Entrainment of solid-state laser arrays

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We find that the natural antiphasing tendency in linear solid-state laser arrays can be overcome by an injected field, even if the N elements are not identical. We derive a condition for full entrainment that agrees well with numerical simulations using experimentally accessible parameters. The resulting output intensity saturates near the maximum coherent value of N^2 times that of a single laser. We find that the entrained output can be modulated in a prescribed manner by a suitable choice of the injected field.

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I. INTRODUCTION

Laser arrays are promising for applications that require high optical power from a compact source [1,2]. The most efficient mode of operation is when the elements are synchronized so that the output constructively interferes and the light intensity is maximized. Both semiconductor [3–6] and solid-state [7–9] arrays are subjects of current research, and aspects of their dynamical behavior have been reported previously. Solid-state media, such as neodymium-doped yttrium aluminum garnet (Nd:YAG) [8], offer certain advantages for synchronization: unlike semiconductor lasers they do not suffer from linewidth enhancement.

Quite apart from their potential applications, laser arrays provide an intriguing class of nonlinear dynamical systems with many degrees of freedom. Of particular current interest is the spontaneous emergence of dynamically ordered behavior wherein mutually interacting elements execute synchronized oscillations. This phenomenon is believed to be important in a variety of physical and biological systems [10], yet our current theoretical understanding of the subject is in its infancy. It will turn out that the laser array dynamics considered in this paper reduces, under certain conditions, to perhaps the most widely studied class of coupled nonlinear oscillators, so-called phase models [11].

In this paper we study the entrainment of frequency and phase in a one-dimensional array of N solid-state lasers. Sufficient motivation for this work likes in the experiments of Goldberg *et al.* [12], who used less than 3 mW of injected power to generate 105 mW output power from a ten-element laser diode array. The basic physics of our system can be summarized as follows. Each laser has its own internal nonlinear dynamics, governed by its complex electric field and atomic inversion (the polarization is adiabatically eliminated). Adjacent lasers are evanescently coupled through overlap of their electric fields. The dynamical equations describing the time evolution of coupled lasers were introduced by Basov, Belenov, and Letokhov [13], Perel and Rogova [14], and Spencer and Lamb [15]. In addition, we consider each laser to be driven by an externally injected field, which we discuss further below. The point of view described here is that of N individual lasers governed by equations for their own dynamical variables and coupled by evanescent overlap. This approach is valid only in the limit of weak coupling since generally one should start by determining the global supermodes of the system and then write down amplitude equations for their excitation [5]. Nevertheless, the weak-coupling limit is a physically important one for the discussion of semiconductor and solid-state arrays. Recent experiments on two coupled solid-state (Nd:YAG) lasers have shown [8] that the effect of the coupling is to frequency entrain the elements, but the lasers are pushed out of phase. This is also true in semiconductor arrays [12] and can be compensated for by use of a π phase shifter. Early studies of the phase locking of laser arrays [16] developed a supermode theory, but did not address the stability of the in-phase state: an in-phase solution for the array exists, but is unstable with respect to the out-of-phase state.

Dynamical studies of injection locking in semiconductor arrays have been made previously [5]. These include the effects of linewidth enhancement and other specific features not relevant to the solid-state arrays of interest here. We show theoretically that the natural dephasing effects of evanescent coupling can be overcome by injecting a common field to each laser. Our goal is to achieve a frequency entrained and nearly in-phase operation so that maximum constructive interference is achieved. In so doing the array must overcome the intrinsic coupling of elements responsible for destabilizing the in-phase state. Moreover, any variation in the parameters across the array introduces an element of disorder, which can further hinder synchronization. We find that it is possible to overcome both the intrinsic antiphasing and a degree of disorder with a monochromatic external field to which each laser is entrained in frequency and phase. Moreover, we find it is possible to entrain the array with nonmonochromatic fields, such as an amplitude modulated field, and in this way simultaneously achieve constructive interference and an output that is modulated according to the frequency spectrum of the injected field.

The paper is organized as follows. In Sec. II we discuss the theoretical model for the laser array and derive an ap-

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proximate description based on coupled equations for the laser phases. This model is analyzed in Sec. III and a simple analytic criterion for stability of the in-phase state is derived for the cases of monochromatic and periodically modulated injected fields. In Sec. IV we present numerical simulations of the full laser array equations to test the predictions of Sec. III: good agreement is found. The effects of disorder and the entrainment of arrays via quasiperiodic injection fields are also illustrated. In Sec. V we summarize our conclusions.

