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Radiative recoil correction to the Lamb shift

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A simplified calculation of the radiative recoil correction to hydrogenic energy levels in $m^2/M[\alpha\,(Z\,\alpha)^5/n^3]$ order is presented. The method is based on evaluation of the proton kinetic energy term in the electron state. The result obtained is in disagreement with a previous calculation of Bhatt and Grotch [Ann. Phys. (N.Y.) **178**, 1 (1987)]. We also analyze the nuclear self-energy contribution and the corresponding definition of the nuclear mean square charge radius.

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I. INTRODUCTION

The hydrogen atom is the simplest stable two-body system. Precise knowledge of its energy levels allows for many QED tests and also for the determination of physical constants such as the Rydberg constant or a nuclear radius. Since the electron-proton mass ratio is only 1/1836, in a first approximation for the energy levels one treats the proton as a static source of the Coulomb field. In this external field approximation one solves the Dirac equation and then calculates all the radiative corrections [1]. Remaining corrections are due to the movement of the nucleus. Each photon exchange between the electron and the proton is associated with a change of the proton kinetic energy. In the nonrelativistic limit, the proton mass dependence is accounted for by the reduced mass of the two-body system. The relativistic treatment is much more complicated, and in general it is not even possible to write a two-body Hamiltonian. So, as a starting point one considers the approximated effective Hamiltonian [2], which is

$$H_{\rm eff} = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta\,m + \frac{\mathbf{p}^2}{2M} + V_{\rm eff}\,, \tag{1}$$

$$V_{\rm eff}(\mathbf{p},\mathbf{p}') = -\frac{Z\,e^2}{(\mathbf{p}-\mathbf{p}')^2}\left(1+\frac{\boldsymbol{\alpha}_\perp\cdot(\mathbf{p}+\mathbf{p}')}{2M}\right)\,, \tag{2}$$

where $\boldsymbol{\alpha}$ are the Dirac matrices and $\boldsymbol{\alpha}_\perp$ is the component perpendicular to $\mathbf{p}-\mathbf{p}'$, and m and M are the electron and the nuclear mass, respectively. The higher order terms not accounted for by (1) and (2) require a separate treatment and have been calculated in order $(Z\,\alpha)^5$ in [3] and in order $(Z\,\alpha)^6$ in [4]. The second class of recoil corrections is related to the finite nuclear mass effects in the electron self-energy and the vacuum polarization. In order $\alpha\,(Z\,\alpha)^4$ these corrections can be obtained by a simple rescaling of the wave function at the origin by a factor $(\mu/m)^3$ for all the terms except for the spin-orbit term for which the factor is $(\mu/m)^2$, and by changing the argument of $\ln(Z\,\alpha)^{-2}$ to $\ln\left(\frac{m}{\mu}(Z\,\alpha)^{-2}\right)$. The additional correction in order $\alpha\,(Z\,\alpha)^5$ beyond these replacement rules will be treated in this paper. These corrections have previously been studied by Bhatt and Grotch in a series of papers [5,6]. We present a simplified method and obtain a result which, however, is in disagreement with the earlier works. Our method is based on evaluation of the matrix element of the proton kinetic energy operator

$$\Delta E = \left\langle \psi_R\left|\frac{(\mathbf{P}+Z\,e\,\mathbf{A})^2}{2\,M}\right|\psi_R\right\rangle\,, \tag{3}$$

on an electron state in a Coulomb field centered at the position R of the proton. In a former paper [4] we have derived from (3) the exact (in $Z\,\alpha$) expression for pure recoil corrections. These formulas have been derived previously by Shabaev in [7] using the Bethe-Salpeter approach. In slightly rewritten form they are the Coulomb contribution

$$E_C = \frac{1}{2\,M}\int\frac{d\omega}{2\,\pi\,i}\,\langle\bar\phi^S|p^i(V)\gamma^0\,S_F(E+\omega)p^i(V)\,\gamma^0\,|\phi^S\rangle \times\left(\frac{1}{(\omega-i\,\epsilon)^2}+\frac{1}{(\omega+i\,\epsilon)^2}\right)\,, \tag{4}$$

where $V=-\frac{Z\,\alpha}{r}$ and $p^i(V)=[p^i,V]$, the single transverse contribution

$$E_T = -\frac{Z\,e^2}{M}\int\frac{d\omega}{2\,\pi\,i}\int\frac{d^3k}{(2\,\pi)^3}\,\delta_\perp^{ij}\,\frac{1}{\omega^2-k^2} \times\left\{\left\langle\bar\phi_S\left|\gamma^j\,e^{i\,k\,r}\,S_F(E_S+\omega)\,\gamma^0\,\frac{p^i(V)}{(-\omega-i\,\epsilon)}\right|\phi_S\right\rangle +\left\langle\bar\phi_S\left|\gamma^0\,\frac{p^i(V)}{(\omega-i\,\epsilon)}\,S_F(E_S+\omega)\,\gamma^j\,e^{i\,k\,r}\right|\phi_S\right\rangle\right\}, \tag{5}$$

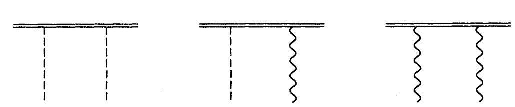

FIG. 1. Diagrams representing pure recoil corrections to the Lamb shift. Double lines denote electron propagator in the Coulomb field; wave and dashed lines denote the transverse and Coulomb photon, respectively.

