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q analogs of the radial Schrödinger equation in N space dimensions

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Transparent closed forms of the radial $SO_q(N)$ Schrödinger equation acting on the noncommutative quantum Euclidean space are written both for $l=0$ and $l\neq 0$. Starting from the s-wave equation leads to the derivation of an alternative q analog of the radial Schrödinger equation in N space dimensions exhibiting a modified symmetry, now in terms of wave functions depending on the radial coordinate only. The harmonic oscillator and the Coulomb system are treated as concrete examples. These results open the way to the derivation of q-deformed $(1/N)$-energy formulas for arbitrary spherically symmetrical potentials. Comparisons with exact $SO_q(N)$ results, as well as further interpretations, have also been made.

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I. INTRODUCTION

The q quantization of Lie algebras [1–4] has attracted much interest (see, e.g., Refs. [5,6]). Euclidean spaces in which the coordinates x^i $(i=1,2,\ldots,N)$ are subject to nontrivial commutation relations have also been considered [7,8]. Such commutation relations remain invariant under linear coordinate transformations in so far as the transformation matrices themselves exhibit the symmetry structure of a related quantum group. The differential calculus has also been applied to quantum groups [9] as well as to functions depending on the noncommutative coordinates referred to above [10]. This results in nontrivial differentiation rules [11–15], which lead to corresponding q deformations of Schrödinger and other wave equations. Considering spherically symmetrical potentials acting in N space dimensions, we then have to realize that the underlying $SO(N)$ symmetry [16,17] gets replaced by the $SO_q(N)$ symmetry, as discussed before in the case of the Schrödinger equation [11]. Accordingly, the metric tensor C^{ij} of the N-dimensional Euclidean quantum space R_q^N and the R matrix of the quantum group $SO_q(N)$ are interrelated. This also means that the square of the length

$$r^2=x^ix_i=C^{ji}x_ix_j \tag{1}$$

is a central element of the algebra, in the sense that $[r^2,x^i]=0$. The harmonic oscillator and the Coulomb system have then been analyzed in N [11,12] and $N=3$ [18] space dimensions, respectively. The free particle in polar coordinates has also been discussed [14].

We have to recognize, however, that to date many questions bearing on q deformations in quantum mechanics are still open for further clarification [19]. Coming back to spherically symmetrical potentials, one is faced with the derivation of an explicit q analog of the radial Schrödinger equation in N space dimensions [20,21],

$$\left[-\partial^2-\frac{N-1}{r}\partial+\frac{1}{r^2}l(l+N-2)+V(r)\right]\psi_l=E_l\psi_l\ , \tag{2}$$

where $r=|\vec{x}|$, $\partial=\partial/\partial r$, and $N\geq 2$. In order to simplify the notation, the same notation for the radial coordinate has been preserved. Accordingly, we have to look for a closed and reasonably tractable formula for the radial constituent of the q Laplace operator $\Delta_q=\partial^i\partial_i$, where $\partial^i=C^{ij}\partial_j$. Note that in none of the previous papers mentioned above does the explicit q analog of Eq. (2) appear. For this purpose we shall proceed by analyzing the rotational excitations, now without invoking the whole mathematical machinery of the noncommutative x^i coordinates. More exactly, we shall perform a simplified description of rotational excitations just in terms of r-dependent states like $r^l\psi_0$, where ψ_0 stands for the $SO_q(N)$ ground state. We then find that under such simplified assumptions the original $SO_q(N)$ energies derived before [11,12,18] get modified, in a well-defined manner, by the potential-independent quantum q defect

$$\delta_q=(q-1)[[l]]_q[[l+N-2]]_q\ , \tag{3}$$

relying on Eqs. (37)–(38), in which the q number reads [22]

$$[[l]]_q=\frac{q^l-1}{q-1}\ , \tag{4}$$

which tends to l as $q\rightarrow 1$. Keeping in mind the invariant form of this defect, we emphasize that the radial description referred to above provides both a useful and transparent method for the derivation of an explicit q analog to Eq. (2). In this respect, our main emphasis in this paper is on the derivation of an explicit q analog of Eq. (2). This opens the way to the definition of novel q analogs of $1/N$ energy formulas for arbitrary spherically symmetrical potentials.