II. THEORETICAL MODEL

The equations for a linear array of N lasers with the polarization adiabatically eliminated are given by

$$\dot{E}_j(t)=(G_j-\xi_j+i\delta_j)E_j+\kappa(E_{j+1}+E_{j-1})+E_e(t)\ , \quad (1a)$$

$$\dot{G}_j(t)=\frac{\tau_c}{\tau_f}[p_j-(1+|E_j|^2)G_j]\ , \quad (1b)$$

where $j=1,\ldots,N$ and with free end boundary conditions $E_0(t)=E_{N+1}(t)=0$. The variables E_j and G_j are the dimensionless complex electric field and gain for the jth laser. All times and frequencies are scaled relative to the cavity round-trip time τ_c and τ_f is the fluorescence time of the laser medium; ξ_j and p_j are, respectively, the dimensionless cavity decay and pump rates for the jth laser, κ is the evanescent coupling constant between adjacent lasers, and $E_e(t)$ is the slowly varying amplitude of the external field that drives each laser [5,17]. Equations (1a) and (1b) are written in a frame rotating at the frequency ω_e, at which the external field has a nonzero Fourier component. This frequency is tuned in such a way as to minimize the detuning from the cavity resonances. In practice, the output power emitted from an array depends on how an external field is tuned to the cavities [18]. The detuning $\delta_j=\omega_e-\omega_{cj}-G_j\Delta\omega_j\approx\omega_e-\omega_{cj}$, where ω_{cj} is the cavity resonance frequency for laser j and $\Delta\omega$ is the atomic detuning from ω_e in units of the polarization decay rate. For solid-state lasers we ignore the latter dynamic contribution to the detuning. A variation in detuning among the lasers would result from a variation in cavity lengths for the laser elements. However, we have in mind a single cavity containing the array. An additional contribution could arise from a careful analysis of the intracavity propagation and the influence of evanescent coupling. In the following we allow for a small spread in detunings as a way to test the robustness of the entrainment mechanism to a physically reasonable parameter spread.

Substituting $E_j(t)=\sqrt{I_j(t)}\exp[i\phi_j(t)]$, where $I_j(t)$ and $\phi_j(t)$ are the intensity and the phase of laser j, $E_e(t)=\sqrt{I_e}f(t)$, and assuming that all the lasers have the same losses ($\xi_j=\xi$) and pump rates ($p_j=p$) gives

$$\begin{aligned}\dot{I}_j&=2(G_j-\xi)I_j+2\sqrt{I_eI_j}\mathrm{Re}[f(t)e^{-i\phi_j}]\\&\quad+2\kappa[\sqrt{(I_{j+1}I_j)}\cos(\phi_{j+1}-\phi_j)\\&\quad+\sqrt{(I_{j-1}I_j)}\cos(\phi_{j-1}-\phi_j)]\ ,\end{aligned} \quad (2a)$$

$$\begin{aligned}\dot{\phi}_j&=\delta_j+\kappa\left[\left[\frac{I_{j+1}}{I_j}\right]^{1/2}\sin(\phi_{j+1}-\phi_j)\right.\\&\quad\left.+\left[\frac{I_{j-1}}{I_J}\right]^{1/2}\sin(\phi_{j-1}-\phi_j)\right]\\&\quad-\left[\frac{I_e}{I_j}\right]^{1/2}\mathrm{Im}[f(t)e^{-i\phi_j}]\ ,\end{aligned} \quad (2b)$$

$$\dot{G}_j=\frac{\tau_c}{\tau_f}(p-G_j-G_jI_j)\ . \quad (2c)$$

These are the equations we use in later numerical simulations. When the amplitude of an injected field I_e is small enough, intensity variations across the array may be self-consistently ignored provided $|I_j(t)-I|/I\ll 1$, where the I are the independent laser ($I_e=\kappa=0$) stationary intensities $I=p/\xi-1$. This condition is satisfied in our numerical simulations. Simulations also show that this approximation breaks down when the coupling and detunings become large enough. Assuming this is not the case, the result is that the phase variables decouple from the intensities and gains [19]:

$$\begin{aligned}\dot{\phi}_j&=\delta_j+\kappa[\sin(\phi_{j+1}-\phi_j)+\sin(\phi_{j-1}-\phi_j)]\\&\quad-A_e\mathrm{Im}[f(t)e^{-i\phi_j}]\ ,\end{aligned} \quad (3)$$

where $A_e=\sqrt{I_e/I}$ is a measure of the injection field amplitude and $f(t)$ is its temporal profile. The free end boundary conditions imply that $\phi_0(t)=\phi_1(t)$ and $\phi_N(t)=\phi_{N+1}(t)$. Although a complete first-principles theory for the determination of κ is lacking for solid-state laser arrays at the present time (for the case of multistripe semiconductor arrays see Ref. [5]), experiments on evanescently coupled Nd:YAG arrays in the absence of an external field have shown the physical effects of the coupling to be consistent with $\kappa<0$. We assume this to be the case in what follows.