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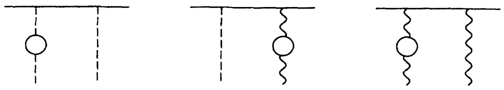

FIG. 2. Diagrams representing vacuum polarization-recoil corrections to the Lamb shift. A continuous line denotes the electron propagator.

where $\delta_\perp^{ij} = \delta^{ij} - \frac{k^i k^j}{k^2}$, and the seagull (double transverse) contribution

$$E_S = -\frac{Z^2 e^4}{M} \int \frac{d\omega}{2\pi i} \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \times \frac{1}{\omega^2 - k_1^2} \frac{1}{\omega^2 - k_2^2} \delta_{1\perp}^{ki} \delta_{2\perp}^{kj} \times \langle \bar{\phi}_S | \gamma^i e^{-i k_1 r} S_F(E_S - \omega) \gamma^j e^{i k_2 r} | \phi_S \rangle . \quad (6)$$

These expressions could be represented by three Feynman diagrams with modified rules for the Coulomb vertex, as shown in Fig. 1. The pole contribution from the above integral gives the expectation value of the proton kinetic energy and of the Breit interaction. The radiative recoil correction corresponds to drawing a photon loop in all the combinations on the electron line, or adding a fermion loop to the photon propagator. To obtain the corrections in order $(Z\alpha)^5$ beyond the reduced mass scaling of the wave function at the origin, we put the external electron legs on mass shell and calculate the diagrams presented in Figs. 2 and 3.

This paper is organized as follows. In the next section the rederivation of the known mass dependence of the $\alpha(Z\alpha)^4$ correction is presented. In the third section the vacuum polarization-recoil correction is calculated. In the fourth section we calculate the so-called pole contribution, in the fifth we finish the calculation of the remaining Coulomb and seagull corrections. In the sixth section we analyze the nuclear self-energy and the definition of its charge radius.

II. THE LEADING ORDER CONTRIBUTION

The recoil corrections to first order in m/M have been given in Eqs. (4)–(6). For the leading order contribution in order $\alpha(Z\alpha)^4$ it is, however, necessary to derive the full mass dependence. We start with the Grotch-Yennie equation [2], which incorporates the nuclear kinetic energy and the Breit interaction,

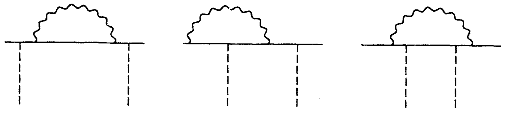

FIG. 3. Diagrams representing double Coulomb recoil corrections to the Lamb shift. The diagrams for the double transverse (seagull) correction are similar. The wave line denotes here a complete photon propagator.

$$\left\{ \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m + \frac{\mathbf{p}^2}{2M} + V \right\} \psi = E \psi , \quad (7)$$

where

$$V = e A^0 = -\frac{Z\alpha}{r} , \quad (8)$$

$$e A^j = \frac{1}{M} p^i \frac{Z\alpha}{r} \left(\delta^{ij} + \frac{r^i r^j}{r^2} \right) . \quad (9)$$

The radiative recoil correction is obtained from the electron self-energy with the modified electron propagator

$$\Delta E = e^2 \int \frac{d^4 k}{(2\pi)^4 i} \frac{g_{\mu\nu}}{k^2} \times \left\langle \bar{\psi} \left| \gamma^\mu \frac{1}{\not p - \not k - e \not A - m - \beta \frac{\mathbf{p}^2}{2M}} \gamma^\nu \right| \psi \right\rangle - \langle \delta m \rangle . \quad (10)$$

The pole contribution of Eqs. (4) and (5) with a radiative photon incorporated on the electron line is equal to the first term of (10) in the m/M expansion. For the evaluation of (10) we use the method we developed in [8], which in this case is very similar to the original calculation of the leading self-energy contribution. The contour of the ω integration is deformed, as depicted in Fig. 4, and divided into two parts. On the low-energy part we can use nonrelativistic and dipole approximation. The mass dependence in the kinetic energy part of the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{p^2}{2M} + V - \frac{e}{m} \mathbf{p} \cdot \mathbf{A} \quad (11)$$

can be hidden in the reduced mass of the electron-nucleus system $\mu = m M/(m+M)$,

$$H = \frac{p^2}{2\mu} + V - \frac{e'}{\mu} \mathbf{p} \cdot \mathbf{A} , \quad (12)$$

where $e' = e \frac{\mu}{m}$. This Hamiltonian has the same form as the standard one, thus we can write immediately the correction to the energy that comes from the low-energy part,

