and with the definition of the characteristic functions in P representation. It is referred to as *symmetric Fourier transformation* because the inverse transform is calculated according to exactly the same formula as the direct transform, so we have $\tilde{\tilde{\varphi}}=\varphi$. According to this definition, it can be shown that [9]

$$\tilde{\delta}^{(k,l)}(w,\bar{w})=(\bar{w})^k(-w)^l,\quad k,l=0,1,2,\ldots . \tag{3.8}$$

Using Eq. (3.8) in Eq. (3.6) and changing the order of summations, we obtain the Fourier transform of the $P(z,\bar{z})$ function as

$$\begin{aligned}\tilde{P}(w,\bar{w})&=\sum_{i=0}^{\infty}\frac{1}{i!}\begin{bmatrix}m\\ i\end{bmatrix}(1-a)^i\\ &\quad\times\sum_{k=i}^{\infty}\frac{1}{(k-i)!}\left[\frac{a}{e^x-1}\right]^{k-i}(-|w|^2)^k\\ &=\exp\left[-\left[\frac{a}{e^x-1}\right]|w|^2\right]L_m((1-a)|w|^2)\ ,\end{aligned} \tag{3.9}$$

where

$$L_m(x)=\sum_{i=0}^{\infty}\frac{1}{i!}\begin{bmatrix}m\\ i\end{bmatrix}(-x)^i \tag{3.10}$$

is the Laguerre polynomial.

It is well known that the P function for the thermal (blackbody) state is

$$P_{\rm th}(z,\bar{z})=(e^x-1)\exp[-(e^x-1)|z|^2] \tag{3.11}$$

with Fourier transform

$$\tilde{P}_{\rm th}(z,\bar{z})=\exp[-|z|^2/(e^x-1)]\ . \tag{3.12}$$

It is also established in Ref. [9] that the P function for the number (Fock) state of definite photon number m is

$$P_m(z,\bar{z})=\sum_{k=0}^{m}\frac{1}{k!}\begin{bmatrix}m\\ k\end{bmatrix}\delta^{(k,k)}(z,\bar{z}) \tag{3.13}$$

with Fourier transform

$$\tilde{P}_m(w,\bar{w})=L_m(|w|^2)\ . \tag{3.14}$$

On the other hand, it is well known that the product of the Fourier transforms of two functions is the Fourier transform of the convolution product of the two functions. It is also well known that the convolution product of the P functions of two quantum states is the P function of the mixture of the two quantum states. With these facts in mind and comparing Eqs. (3.12) and (3.14) with Eq. (3.9), we can see that the Fourier transform of the P function for the gray-body state clearly reflects the fact that it is the mixture of the thermal state and the number state in the proportion of a to $1-a$.

Using Eqs. (3.11) and (3.13) and the convolution theorem of Fourier transforms, we can obtain another expression for the P function of the gray-body state as a convolution integral

$$P_m(z,\bar{z})=\frac{1}{\pi}\int d^2v\left\{\frac{e^x-1}{a}\exp\left[-\frac{e^x-1}{a}(z-v)(\bar{z}-\bar{v})\right]\right\}\sum_{k=0}^{m}\frac{(1-a)^k}{k!}\begin{bmatrix}m\\ k\end{bmatrix}\delta^{(k,k)}(v,\bar{v})\ . \tag{3.15}$$

By repeated integrations by parts, we can convert the last expression in terms of distributions into a regular function. But it is easier to obtain the regular function by the symmetric Fourier transformation of Eq. (3.9) directly.

Since our Fourier transform is symmetric, we can use the result of the Appendix on Eq. (3.9) to obtain the (inverse) Fourier transform of $\tilde{P}(w,\bar{w})$. Substitution of $a/(e^x-1)$ for γ and $1-a$ for β in Eq. (A6) gives

$$P(z,\bar{z})=\frac{e^x-1}{a}\left[-\frac{e^x-ae^x-1}{a}\right]^m\exp\left[-\left[\frac{e^x-1}{a}\right]|z|^2\right]L_m\left[\frac{(1-a)(e^x-1)^2|z|^2}{a(e^x-ae^x-1)}\right]\ . \tag{3.16}$$

This is the P function for the gray-body state expressed as an ordinary function.