Before further analysis we can gain some insight into the competing physical mechanisms by considering various limits in Eq. (3). Consider first the case of zero driving field and zero detuning ($A_e=\delta_j=0$), which corresponds to laser oscillation at cavity resonance. The system them admits a family of fixed points, by taking $\phi_j-\phi_{j-1}=0$ or π independently for each j. However, for $\kappa<0$ the only stable states are the out-of-phase "antiferromagnetic" states where $\phi_j-\phi_{j-1}=\pi$ for all j. By contrast, the in-phase state ($\phi_j=\phi_0$ for all j) is unstable: setting $\phi_j=\phi_0+\eta_j$ and linearizing Eq. (6) for small deviations η_j yields $\dot{\eta}_j=\kappa(\eta_{j+1}-2\eta_j+\eta_{j-1})$. Since for $\kappa<0$ all N eigenvalues are positive, the coupling produces a "negative diffusion," which amplifies any difference between neighboring elements. In the opposite limit of zero coupling and monochromatic injected field [$\kappa=0$ and $A_e>0$ with $f(t)=\mathrm{const}$], the laser phases will tend to lock to that of the drive provided the amplitude A_e is sufficiently large to overcome any spread in detunings δ_j.

Recapping, we see that Eq. (3) contains three competing effects. The evanescent coupling drives the system away from the in-phase state towards the out-of-phase state, a spread in detunings tends to destroy the frequency and phase locking, and the injected field tends to entrain the lasers to a common frequency and phase.

III. ANALYSIS OF THE PHASE MODEL

The next step is to try to understand, on the basis of Eq. (3), the circumstances under which the injected field can overcome the effects of the coupling and any spread in detunings. This is illustrated by considering a monochromatic and an amplitude modulated injection field, respectively.

A. Monochromatic injection field

In this case we set $f(t)=1$ and Eq. (3) simplifies to

$$\dot{\phi}_j=\delta_j+\kappa[\sin(\phi_{j+1}-\phi_j)+\sin(\phi_{j-1}-\phi_j)]-A_e\sin(\phi_j)\ . \tag{4}$$

If we consider the limit of identical laser elements such that $\delta_j=0$ for all j, then Eq. (4) has the fixed point solution $\phi_j=0$, which is unstable in the presence of coupling for $A_e=0$. This in phase state is stabilized by injection of a large enough field as we now show. Linearizing Eq. (4) around the in phase solution leads to

$$\dot{\phi}_j=\kappa(\phi_{j+1}-2\phi_j+\phi_{j-1})-A_e\phi_j \tag{5}$$

and the ansatz $\phi_j=x_{j\mu}e^{\lambda_\mu t}$ gives the eigenvalues $\lambda_\mu=-A_e-4\kappa\sin^2(\mu\pi/N)$, $\mu=0,1,\ldots,N-1$. The in-phase state is thus locally stable provided $A_e>4|\kappa|$ for $\kappa<0$. For nonidentical lasers the strict in-phase state is no longer a solution of Eq. (4); nevertheless, for small disorder we expect that a nearby solution exists and will be stabilized with the condition $A_e>4|\kappa|$ holding approximately.

B. Periodic amplitude modulated injection field

Consider an injected field with frequency components at ω_e and $\omega_e+2\delta_e$, where $\delta_e\ll\xi\ll\omega_e$. This physically represents the sinusoidal amplitude modulation at frequency δ_e of a carrier wave at frequency $\omega_e+\delta_e$. In this case we take $f(t)=\frac{1}{2}(1+e^{-i2\delta_e t})$. The factor $\frac{1}{2}$ ensures that the injected field has the same peak amplitude as that considered in Sec. III A. Equation (3) then becomes

$$\dot{\phi}_j=\delta_j+\kappa[\sin(\phi_{j+1}-\phi_j)+\sin(\phi_{j-1}-\phi_j)] -\tfrac{1}{2}A_e[\sin(\phi_j)+\sin(\phi_j+2\delta_e t)]\ . \tag{6}$$

We now treat the restricted but important case when the coupling, the driving, and the detunings are all of the same order of magnitude $A_e\sim|\kappa|\sim\delta_j\sim 0(\epsilon)$, where ϵ is a small parameter. Equation (4) is then of the form $\dot{\phi}_j=\epsilon f_j(\phi_1,\phi_2,\ldots,\phi_N,t,\epsilon)$, with f_j periodic in time t. The averaging theorem [20] states that we can replace the right-hand side of Eq. (6) by its average over one period correct to $O(\epsilon)$:

$$\dot{\phi}_j=\delta_j+\kappa[\sin(\phi_{j+1}-\phi_j)+\sin(\phi_{j-1}-\phi_j)] -\tfrac{1}{2}A_e\sin(\phi_j)+O(\epsilon^2)\ . \tag{7}$$

Moreover, the existence and stability of fixed points of the averaged system Eq. (7) implies corresponding periodic solutions for the nonautonomous system Eq. (6). Equation (7) is identical to Eq. (4) up to the factor $\frac{1}{2}$ discussed above. For identical array elements the in-phase fixed point of Eq. (7) is stable for $A_e>8|\kappa|$, which implies a corresponding stable periodic solution of Eq. (6). The consequences of this result are clear: the resonant sideband at ω_e entrains the lasers to the in-phase state while the output is periodically modulated by the sideband frequency component at $\omega_e+2\delta_e$. It is straightforward to extend the argument to more general periodic functions