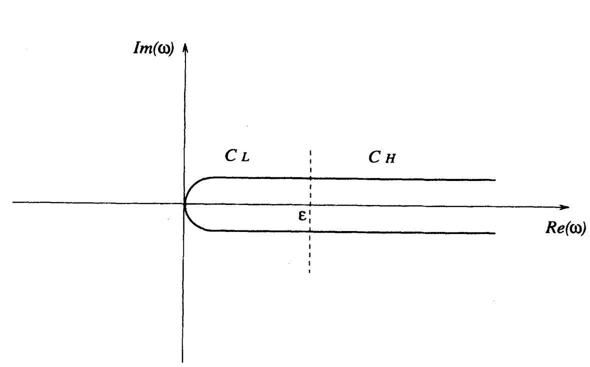

FIG. 4. Contour of integration used in the reevaluation of the leading radiative recoil correction. ϵ is an artificial parameter that divides the contour into two parts C_L and C_H.

$$E_L = \frac{1}{6}\frac{e'^2}{\mu^2}\int_0^\epsilon \omega\, d\omega \left\langle \phi \left| \mathbf{p}\frac{1}{H-(E-\omega)}\mathbf{p}\right|\phi\right\rangle \tag{13}$$

$$= \frac{\mu^3}{m^2}\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\left[\frac{4}{3}\ln\left(\frac{2\epsilon}{\mu\,(Z\alpha)^2}\right)\delta_{l0} - \frac{4}{3}k(n,l)\right], \tag{14}$$

where $k(n,l)$ is the Bethe logarithm. On the C_H contour we have the high-energy part. It is calculated using the fact that in the first approximation the radiative corrections are equivalent to the modification of the vertex function (i.e., the interaction of the electron with the external field)

$$\gamma_\rho A^\rho \to \left(\gamma_\rho\left\{1+\frac{\alpha\, q^2}{3\pi m^2}\left[\ln\left(\frac{m}{2\epsilon}\right)+\frac{11}{24}-\frac{1}{5}\right]\right\} + \frac{i}{2m}\frac{\alpha}{2\pi}\sigma_{\rho\nu}\, q^\nu\right)A^\rho(q)\,, \tag{15}$$

where A^0 and $\mathbf{A}$ are given by Eqs. (8) and (9). The correction to the energy is given by the expectation value of γA on the electron state, for which (7) holds,

$$E_H = e\,\alpha\left\langle\psi\left|\frac{1}{3\pi m^2}\left[\ln\left(\frac{m}{2\epsilon}\right)+\frac{11}{24}-\frac{1}{5}\right]\Delta A^0(r) + \frac{i}{4\pi m}\boldsymbol{\gamma}\,\mathbf{E}(r) + \frac{i}{4\pi m}\gamma^0\,\mathbf{A}\times\mathbf{q}\cdot\boldsymbol{\sigma}\right|\psi\right\rangle. \tag{16}$$

Since $\Delta A^0(r) = Z\,e\,\delta^3(r)$, the first term is given by the wave function at the origin,

$$E_H^1 = \frac{\mu^3}{m^2}\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\frac{4}{3}\left[\ln\left(\frac{m}{2\epsilon}\right)+\frac{11}{24}-\frac{1}{5}\right]\delta_{l0}\,. \tag{17}$$

The second term is

$$E_H^2 = \frac{Z\alpha^2}{8\pi m^2}\left\langle\phi\left|4\pi\,\delta^3(r)+4\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}\right|\phi\right\rangle = \frac{\mu^3}{m^2}\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\frac{C_{jl}}{2\,(2\,l+1)}\,, \tag{18}$$

where

$$C_{jl} = \begin{cases} \frac{1}{l+1} & \text{for } j = l+\frac{1}{2} \\ -\frac{1}{l} & \text{for } j = l-\frac{1}{2}. \end{cases} \tag{19}$$

The third term is

$$E_H^3 = \frac{Z\alpha^2}{2\pi m M}\left\langle\phi\left|\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}\right|\phi\right\rangle = \frac{\mu^3}{mM}\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\frac{C_{jl}}{2\,(2\,l+1)}(1-\delta_{l0})\,. \tag{20}$$

The sum of the low-energy part E_L and the high-energy part E_H does not depend on the artificial cutoff ϵ and is

$$\Delta E = E_L + E_H = m\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\left\{\left[\left(\frac{4}{3}\ln\frac{m}{\mu\,(Z\alpha)^2}+\frac{10}{9}-\frac{4}{15}\right)\delta_{l0} - \frac{4}{3}k(n,l)\right]\frac{\mu^3}{m^3} + \frac{C_{jl}}{2(2\,l+1)}(1-\delta_{l0})\frac{\mu^2}{m^2}\right\}. \tag{21}$$

This expression of order $\alpha\,(Z\alpha)^4$ is valid for the arbitrary mass ratio.