II. THE s-WAVE $SO_q(N)$ SCHRÖDINGER-EQUATION

All that one needs in order to perform a radial description of the Schrödinger equation on R_q^N is to apply the q-deformed differentiation rules [23]

$$\partial^ir^n=\frac{\mu}{q+1}[[n]]_qx^ir^{n-2}+q^nr^n\partial^i\ , \tag{5}$$

and

$$\partial^i x_i = \frac{\mu}{q+1}[[N]]_q + q^N x^i \partial_i \ , \tag{6}$$

where $\mu=\mu(N)=1+q^{2-N}$ and $\partial_i=\partial/\partial x^i$. We can then deal with the spherically symmetrical ground-state wave function by using the radial derivative

$$\tilde{\partial} r^n = \frac{\mu}{q+1}[[n]]_q r^{n-1} + q^n r^n \tilde{\partial} \ , \tag{7}$$

which relies on Eq. (5). For the sake of discrimination, this derivative has been denoted by $\tilde{\partial}$. Under such circumstances, the s-wave $SO_q(N)$ Schrödinger equation reads

$$\left[-q^{N-1}\tilde{\partial}^2 - \frac{\mu}{q+1}[[N-1]]_q \frac{1}{r}\tilde{\partial} + V(r) \right] \psi_q^{(0)} = E_q^{GS} \psi_q^{(0)} \ , \tag{8}$$

which reproduces the $l=0$ form of Eq. (2) in terms of the $q \to 1$ limit, as one might expect. Indeed, accounting for the ground-state

$$\psi_q^{(0)} = \psi_q^{(0)}(r,M,\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n r^{Mn}}{[[Mn]]_q!} \left[\frac{q+1}{\mu} \right]^n \ , \tag{9}$$

one finds

$$\frac{\mu\alpha}{q+1} r^{M-2} (q^{N-1}[[M-1]]_q + [[N-1]]_q) - \alpha^2 r^{2M-2} q^{N+M-2} + V(r) = E_q^{GS} \ , \tag{10}$$

by virtue of Eq. (8), where α and M are positive parameters. This shows that the only solutions are given by the harmonic-oscillator potential $V(r)=\omega^2 r^2$ and the Coulomb potential $V(r)=-Z/r$, in which case $M=2$ and $M=1$, respectively. The corresponding ground-state energies are then given by

$$E_q^{GS} = \frac{\mu\omega}{q+1} \frac{[[N]]_q}{q^{N/2}} \ , \tag{11}$$

and

$$E_q^{GS} = -\frac{(q+1)^2}{\mu^2} \frac{Z^2 q^{N-1}}{[[N-1]]_q^2} \ , \tag{12}$$

which proceed via $\alpha=\omega q^{-N/2}$ and

$$\alpha = \frac{q+1}{\mu} \frac{Z}{[[N-1]]_q} \ , \tag{13}$$

respectively. One sees that Eq. (11) reproduces exactly the ground-state energy of the Schrödinger equation on R_q^N with a harmonic oscillator [11], whereas Eq. (12) agrees with the $N=3$ result obtained for the Coulomb potential [18].

Further, we have to remember that the principal quantum numbers characterizing the above potentials come from the expansion parameter of the $1/N$ method as (see, e.g., Ref. [24])

$$d_0 = l + \frac{N-2}{2} + (n_r + \tfrac{1}{2})\sqrt{2-\nu} \ , \tag{14}$$

which proceeds to first $1/N$ order via $V(r) \sim 1/r^\nu$, such that $\nu=-2$ and $\nu=1$, respectively. The radial quantum number has been denoted by n_r. We then have to realize that the above formula sheds some light on the nontrivial generalization of Eqs. (11) and (12) to higher levels via $N \to 2d_0$ ($\nu=-2$) and $N-1 \to 2d_0$ ($\nu=1$).

III. THE RADIAL DESCRIPTION OF ROTATIONAL EXCITATIONS

Our next step is to perform a transparent radial description of rotational excitations. This differs from the original $SO_q(N)$ approach, in which the manifold of x^i coordinates is subject to quite detailed algebraic structures. To this end let us consider the harmonic-oscillator wave function

$$\psi_q^{(l)}(r) = r^l \psi_q^{(0)}(r,2,\tilde{\alpha}) \ , \tag{15}$$

for which $n_r=0$, where $\tilde{\alpha}$ denotes a rescaled coupling which should be fixed later. We remark that Eq. (15) is reminiscent of the SO(N) symmetry exhibited by Eq. (2). We shall then put forward the modified radial version of the Schrödinger equation on R_q^N as