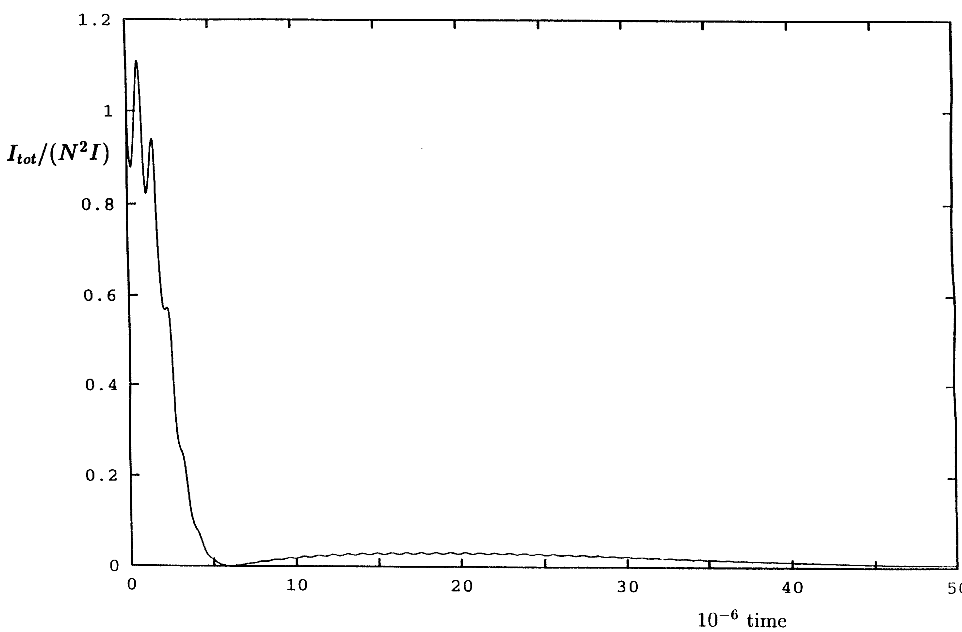


FIG. 1. Time series of the total intensity I_{tot} for $N=10$ lasers array normalized to N^2I, where I is the single laser intensity for $I_e=0$. The initial condition is the in-phase state. The other parameters are $\xi=0.01$, $\kappa=-2.5\times10^{-6}$, $p=0.015$, $\tau_c/\tau_f=5.0\times10^{-7}$, and $I=0.5$. Detunings δ_j where chosen at random in an interval $\pm3.75\times10^{-6}$ and the time is in units of the cavity round-trip time τ_c.

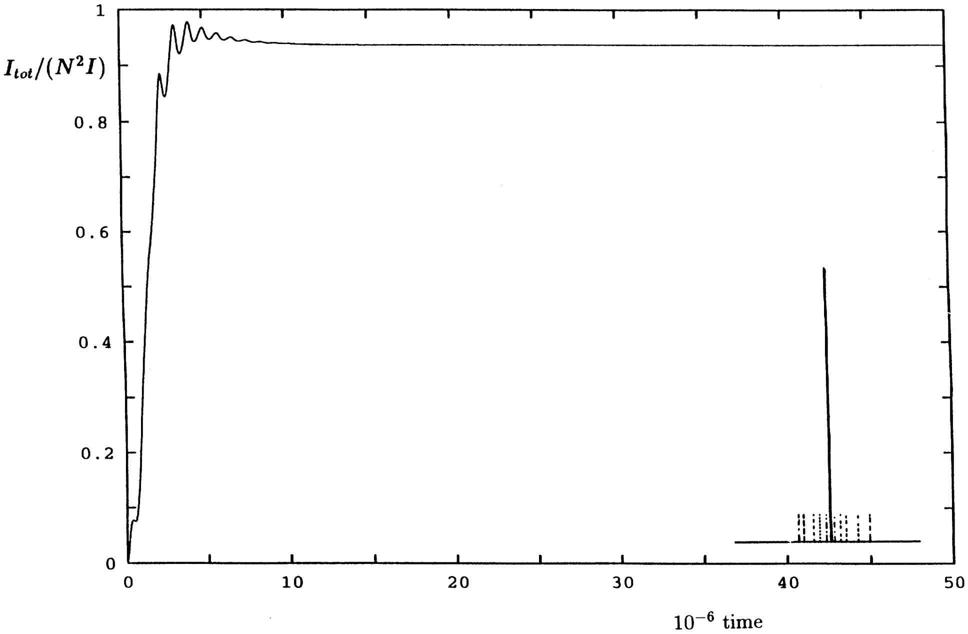


FIG. 2. Time series of the total intensity $I_{\rm tot}$ normalized to N^2I, where I is the single laser intensity when monochromatic field was injected to $N=10$ lasers array. The amplitude of the injected field is $A_e/4\kappa=1$ and the initial condition is the out-of-phase state. The other parameters are in Fig. 1. The inset shows the frequency components of the injection field (solid line) and the cavity resonances (dotted lines).

by writing $f(t)$ in the form of a Fourier series. Provided a resonant Fourier component can entrain the phases of the lasers, the residual nonresonant components adiabatically modulate the in-phase state. This presents a way to simultaneously entrain the array to maximize the coherent output intensity and modulate it for the purpose of applications, for example, communications where the modulation carries the information to be transmitted.