III. RECOIL WITH VACUUM POLARIZATION CORRECTION

This correction is described by the diagrams presented in Fig. 2. The leading order term is already included in Eq. (21). The part of this correction in order $\frac{m}{M}\alpha\,(Z\alpha)^5$ is incorporated by the reduced mass scaling of the known correction in order $\alpha\,(Z\alpha)^5$,

$$E = \frac{\mu^3}{m^2}\frac{\alpha}{\pi}\frac{(Z\alpha)^5}{\pi n^3}4\pi^2\left(\frac{139}{128}+\frac{5}{192}-\frac{1}{2}\ln(2)\right), \tag{22}$$

where the term 5/192 comes from the vacuum polarization. For the evaluation of remaining vacuum polarization terms we assume that the photon has some mass ρ which is next integrated with the weight obtained from the imaginary part of the photon self-energy diagram,

$$\Delta E_{\rm VP} = \frac{\alpha}{3\pi}\int_4 \frac{d(\rho^2)}{\rho^2}\sqrt{1-\frac{4}{\rho^2}}\left(1+\frac{2}{\rho^2}\right)E(\rho)\,. \tag{23}$$

Where it does not lead to confusion, we set electron mass $m = 1$. Our formulas (4)–(6) have been derived for a massless photon in the Coulomb gauge. The extension to massive photons is obtained by introducing mass to the photon propagator

$$-V(\mathbf{k}) = \frac{Z\,e^2}{\mathbf{k}^2} \to \frac{Z\,e^2}{\mathbf{k}^2+\rho^2}, \tag{24}$$

$$\frac{1}{\omega^2-\mathbf{k}^2}\left(\delta^{ij}-\frac{k^i\,k^j}{\mathbf{k}^2}\right) \to \frac{1}{\omega^2-\mathbf{k}^2-\rho^2}\times\left(\delta^{ij}-\frac{k^i\,k^j}{\mathbf{k}^2+\rho^2}\right). \tag{25}$$

The correction in order $\alpha\,(Z\alpha)^5$ beyond the reduced mass scaling is obtained by putting the electron wave functions on mass shell in (4)–(6) and replacing S_F by a free electron propagator. The relevant formulas are the following:

$$E_C(\rho) = -\frac{Z^2\,e^4}{M}\phi^2(0)\int\frac{d^4k}{(2\pi)^2\,i}\left(\frac{1}{(\omega-i\,\epsilon)^2}+\frac{1}{(\omega+i\,\epsilon)^2}\right)\mathrm{Tr}\left[\gamma^0\frac{\mathbf{k}}{\mathbf{k}^2}\frac{1}{\not t-\not k-m}\frac{\mathbf{k}}{\mathbf{k}^2+\rho^2}\gamma^0\frac{(\gamma^0+I)}{4}\right], \tag{26}$$

$$E_T(\rho) = -\frac{Z^2 e^4}{M}\,\phi^2(0) \int \frac{d^4k}{(2\pi)^2\, i} \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2+\rho^2}\right) \frac{1}{k^2-\rho^2} \left\{ \mathrm{Tr}\left[\gamma^i \frac{1}{\not t - \not k - m}\,\gamma^0 \frac{1}{(\omega - i\,\epsilon)} \frac{\mathbf{k}^j}{\mathbf{k}^2} \frac{(\gamma^0+I)}{4}\right] + \mathrm{Tr}\left[\frac{1}{(\omega + i\,\epsilon)} \frac{\mathbf{k}^j}{\mathbf{k}^2}\,\gamma^0 \frac{1}{\not t - \not k - m}\,\gamma^j \frac{(\gamma^0+I)}{4}\right]\right\}, \tag{27}$$

$$E_S(\rho) = -2\,\frac{Z^2 e^4}{M}\,\phi^2(0) \int \frac{d^4k}{(2\pi)^2\, i} \frac{1}{k^2} \frac{1}{k^2-\rho^2} \left(\delta^{ki} - \frac{k^k k^i}{\mathbf{k}^2}\right) \left(\delta^{kj} - \frac{k^k k^j}{\mathbf{k}^2+\rho^2}\right) \mathrm{Tr}\left[\gamma^i \frac{1}{\not t - \not k - m}\,\gamma^j \frac{(\gamma^0+I)}{4}\right], \tag{28}$$

where $\omega = k^0$ and $t = (m,0,0,0)$. The integration was done first in ω, then in $\mathbf{k}$. Since E_C contributes also to the lower order the $1/k$ divergence appears in (26), which is eliminated by a simple subtraction of the divergent term. After these integrations the sum of E_i is

$$E(\rho) = E_C(\rho) + E_T(\rho) + E_S(\rho)$$
$$= \frac{Z^2 e^4}{M}\phi^2(0)\frac{1}{2\pi^2}\left\{\frac{1}{4} - \frac{2}{\rho^2} + \frac{1}{2}\,\ln(\rho)\left(1 - \frac{\rho^2}{2}\right) - \left(\frac{4}{\rho^3} + \frac{\rho}{2}\right)\sqrt{1 - \frac{\rho^2}{4}}\,\arccos\left(\frac{\rho}{2}\right)\right\}. \tag{29}$$