$$\left[-q^{N-1}\tilde{\partial}^2 - \frac{\mu}{q+1}[[N-1]]_q \frac{1}{r}\tilde{\partial} + \frac{\mu^2}{(q+1)^2} \frac{1}{r^2}[[l]]_q[[l+N-2]]_q + V(r) \right] \psi_q^{(l)} = E_q^{(l)} \psi_q^{(l)} \ , \tag{16}$$

for reasons which will be made clear in a moment. We have also to anticipate that proceeding via Eq. (15) may eventually produce a certain alteration of the original $SO_q(N)$ energy result. Indeed, combining Eqs. (15) and (16) gives the energy

$$E_q^{(l)} = \frac{\mu\tilde{\alpha}}{q+1}([[2l+N]]_q + \delta_q) \ , \tag{17}$$

when

$$\tilde{\alpha} = \frac{\omega}{q^{l+N/2}} \ . \tag{18}$$

On the other hand, the original $SO_q(N)$ energy formula is [12]

$$E_q(n_0,N) = \frac{\mu\omega}{q+1} \frac{[[2n_0+N]]_q}{q^{n_0+N/2}} \ , \tag{19}$$

where $n_0=0,1,2,\ldots$. Making the identification $n_0=l+2n_r$, one realizes immediately that the appearance of the quantum q defect proceeds in terms of the substitution

$$[[2d_0]]_q \to [[2d_0]]_q + \delta_q \ , \tag{20}$$

in which δ_q is reminiscent of the $SO_q(N)$ symmetry. It is understood that Eq. (20) should work for both $n_r=0$ and $n_r \neq 0$, as the n_r dependence definitely concerns d_0. This latter statement agrees with the energy result which has been established recently by solving the Schrödinger equation on R_q^N in terms of q polynomials and q exponential functions [25]. However, the point is that repeating the same procedure in the case of the Coulomb potential one finds that Eq. (20) is confirmed, now for $\nu=1$. Accordingly, the q-deformed energy is

$$E_q^{(l)}=-\frac{(q+1)^2}{\mu^2}\frac{Z^2q^{2d_0}}{([[2d_0]]_q+\delta_q)^2}\ , \tag{21}$$

which generalizes Eq. (4.20) in Ref. [18] to N space dimensions as soon as one puts $\delta_q=0$. Thus both Eqs. (17) and (21) can be traced back to the original $\mathrm{SO}_q(N)$ results just by removing δ_q. This shows that Eq. (20) proceeds isomorphically, as one might expect. Such symmetry attributes are then able to support the relevance of Eq. (16) as a valuable candidate for the q analog of Eq. (2).

It is also of interest to establish the conditions under which a q analog of the reduced radial equation

$$\left[-\partial^2+\frac{1}{r^2}\left[l+\frac{N-3}{2}\right]\left[l+\frac{N-1}{2}\right]+V(r)\right]\varphi_l=E_l\varphi_l \tag{22}$$

is able to be implied by Eq. (16), too. Using the usual interconnection

$$\psi_q^{(l)}(r)=r^{(1-N)/2}\varphi_q^{(l)}(r) \tag{23}$$

and inserting Eq. (23) into Eq. (16), one finds, surprisingly enough, that the first derivative $\tilde{\partial}\varphi_q^{(l)}$ is eliminated if $N=3$ only. One would then obtain the equation

$$\left[-\tilde{\partial}^2+\frac{1}{q^2r^2}[[l]]_q[[l+1]]_q+V(r)\right]\varphi_q^{(l)}=E_q^{(l)}\varphi_q^{(l)}\ , \tag{24}$$

for $N=3$, in which case $\mu(3)=(1+q)/q$. However, the general radial equation can be readily derived using $\psi_q^{(l)}(r)=r^a\varphi_q^{(l)}(r)$ instead of Eq. (23), where $q^a=\mu/(q+1)$. Then $a=-1$ for $N=3$ and $a\to(1-N)/2$ as $q\to1$.