IV. NUMERICAL RESULTS

In order to test the predictions of Sec. III we have numerically integrated the full set of laser equations Eqs. (2) using parameter values from the experiments on Nd:YAG arrays [8]. Comparing the phase evolution from Eqs. (2b) and (3) indicates that as far as phase dynamics is concerned, intensity variations across the array can be self-consistently ignored and the condition $|I_j(t)-I|/I=0(\epsilon)$ is satisfied for the parameters used. The phase entrainment amplitudes as well as the time series for the phases (for a variety of amplitudes of the injected field) calculated from the full laser equations Eqs. (2) are in good agreement with the predictions of the phase model of Secs. III A and III B.

Figure 1 shows the total array intensity ($I_{\rm tot}=|\sum_{j=1}^{N}E_j|^2$) as a function of time with no injected

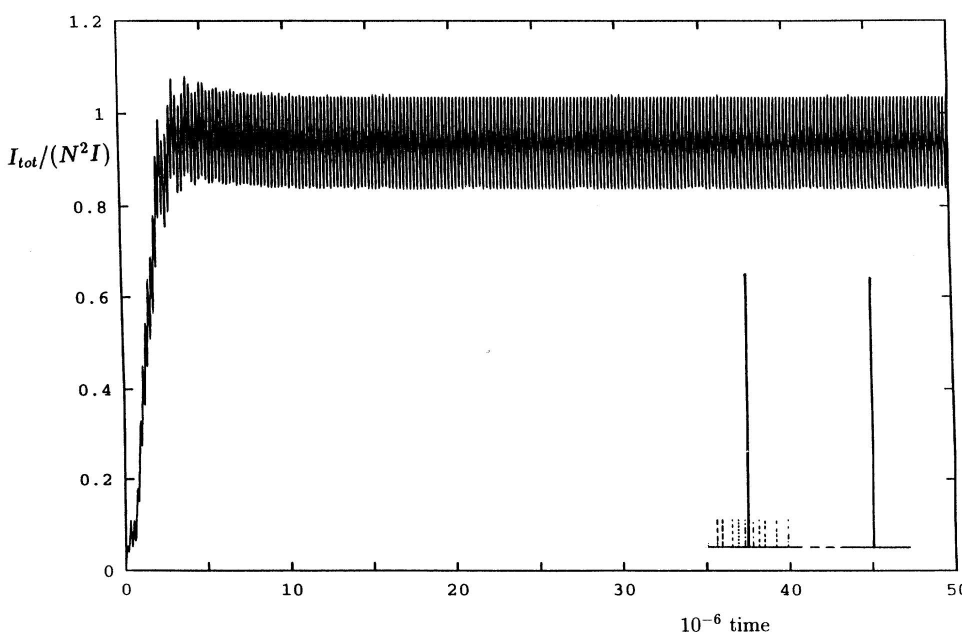
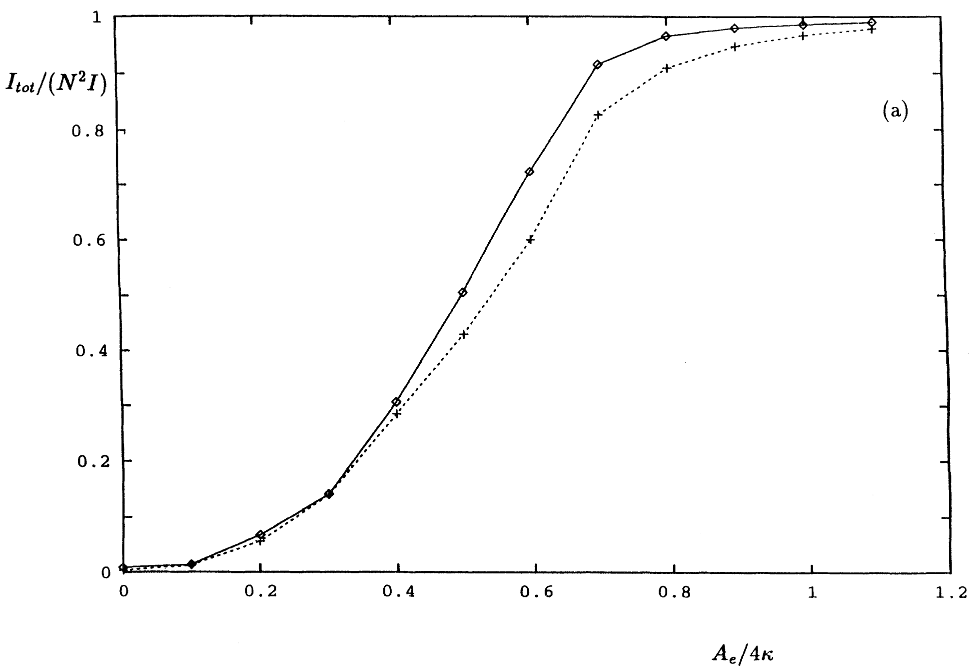