The correction to the energy as given in (23), after ρ integration, is

$$\Delta E_{\rm VP} = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\, n^3}\left(-\frac{70}{27} + \frac{2}{9}\pi^2\right). \tag{30}$$

IV. RADIATIVE RECOIL CORRECTION FROM THE POLE

The correction from the pole at $\omega = 0$ (see Fig. 5) in order $\alpha\,(Z\,\alpha)^5$ comes only from the Coulomb term and is given by

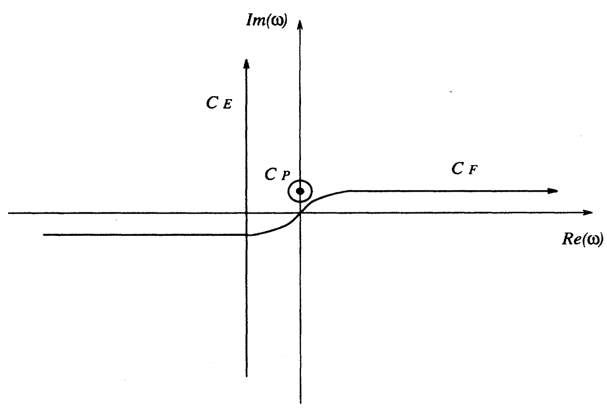

FIG. 5. Contours of integration. The pole contribution denoted by the circle C_P has a counterclockwise direction. The contour C_E was used for the numerical integration.

$$\Delta E_P = -\frac{Z^2 e^4}{2\,M}\,\phi^2(0)\int \frac{d^4p}{(2\pi)^4\, i}\,T^{00}(p)\,\frac{1}{\mathbf{p}^2} \times\left(\frac{1}{(p_0 - 1 - i\,\epsilon)^2} - \frac{1}{(p_0 - 1 + i\,\epsilon)^2}\right) \tag{31}$$

$$= -\frac{\alpha\,(Z\,\alpha)^5}{M\,n^3}\,8\int\frac{d^3p}{(2\pi)^3}\,\frac{f(p^2)}{p^2}, \tag{32}$$

where

$$T^{\mu\nu}(p) = \mathrm{Tr}\left[\gamma^\mu\left(\Lambda(t,p,t) + 2\,\Gamma(t,p)\,\frac{1}{\not p - m} + \Sigma(p)\,\frac{1}{(\not p - m)^2}\right)\gamma^\nu\,\frac{(\gamma_0 + I)}{4}\right], \tag{33}$$

$$f(p^2) = \frac{\pi}{\alpha}\,\frac{\partial}{\partial p_0}\,T^{00}\bigg|_{p_0=1}, \tag{34}$$

and Λ, Γ, Σ are double-vertex, single-vertex, and self-energy functions, respectively, with t on mass shell $t = (m,0,0,0)$. The single transverse term does not contribute in order $\alpha\,(Z\,\alpha)^5$. The function $f(p^2)$ in the above behaves for small p like $1/p^2$. We subtract this divergent term since it gives a contribution in the known lower order. The integral (32) expressed in terms of the f^A function (the imaginary part of f on the branch cut)

$$f(p^2) = -\int_0^\infty d(q^2)\,\frac{1}{p^2+q^2}\,f^A(q^2) \tag{35}$$

takes the form

$$\Delta E_P = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{n^3}\,4\int_0^\infty dq\, f^A(q^2). \tag{36}$$

The function f^A can be calculated using the standard Feynman parameter technique or the Cutkosky rules [9]. The result is

$$f^A(q^2) = J^A\left(\frac{1}{2} - \frac{8}{q^4} + \frac{6}{q^2}\right) + \frac{1}{2} + \frac{8}{q^4} - \frac{26}{3\,q^2} + \frac{1}{2\,(1+q^2)^2} + \frac{5}{2\,(1+q^2)} + \sqrt{1 - \frac{4}{q^2}}\left(-\frac{1}{2} - \frac{2}{q^4} + \frac{5}{2\,q^2} - \frac{1}{2\,(q^2-4)}\right) \times\Theta(q-2), \tag{37}$$

where

$$J^A = \frac{1}{q}\left[\arctan q - \Theta(q-2)\arccos\left(\frac{2}{q}\right)\right]. \qquad (38)$$

The final q integration gives a result

$$\Delta E_P = \frac{m^2}{M}\,\alpha\,\frac{(Z\alpha)^5}{n^3}\left(-\frac{79}{32} + 2\ln(2)\right). \qquad (39)$$