IV. NONCLASSICAL DEPTH

A continuous parameter τ was recently introduced [6] into the convolution transformation of the P function to define a general distribution function as

$$R(z,\bar{z},\tau)=\frac{1}{\pi\tau}\int\exp\left[-\frac{1}{\tau}|z-w|^2\right]P(w,\bar{w})d^2w\ . \tag{4.1}$$

We shall call $R(z,\bar{z},\tau)$ the R function. The original P and Q functions are two limiting cases of the R function with $\tau=0$ and 1, respectively.

It should be pointed out that similar continuous parameters have been introduced before: the ε parameter by Graham *et al.* in 1968 [11] and the s parameter by Cahill and Glauber in 1969 [10]. Our τ parameter is related to these two previously introduced parameters as $\tau=1-\varepsilon$ and $\tau=(1-s)/2$. Our motivation for introducing this τ parameter is to define a measure of how nonclassical a quantum state is. For this purpose, the τ parameter seems to be more natural than the other two.

It is well known that the origin of the nonclassical effects is that the P function of a quantum state is not positive definite; hence it is called quasidistribution function. On the other hand, the Q function is always a positive definite regular function. The reason that the Q function behaves better than the P function is because a convolution transformation can be viewed as a *moving average;* so it has the effect of making the transformed function smoother. The smoothing effect of the convolution transformation of Eq. (4.1) is enhanced as τ increases. If τ is large enough so that the R function becomes acceptable as a classical distribution function, i.e., it is a positive definite regular function and normalizable, then we say that the smoothing operation is complete. Let Ω denote the set of all the τ that will complete the smoothing of the P function of a quantum state and let the greatest lower bound, or infimum, of all the τ in Ω be denoted by

$$\tau_m \equiv \inf_{\tau\in\Omega} (\tau) . \tag{4.2}$$

We have proposed recently [6] to define τ_m as the nonclassical depth of the quantum state.

According to this definition, we have $\tau_m=0$ for the coherent state $|\alpha\rangle$ because its P function is a delta function; this is very reasonable since the coherent state is known to be on the borderline between classical and nonclassical states. On the other hand, for $\tau=1$ we have $R(z,\bar{z},1)=Q(z,\bar{z})$, which is always acceptable as a classical distribution function for any quantum state; hence, 1 is an upper bound for τ_m. Therefore, we can specify the range of τ_m to be

$$0\le\tau_m\le1 . \tag{4.3}$$

Applying the convolution theorem of Fourier transforms to Eq. (4.1), we can express the Fourier transform of the R function as

$$\tilde{R}(w,\bar{w},\tau)=\exp(-\tau w\bar{w})\tilde{P}(w,\bar{w}) . \tag{4.4}$$

Substitution of Eq. (3.9) into Eq. (4.4) gives

$$\tilde{R}(w,\bar{w},\tau)=\exp\left[-\left[\frac{a}{e^x-1}+\tau\right]|w|^2\right]L_m((1-a)|w|^2) , \tag{4.5}$$

which is still the product of an exponential function and a Laguerre polynomial with arguments proportional to $|w|^2$; so we can again use the formula derived in the Appendix to obtain the R function is

$$R(z,\bar{z},\tau)=\frac{e^x-1}{\tau(e^x-1)+a}\left[-\frac{(e^x-1)-ae^x-\tau(e^x-1)}{\tau(e^x-1)+a}\right]^m\exp\left[-\left[\frac{e^x-1}{a}\right]|z|^2\right]$$
$$\times L_m\left[\frac{(1-a)(e^x-1)^2|z|^2}{[\tau(e^x-1)+a][(e^x-1)-ae^x-\tau(e^x-1)]}\right] . \tag{4.6}$$

The Laguerre polynomial can be positive definite only if its argument is always negative, and the only way for the latter to be negative is

$$(e^x-1)-ae^x-\tau(e^x-1)<0 . \tag{4.7}$$

When this condition is satisfied, the R function of Eq. (4.6) is a positive definite regular function. Therefore, the nonclassical depth of the gray-body state is

$$\tau_m=1-a-\frac{a}{e^x-1} . \tag{4.8}$$

Examination of this expression indicates that, as a varies from 0 to 1, τ_m varies from 1 to $-1/(e^x-1)$; and the transition from a positive value to a negative one occurs at

$$a=1-e^{-x} , \tag{4.9}$$

which means that the gray-body state is a nonclassical state when $a<1-e^{-x}$ and becomes a classical state when $a>1-e^{-x}$. The negative value for τ_m mentioned above is out of the range given in Eq. (4.3), so we should be cautious about its meaning.