FIG. 3. Time series of the total intensity $I_{\rm tot}$ normalized to N^2I, where I is the single laser intensity when the sinusoidal amplitude modulated field was injected to $N=10$ lasers array. The amplitude of the injected field is $A_e/8\kappa=1$, the initial condition is the out-of-phase state, $\delta_e=1.5\times10^{-4}$, and the other parameters are as in Fig. 1. The inset shows the frequency components of the injection field (solid line) and the cavity resonances (dotted lines).

field, for an $N=10$ array. For simplicity we do not include the spatial distribution of lasers in the definition of intensity here, i.e., Fig. 1 shows only the temporal coherence properties. The vertical axis has been scaled to N^2I to provide an approximate measure of the maximum coherent output (where $I=p/\xi-1$ is the intensity of a single laser in the uncoupled limit). The initial conditions chosen correspond to an in-phase state. We observe a rapid reduction of the array intensity as the dynamics drives the system away from the in-phase state and toward the out-of-phase state, as expected for $\kappa<0$. If there is no disorder to total intensity will fall to zero, reflecting the complete destructive interference of the out-of-phase state. However, we have included a spread in the detunings δ_j, so there is a small residual output. Figure 2 shows the effect of switching on a monochromatic injection field: after a transient, the array settles into an entrained (nearly in-phase) state with a high degree of constructive interference. Note that the simulation also tells us something else: the initial condition in Fig. 2, which corresponds to the stationary solution of the undriven array, lies in the basin of attraction of the



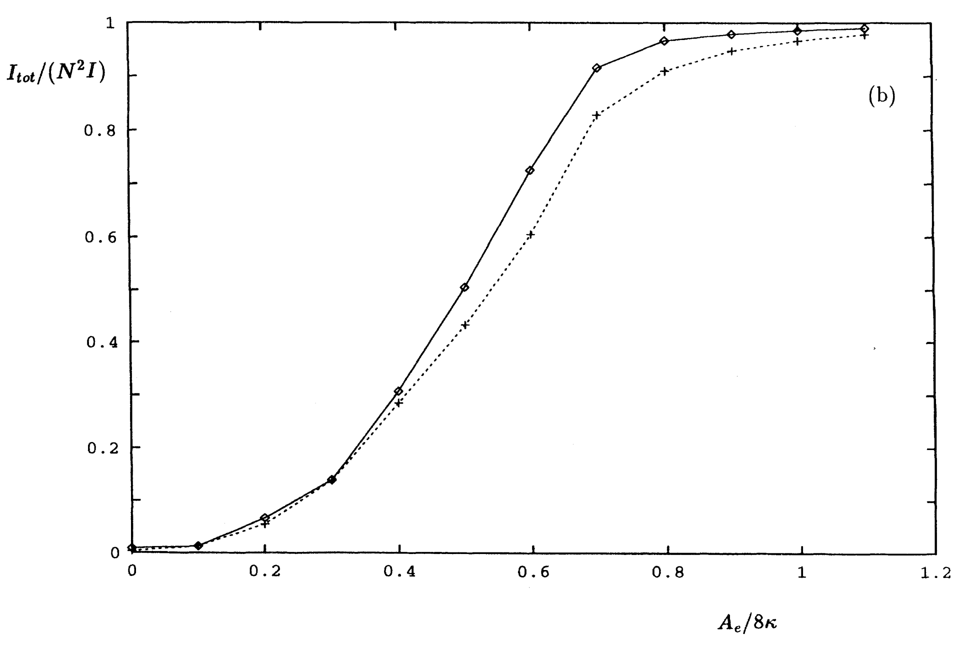


FIG. 4 (a) Time-averaged intensity divided by N^2I as a function of the injected field amplitude $A_e/4\kappa$ (monochromatic field), for $N=10$ (the upper curve) and $N=20$ (the lower curve). Other parameter values are as quoted in Fig. 3. Detunings δ_j were chosen at random in an interval $\pm 3.75\times10^{-6}$. (b) Time-averaged intensity divided by N^2I as a function of the injected field amplitude $A_e/8\kappa$ (the sinusoidal modulated field), for $N=10$ (the upper curve) and $N=20$ (the lower curve). Other parameter values are as quoted in Fig. 3. Detunings δ_j were chosen at random in an interval $\pm 3.75\times10^{-6}$.