V. DOUBLE COULOMB AND SEAGULL CONTRIBUTION

The correction in $\alpha\,(Z\alpha)^5$ order is obtained by adding a one photon loop on the electron line according to graphs shown in Fig. 3. The expressions derived for the Coulomb and seagull contributions are

$$\Delta E_C = -\frac{Z^2 e^4}{M}\phi^2(0)\int\frac{d^4q}{(2\pi)^4 i}\,T^{00}\,\frac{1}{\mathbf{q}^2}\,\frac{1}{(q_0+i\epsilon)^2}, \qquad (40)$$

$$\Delta E_S = -\frac{Z^2 e^4}{M}\phi^2(0)\int\frac{d^4q}{(2\pi)^4 i}\left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right)\times T^{ij}\,\frac{1}{(q^2)^2}, \qquad (41)$$

where $T^{\mu\nu}$ is defined in (33). The corresponding single transverse contribution vanishes. Since the tensor

$$T_s^{\mu\nu}(q) = \frac{T^{\mu\nu}(q) + T^{\mu\nu}(-q)}{2} \qquad (42)$$

fulfills the continuity condition

$$q_\mu\,T_s^{\mu\nu}(q) = 0, \qquad (43)$$

all its components

$$T_s^{\mu\nu} = t^\mu t^\nu f_1 + q^\mu q^\nu f_2 + (t^\mu q^\nu + q^\mu t_\nu) f_3 + g^{\mu\nu} f_4 \qquad (44)$$

can be expressed in terms of T_s^{00} and $T^{\mu}_{s\mu}$, for example,

$$\left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right) T_s^{ij} = -\left(T^{\mu}_{s\mu} + \frac{q^2}{\mathbf{q}^2}\,T_s^{00}\right). \qquad (45)$$

Since the integral in Eq. (41) is symmetric in q, we could make the replacement in (41)

$$\left(\delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}\right) T^{ij} \to -\left(T^{\mu}_{\mu} + \frac{q^2}{\mathbf{q}^2}\,T^{00}\right). \qquad (46)$$

T^{00} and T^μ_μ are, according to (33), the one-loop corrections to the electron factor. We have calculated them analytically using the Feynman parameter approach. The result is

$$\begin{aligned} T^{00} = \frac{\alpha}{4\pi}\Bigg[&-3 + p^2 - \frac{8}{p^2-1} - \frac{5}{2q^2} + \frac{3p^2}{q^2} - \frac{p^4}{2q^2} - \frac{q^2}{2} + \frac{2q^2}{p^2-1} \\ &+ J\left(-13 - 3p^2 - \frac{16}{p^2-1} + 6q^2 + \frac{12q^2}{p^2-1} - \frac{2q^4}{p^2-1}\right) \\ &+ \left(-15 + p^2 - \frac{56}{p^2-1} + \frac{10}{q^4} - \frac{12p^2}{q^4} + \frac{2p^4}{q^4} + \frac{60}{q^2} - \frac{4p^2}{q^2} + \frac{96}{(p^2-1)\,q^2} - q^2 + \frac{8q^2}{p^2-1}\right)\frac{\arcsin(\frac{q}{2})}{\sqrt{1-\frac{4}{q^2}}} \\ &+ \left(-8 - \frac{1}{2p^2} + \frac{p^2}{2} - \frac{16}{p^2-1} - \frac{q^2}{2} + \frac{q^2}{2p^2} + \frac{4q^2}{p^2-1}\right)\ln(1-p^2)\Bigg], \end{aligned} \qquad (47)$$

$$\begin{aligned} T^\mu_\mu = \frac{\alpha}{4\pi}\Bigg[&9 + p^{-2} - \frac{8}{p^2-1} - \frac{q^2}{p^2} - \frac{4q^2}{p^2-1} + J\left(-10 + 2p^2 - \frac{16}{p^2-1} - 4q^2 + \frac{4q^4}{p^2-1}\right) \\ &+ \left(\frac{32}{p^2-1} - \frac{24}{q^2} + \frac{96}{(p^2-1)\,q^2} - \frac{20q^2}{p^2-1}\right)\frac{\arcsin(\frac{q}{2})}{\sqrt{1-\frac{4}{q^2}}} \\ &+ \left(-1 + p^{-4} - \frac{16}{p^2-1} - \frac{q^2}{p^4} + \frac{q^2}{p^2} - \frac{8q^2}{p^2-1}\right)\ln(1-p^2)\Bigg], \end{aligned} \qquad (48)$$

where $p = q + t$,

$$J = \int\frac{d^4k}{\pi^2 i}\,\frac{1}{k^2}\,\frac{1}{(t-k)^2-1}\,\frac{1}{(p-k)^2-1} \qquad (49)$$

$$= \int_0^1 du\,\frac{1}{1-u(1-u)\,q^2 - u(1-p^2)}\,\ln\left(\frac{1-u(1-u)q^2}{u(1-p^2)}\right) \qquad (50)$$