V. NEGATIVE BINOMIAL STATE

We now focus our attention on the critical situation when the condition of Eq. (4.9) is satisfied.

First of all, we have

$$\sum_{i=0}^{m}\frac{(m+n-i)!}{i!(m-i)!(n-i)!}\left[\frac{(e^x-1+a)(1-a-e^{-x})}{a^2}\right]^i=\frac{(m+n)!}{m!n!} , \tag{5.1}$$

because the summand vanishes except when $i=0$. The condition of Eq. (4.9) can also be written as

$$e^x=1/(1-a) . \tag{5.2}$$

Substitution of Eqs. (5.1) and (5.2) into Eq. (2.2) yields a much simpler photon-number distribution

$$p_a^c(n|m)=\frac{(m+n)!}{m!n!}\frac{(1-a)^n}{(2-a)^{m+n+1}} , \tag{5.3}$$

which represents a special case of the negative binomial state discussed recently by Matsuo [12] and by Gray, Srinivasan, and Lee [13].

Using Eq. (5.2) in Eqs. (2.12) and (2.13), we obtain the factorial moments and the Q parameter as

$$\langle n^{(k)}\rangle^c=(1-a)^k(m+k)!/m! \tag{5.4}$$

and

$$Q^c=1-a \ , \tag{5.5}$$

respectively.

Using Eq. (5.2) in Eq. (3.16), we obtain the P function for the negative binomial state as

$$P^c(z,\bar{z})=\frac{1}{m!}\frac{|z|^{2m}}{(1-a)^{m+1}}\exp\left[-\frac{|z|^2}{1-a}\right] , \tag{5.6}$$

which is in agreement with the expression given in Refs. [12] and [13] for the special case when the continuous parameter is an integer ($\lambda=m+1$).

It is well known that the coherent state is right on the borderline between classical and nonclassical states because its nonclassical depth $\tau_m=0$ [9] and its Q parameter also vanishes. We were under the impression that the coherent state is unique in being "the" borderline state. We now recognize another borderline state.

We can see more clearly why the negative binomial state is a borderline state by focusing our attention on the following factor of the P function given in Eq. (3.16):

$$F(|z|^2)=\left[-\frac{e^x-ae^x-1}{a}\right]^m L_m\left[\frac{(1-a)(e^x-1)^2|z|^2}{a(e^x-ae^x-1)}\right] . \tag{5.7}$$

For $a<1-e^{-x}$, $F_m(|z|^2)$ is a polynomial in $|z|^2$ with alternative signs in its coefficients; and it is well known that the Laguerre polynomial $L_m(x)$ has m real positive roots; therefore, $F_m(|z|^2)$ cannot be positive definite. On the other hand, for $a>1-e^{-x}$, $F_m(|z|^2)$ is a polynomial in $|z|^2$ with all positive coefficients, hence, it is positive definite. However, at $a=1-e^{-x}$, all the coefficients of $F_m(|z|^2)$ vanish except for the highest-order one.

Unfortunately, the Q parameter given in Eq. (5.5) does not reflect the borderline-state nature of the negative binomial state. It should be very interesting if we can find some more practical ways to demonstrate the nonclassical effects in quantum states in the neighborhood, but on the nonclassical side, of the negative binomial state.