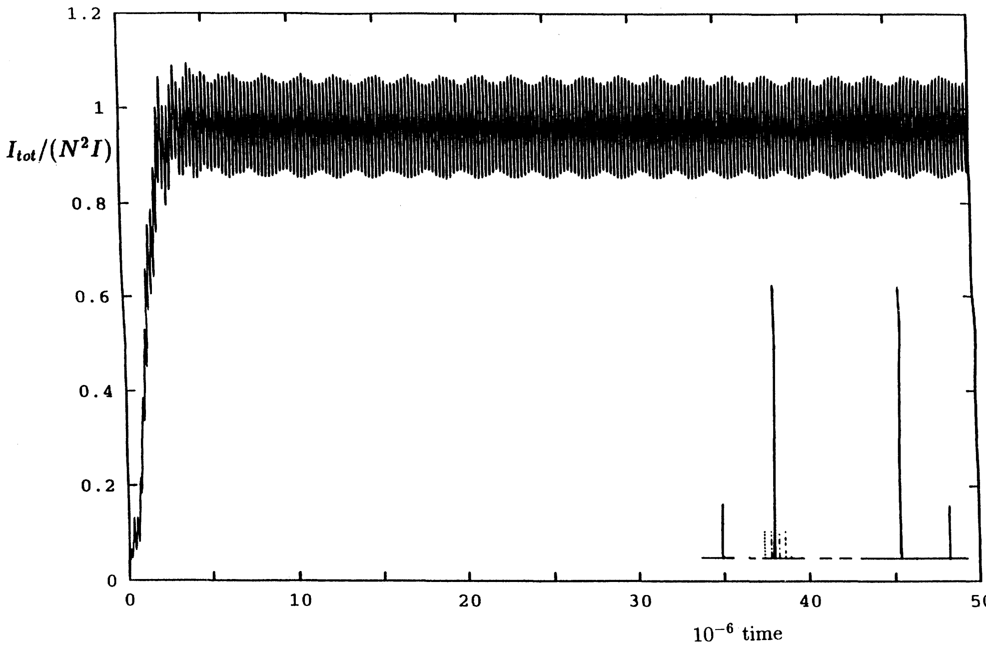


FIG. 5. Time series of the total intensity I_{tot} normalized to N^2I, where I is the single laser intensity when the two-frequency sinusoidal amplitude modulated field was injected into the $N=10$ lasers array. The amplitude of the injected field is $A_e/8\kappa=1$ and the initial condition is the out-of-phase state. The other parameters are as in Fig. 3. The inset shows the frequency components of the injection field (solid line) and the cavity resonances (dotted lines).

entrained state.

In Fig. 3 we illustrate the effects of injecting a sinusoidal amplitude modulated field. In agreement with the predictions of Sec. III B, the array is attracted to an entrained and periodically modulated state. In Fig. 4 we show how the final output intensity varies with the external injection amplitude for $N=10$ and 20 arrays, with monochromatic and amplitude modulated injected fields, respectively. In both cases we observe saturation at values that agree well with the criteria derived from the phase model with zero detunings, $A_e=4|\kappa|$ for the monochromatic field [Fig. 4(a)] and $8|\kappa|$ for the sinusoidal amplitude modulated field [Fig. 4(b)]. An appreciable boost in array intensity is achieved even for drive amplitudes less than the saturation value.

Finally, our simulations show that it is possible to entrain the array output to more complicated injection fields ($f(t)=\frac{1}{2}[(1+\alpha)+\exp(-2i\delta_e t)+\alpha\exp(-2i\beta\delta_e t)]$, $\alpha=0.1$, and $\beta=1.111\ldots$). Figure 5 shows the total intensity versus time for a two-frequency quasiperiodic injection field. The high value (close to unity) of the normalized output demonstrates that the lasers are (nearly) in phase; at the same time the output shows quasiperiodic modulation just like the input field. Although the averaging theorem arguments of Sec. III B do not apply in this case [21], the result is easy to understand on physical grounds: The strong resonant sideband of the injected field entrains the array, while the off-resonant components adiabatically modulate the output about the entrained state. This presumably generalizes to injected fields with additional frequency components and opens up the possibility of achieving prescribed output modulation by control of the injected field spectrum. The restriction phase entrainment puts on the frequency spectrum of the injected field is an interesting topic for further study.

V. CONCLUSION

We have shown how the natural antiphasing tendency in linear arrays of N solid-state lasers can be overcome by an injection field applied to each laser. The resulting entrained state leads to a high degree of constructive interference and consequently a large total intensity. The temporal behavior of the entrained state depends on the character of the injected field: a monochromatic field produces a stationary output, while a periodic or quasiperiodic field produces a corresponding time-dependent output. Simple criteria for achieving full entrainment were derived on the basis of an approximate model and agree well with numerical simulations of the full nonlinear equations, even with disorder in the form of a variation of detunings. For somewhat weaker injected fields the observed enhancement is still considerable. Existing experiments fall within the parameter regime considered here, so these effects should be observable with present technology.