$$\begin{aligned} = \frac{1}{y_2-y_1}\Bigg\{&-L_2\left(1-\frac{1}{y_2}\right) + L_2\left(1-\frac{1}{y_1}\right) + L_2\left(1-\frac{\beta}{y_2}\right) - L_2\left(1-\frac{\beta}{y_1}\right) + L_2\left(1-\frac{1}{\beta y_2}\right) \\ &- L_2\left(1-\frac{1}{\beta y_1}\right) + \ln(1-p^2)\,\ln\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln^2 y_2 - \frac{1}{2}\ln^2 y_1\Bigg\}, \end{aligned} \qquad (51)$$

and

$$y_1 = p_0 - |\mathbf{p}|\,, \tag{52}$$

$$y_2 = p_0 + |\mathbf{p}|\,, \tag{53}$$

$$\beta = -\frac{q^2}{4}\left(1-\sqrt{1-\frac{4}{q^2}}\right)^2\,, \tag{54}$$

and L_2 is a dilogarithmic function. The remaining integration in respect to q is done numerically. We first subtract a common prefactor

$$\Delta E_C = -\frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,\frac{1}{\pi}\int\frac{d^4q}{4\,\pi\,i}\,\Lambda_1\,\frac{1}{\mathbf{q}^2}\,\frac{1}{(q_0+i\,\epsilon)^2}\,, \tag{55}$$

$$\Delta E_S = -\frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,\frac{1}{\pi}\int\frac{d^4q}{4\,\pi\,i}\,\Lambda_2\,\frac{1}{(q^2)^2}\,, \tag{56}$$

where

$$\Lambda_1 = \frac{4\,\pi}{\alpha}\,T^{00}\,, \tag{57}$$

$$\Lambda_2 = \frac{4\,\pi}{\alpha}\left(\delta^{ij}-\frac{q^i\,q^j}{\mathbf{q}^2}\right)T^{ij}\,, \tag{58}$$

perform a Wick rotation, and finally numerically integrate in two dimensions using the Gaussian method. Our results are

$$\Delta E_C = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,(-1)\,2.629\,46(1)\,, \tag{59}$$

$$\Delta E_S = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,0.245\,23(1)\,. \tag{60}$$

VI. PROTON SELF-ENERGY AND FINITE SIZE CORRECTIONS

There is an additional radiative recoil correction that has been neglected so far. It is due to the proton self-energy. If we assume a pointlike proton the contribution of this term to the Lamb shift is [1]

$$\Delta E = \frac{\alpha^5\,\mu^3}{\pi\,n^3\,M^2}\left[\left(\frac{10}{9}+\frac{4}{3}\,\ln\frac{M}{\mu\,\alpha^2}\right)\delta_{l0}-\frac{4}{3}\,k(n,l)\right]. \tag{61}$$

For a *true* proton there is a finite size correction,

$$\Delta E = \frac{2}{3n^3}\,\alpha^4\,\mu^3\,\langle r^2\rangle\,\delta_{l0}\,. \tag{62}$$

The problem is that the proton self-energy is modified by finite size effects, so some corrections are counted twice in the above. To incorporate the correction (61) unambiguously we must precisely specify the nuclear mean square charge radius. Its definition through the Sachs form factor

$$\frac{\langle r^2\rangle}{6} = \left.\frac{\partial G_E(q^2)}{\partial(q^2)}\right|_{q^2=0} \tag{63}$$

is not the best, because the radiative correction to G_E is infrared divergent. We propose thus a different definition using the forward scattering amplitude described by $T^{\mu\nu}$,

$$T^{\mu\nu}(x-x') = -i\,\langle t|T\,j^\mu(x)\,j^\nu(x')|t\rangle\,, \tag{64}$$

where $t=(M,0,0,0)$. For our purpose we consider a deep nonrelativistic limit $q^0\sim\mathbf{q}^2$ around $p^2-M^2=(t+q)^2-M^2\approx 0$ of the dominant T^{00} component. For a pointlike particle without radiative corrections T^{00} is

$$T^{00} = \mathrm{Tr}\left[\gamma^0\,\frac{1}{\not p-M}\gamma^0\,\frac{(\gamma^0+I)}{4}\right]\approx\frac{2\,M}{p^2-M^2}\,. \tag{65}$$

With a finite size particle

$$\gamma^\mu\to\Gamma^\mu = \gamma^\mu\,F_1+i\frac{\sigma^{\mu\nu}}{2M}\,q_\nu\,F_2\,, \tag{66}$$

T^{00} acquires a correction

$$\Delta T^{00}\approx\frac{2\,M}{p^2-M^2}[G_E^2(q^2)-1]\approx\frac{2\,M}{p^2-M^2}\,q^2\,\frac{\langle r^2\rangle_{\rm bare}}{3}\,, \tag{67}$$

where $G_E = F_1+\frac{q^2}{4M^2}\,F_2$. The radiative corrections for a pointlike particle from Eq. (47) are

$$\Delta T^{00} = \frac{\alpha}{\pi\,M}\,\frac{q^2}{p^2-M^2}\left(\frac{10}{9}+\frac{4}{3}\,\ln\frac{M^2}{M^2-p^2}\right). \tag{68}$$