VI. SUMMARY

Using the photon-number distribution given by Bekenstein and Schiffer, we have studied the nonclassical effects in the gray-body radiation. We have derived explicit expressions for the factorial moments of the photon-number distribution and for the P function as a regular, not necessarily positive definite, function. The nonclassical effects have been examined according to Mandel's Q parameter and Lee's nonclassical depth. The Q parameter depends on the number of the incident photons, but the nonclassical depth is independent of it. This discrepancy can be understood in the following sense: While the nonclassical depth is an overall measure of the nonclassicality of a nonclassical state, the sub-Poissonian photon-number distribution as indicated by the negative value of the Q parameter is just one of many possible manifestations of the nonclassical nature of the radiation state; it is quite possible for a nonclassical state to have super-Poissonian photon-number distribution.

The interesting thing about the gray-body radiation is that, as the absorptivity a varies from 0 to 1, it changes from an extremely nonclassical state to an extremely classical one, covering the whole spectrum. The most interesting discovery of this study is that the negative binomial state is a special case of the gray-body radiation at the critical point of the transition from nonclassical to classical states and that the negative binomial state is an example of a borderline state, in addition to the coherent state; the latter had been assumed to be unique in this respect.

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APPENDIX

In this appendix we shall show that the symmetric Fourier transform of the product of an exponential function and a Laguerre polynomial with arguments proportional to $|z|^2$ is of the same form except for a constant factor and different proportional constants in the arguments.

Cahill and Glauber [10] have derived a very useful integral identity in the complex domain

$$\frac{1}{\pi}\int\exp(\alpha\bar{z}+\beta z-\gamma|z|^2)d^2z=\frac{1}{\gamma}\exp\left[\frac{\alpha\beta}{\gamma}\right] . \tag{A1}$$

We are particularly interested in a special case of the identity when $\beta=-\bar{\alpha}$, which gives the Fourier transform of a Gaussian function $f(z,\bar{z})=\exp(-\gamma|z|^2)$ as

$$\begin{aligned}\tilde{f}(\alpha,\bar{\alpha})&=\frac{1}{\pi}\int\exp(\alpha\bar{z}-\bar{\alpha}z-\gamma|z|^2)d^2z\\&=\frac{1}{\gamma}\exp(-\alpha\bar{\alpha}/\gamma) .\end{aligned} \tag{A2}$$

Using the above relation, we can easily obtain the Fourier transform of the function

$$f_k(z,\bar{z})=|z|^{2k}\exp(-\gamma|z|^2) \tag{A3}$$

as

$$\begin{aligned}\tilde{f}_k(\alpha,\bar{\alpha})&=\frac{1}{\gamma}\left[\frac{\partial}{\partial\alpha}\right]^k\left[-\frac{\partial}{\partial\bar{\alpha}}\right]^k\exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]\\&=k!\left[\frac{1}{\gamma}\right]^{k+1}\exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]L_k\left[\frac{\alpha\bar{\alpha}}{\gamma}\right] .\end{aligned} \tag{A4}$$

Using Eq. (A4), we can further derive the Fourier transform of the function

$$g_n(z,\bar{z})=\exp(-\gamma|z|^2)L_n(\beta|z|^2) \tag{A5}$$

as

$$
\begin{aligned}
\tilde{g}_n(\alpha,\bar{\alpha}) &= \frac{1}{\gamma}\exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]\sum_{k=0}^{n}\binom{n}{k}\left[-\frac{\beta}{\gamma}\right]^k\sum_{j=0}^{k}\frac{1}{j!}\binom{k}{j}\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]^j \\
&= \frac{1}{\gamma}\exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]\sum_{j=0}^{n}\frac{1}{j!}\binom{n}{j}\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]^j\sum_{k=j}^{n}\binom{n-j}{k-j}\left[-\frac{\beta}{\gamma}\right]^k \\
&= \frac{1}{\gamma}\exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]\sum_{j=0}^{n}\frac{1}{j!}\binom{n}{j}\left[\frac{\alpha\bar{\alpha}\beta}{\gamma^2}\right]^j\left[1-\frac{\beta}{\gamma}\right]^{n-j} \\
&= \frac{1}{\gamma}\left[1-\frac{\beta}{\gamma}\right]^n\exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]L_n\left[\frac{\alpha\bar{\alpha}\beta}{\gamma(\beta-\gamma)}\right] .
\end{aligned} \tag{A6}
$$

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