Our results suggest an interesting direction for future research. Our analytic work was restricted to the case of no disorder. However, we found that the essential dynamics is captured by a reduced set of phase equations similar to Kuramoto's coupled oscillator model [10]. The self-consistency methods used to study Kuramoto's model could be used to understand quantitatively the interplay between coupling and disorder on the dynamics and how this affects the threshold for full entrainment.

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[1] D. Botez and D. E. Ackley, IEEE Circ. Dev. Mag. Jan. 8 (1986); D. Botez, Opt. Eng. **28**, 828 (1989); IEE Proc. J. **139**, 14 (1992).

[2] V. V. Likhanskii and A. P. Napartovich, Usp. Fiz. Nauk **160**, 101 (1990) [Sov. Phys. Usp. **33**, 228 (1990)].

[3] S. S. Wang and H. G. Winful, Appl. Phys. Lett. **52**, 1774 (1988); H. G. Winful and S. S. Wang, *ibid.* **53**, 1894 (1988); D.T. Nichols and H. G. Winful, J. Appl. Phys. **73**, 459 (1992).

[4] P. Mandel, Li Ruo-ding, and T. Erneux, Phys. Rev. A **39**, 2502 (1989); Li Ruo-ding, P. Mandel, and T. Erneux, *ibid.* **41**, 5117 (1990); Ruo-ding Li and T. Erneux, *ibid.* **46**, 4252 (1992); **49**, 1301 (1994).

[5] P. K. Jakobsen, R. A. Indik, J. V. Moloney, A. C. Newell, H. G. Winful, and L. Raman, J. Opt. Soc. Am. B **8**, 1674 (1991); H. Adachihara, O. Hess, R. Indik, and J. V. Moloney, *ibid.* **10**, 496 (1993); P. Ru, P. K. Jakobsen, J. V. Moloney, and R. A. Indik, *ibid.* **10**, 507 (1993).

[6] W.-J. Rappel, Phys. Rev. E **49**, 2750 (1994).

[7] S. Shakir and W. W. Chow, Phys. Rev. A **32**, 983 (1985).

[8] L. Fabiny, P. Colet, R. Roy, and D. Lenstra, Phys. Rev. A **47**, 4287 (1993).

[9] M. Silber, L. Fabiny, and K. Wiesenfeld, J. Opt. Soc. Am. B **10**, 1121 (1993).

[10] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984); A. T. Winfree, *The Timing of Biological Clocks* (Freeman, New York, 1987); *Dynamic Patterns in Complex Systems,* edited by J. A. Kelso, A. J. Mandell, and M. F. Shlesinger (World Scientific, Singapore, 1988).

[11] See, for example, S. H. Strogatz, C. M. Marcus, R. M. Westervelt, and R. E. Mirollo, Physica D **36**, 23 (1989); N. Kopell and G. B. Ermentrout, Commun. Pure Appl. Math. **39**, 623 (1986).

[12] L. Goldberg, H. F. Taylor, J. F. Weller, and D. R. Scifres, Appl. Phys. Lett. **46**, 236 (1985).

[13] N. G. Basov, E. M. Belenov, and V. D. Letokhov, Zh. Tekh. Fiz. **35**, 1098 (1965) [Sov. Phys. Tech. Phys. **10**, 845 (1965)].

[14] V. I. Perel and I. V. Rogova, Opt. Spektrosk. **25**, 716 (1968) [Opt. Spectrosc. (USSR) **25**, 401 (1968)]; **25**, 943 (1968) [**25**, 520 (1968)].

[15] M. B. Spencer and W. E. Lamb, Jr., Phys. Rev. A **5**, 893 (1972).

[16] D. R. Scifres, R. D. Buhrman, and W. Streifer, Appl. Phys. Lett. **33**, 1015 (1978); K. Otsuka, Electron. Lett. **19**, 723 (1983); J. K. Butler, D. E. Ackley, and D. Botez, Appl. Phys. Lett. **44**, 293 (1984); **44**, 935 (1984).

[17] R. Bonifacio and L. A. Lugiato, Phys. Rev. A **3**, 1129 (1978); L. A. Lugiato, P. Mandel, and L. M. Narducci, *ibid.* **29**, 1438 (1984).

[18] M. K. Chun, L. Goldberg, and J. F. Weller, Opt. Lett. **14**, 272 (1989).

[19] We note that the reduction to a phase model may not apply to semiconductor laser arrays because of the different parameters and linewidth enhancement that couples amplitude and phase. The validity of phase models in this case is an open question for future work.

[20] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, New York, 1983).

[21] An alternative approach to the averaging yields analytical insight into the form of this result [W. Geist and T. A. B. Kennedy (private communication)].