We define $\langle r^2\rangle$ by the following equation that describes the low-energy behavior of the correction to the forward scattering amplitude:

$$\Delta T^{00} = \frac{q^2\,M}{p^2-M^2}\left(\frac{4\,\alpha}{3\,\pi\,M^2}\,\ln\frac{M^2}{M^2-p^2}+\frac{2}{3}\,\langle r^2\rangle\right). \tag{69}$$

We expect that for any nucleus the logarithmic term above will be the same, since it is only related to the fact that the nucleus has a charge, and does not depend on other details such as the spin. There is an arbitrariness in the choice of the constant factor, i.e., what belongs to charge radius, and what to the nuclear self-energy. The proposed definition separates only the logarithmic term from the charge radius. Another class of radiative corrections is related to vacuum polarization from heavier particles like the muon. Since any reasonable experiment will not distinguish that effect from the intrinsic charge distribution, we incorporate all these terms in the charge radius definition.

The associated correction to the energy for the S states has the form

$$\Delta E = \frac{2}{3\,n^3}\,(Z\,\alpha)^4\,\mu^3\langle r^2\rangle + \frac{4\,(Z^2\,\alpha)\,(Z\,\alpha)^4}{3\,\pi\,n^3}\,\frac{\mu^3}{M^2}\left[\ln\left(\frac{M}{\mu\,(Z\,\alpha)^2}\right)-k_0(n)\right]. \tag{70}$$

The small second term in the above equation gives 4.6

kHz for the $1S$ state in hydrogen, which is significant for the hydrogen-deuterium isotope shift [10].

VII. SUMMARY

We have recalculated all the recoil corrections of order $\frac{m^2}{M}\,\alpha\,(Z\,\alpha)^5$ to the Lamb shift. They consist of the known term that comes from scaling of the wave function at the origin,

$$E = \frac{\mu^3}{m^2}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,4\,\pi^2\left(\frac{139}{128}+\frac{5}{192}-\frac{1}{2}\,\ln(2)\right), \quad (71)$$

the vacuum polarization correction

$$\Delta E_{\rm VP} = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\left(-\frac{70}{27}+\frac{2}{9}\,\pi^2\right), \quad (72)$$

the pole contribution

$$\Delta E_P = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,\pi^2\left(-\frac{79}{32}+2\,\ln(2)\right), \quad (73)$$

the double Coulomb correction beyond the pole approximation,

$$\Delta E_C = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,(-1)\,2.629\,46(1)\,, \quad (74)$$

and the seagull correction

$$\Delta E_S = \frac{m^2}{M}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\,0.245\,23(1)\,. \quad (75)$$

We also list the proton self-energy correction beyond the terms included in charge radius,

$$E = \frac{\alpha^5}{\pi\,n^3}\,\frac{\mu^3}{M^2}\,\frac{4}{3}\left[\ln\left(\frac{M}{\mu\,\alpha^2}\right)-k_0(n)\right]. \quad (76)$$

All the corrections ΔE_i beyond the mass scaling of the wave function at the origin sum to

$$\Delta E = \Delta E_{\rm VP}+\Delta E_P+\Delta E_C+\Delta E_S = \frac{m^2}{M}\,\alpha\,\frac{(Z\,\alpha)^5}{n^3}\,(-1)\,1.364\,49\,. \quad (77)$$

This result is not in agreement with the previous calculation of Bhatt and Grotch in [5,6]. First, the vacuum polarization term has been neglected there, which indeed is small. Second, their value for the pole contribution is $\frac{35}{4}\,\ln 2-\frac{7333}{960}$, which differs from our result (73). To check it we have calculated f^A in three independent ways, and the final integration in q was also done numerically. Their result for the Coulomb term $-2.62(4)$ is in agreement with ours, but not that for the seagull correction $-1.48(1)$. In our calculation of $T^{\mu\nu}$, we have checked the imaginary part by an independent calculation using the Cutkosky rules [9]. Any additional term that does not contribute to the imaginary part would give a divergence in integrals (55), (56). The overall shift ΔE is -14 kHz for the $1S$ state, whereas Bhatt and Grotch obtained a value of -20 kHz. This correction vanishes for all the states with $l \neq 0$. We also analyzed the proton self-energy correction and the corresponding definition of the nuclear radius. The additional correction (76) beyond finite size effect is 4.6 kHz for the $1S$ state. In this way we clarified what radius is measured in atomic spectroscopic measurements. This is important in view of the discrepancy between the deuteron radius obtained from the isotope-shift measurement [10] and that from electron scattering experiments. Since a new precise measurement of the Lamb shift in muonic hydrogen is in preparation at Paul Sherrer Institute [11], a reanalysis of the proton structure corrections is necessary for a reliable determination of its charge radius.

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