IV. THE IMPACT ON THE $1/N$ ENERGY FORMULAS

Removing δ_q and comparing Eqs. (19) and (21) with the exact energies $E=2\omega d_0$ and $E=-Z^2/4d_0^2$ of the harmonic oscillator ($\nu=-2$) and the Coulomb system ($\nu=1$), one realizes that the energy deformation implied by the original $\mathrm{SO}_q(N)$ description amounts to considering the "minimal" substitution rule

$$d_0\to\tilde{d}_0=\frac{\mu q^{-d_0}}{2(q+1)}[[2d_0]]_q\ , \tag{25}$$

which can be rewritten equivalently as

$$d_0\to\tilde{d}_0=\frac{\mu}{2q}[d_0]_q\ , \tag{26}$$

where

$$[d_0]_q=\frac{q^{d_0}-q^{-d_0}}{q-q^{-1}}\ . \tag{27}$$

In addition, one has also the modified substitution rule

$$d_0\to\tilde{d}_0^*=\frac{\mu q^{-d_0}}{2(q+1)}([[2d_0]]_q+\delta_q)\ , \tag{28}$$

which corresponds to Eq. (16). We then have to realize that Eqs. (25) and (28) serve as symmetry-endowing relationships which enable us to incorporate the influence of q deformations into the $1/N$ description. On the other hand, d_0 can be established, at least within reasonable degrees of accuracy, for arbitrary spherically symmetrical potentials. This opens the way to performing energy deformations for such potentials by combining Eq. (25) with the $1/N$ energy formulas presented before [24]. The q-deformed $1/N$ energy is then given by

$$\mathcal{E}_q^{(l)}=V(r_0)+\frac{r_0}{2}V'(r_0)\ , \tag{29}$$

in which r_0 has the meaning of a q analog of the minimum location of the effective potential $d_0^2/r^2+V(r)$ emphasized usually. This amounts to the resolution of the algebraic equation

$$2\tilde{d}_0^2=r_0^3V'(r_0)\ , \tag{30}$$

in which $\tilde{d}_0$ is given by

$$\tilde{d}_0=\frac{\mu}{2q}\left[l+\frac{N-2}{2}+(n_r+\tfrac{1}{2})\left[3+r_0\frac{V''(r_0)}{V'(r_0)}\right]^{1/2}\right]_q\ , \tag{31}$$

to first $1/N$ order. In general, all that we did above can also be repeated by considering Eq. (28) instead of Eq. (25). In consequence, the q deformations of the energy cease to be restricted only to a small number of exactly solvable potentials, which may be useful in practice. We have also to mention that Eq. (31) differs from a rather special q deformation of d_0 which has been established before [26] by interpolating between selected $\mathrm{SU}_q(2)$ energy results characterizing the harmonic oscillator [27] and the Coulomb system [28]. The main point is, however, the fact that, having the opportunity to choose between several q deformation schemes, one is favored when dealing with physical applications.

V. CONCLUSIONS AND FURTHER INTERPRETATIONS

Resorting to a radial description of rotational excitations [see Eq. (15)], we have succeeded in establishing explicit q analogs of the usual radial Schrödinger equations (2) and (22), as given by Eqs. (16) and (24). One starts from Eq. (8), which relies on the radial reduction

$$\Delta_qf(r)=\left[q^{N-1}\tilde{\partial}^2+\frac{\mu}{q+1}[[N-1]]_q\frac{1}{r}\tilde{\partial}\right]f(r)\ , \tag{32}$$

of the q Laplacian by an analytic function, which is produced by Eqs. (5) and (6). In contradistinction to the usual $\mathrm{SO}(N)$ description, we found that Eq. (23) works for $N=3$ only. It has also been shown that, under the influence of the radial wave function given by Eq. (15), the q-deformed energy becomes subject to a certain modification with respect to the previous $SO_q(N)$ results [11,12,18,25]. This amounts to the insertion of a potential-independent q defect via Eq. (20). The question of whether other solvable potentials lead, in accord with Eq. (20), to the same q defect deserves further attention. It is also clear that we can deal with the radial derivative $\tilde{\partial}$ in Eq. (16) by means of the rule

$$\tilde{\partial} f(r)=\frac{\mu}{r(q^2-1)}[f(qr)-f(r)] , \tag{33}$$

which comes from Eq. (7). This indicates that Eq. (16) gets transformed into a q difference equation, which may result in more efficient calculations. Taking into account the rescalings needed, one realizes immediately that Eqs. (32) and (33) agree with Eqs. (2.29) and (2.32) in the recent Ref. [25], respectively. Further, the $SO_q(N)$ Schrödinger equation derived in this latter reference [see Eq. (2.33)] exhibits the rather unusual barrierless radial form

$$\left[-q^{2l+N-1}\tilde{\partial}^2-\frac{\mu}{q+1}[[2l+N-1]]_q\frac{1}{r}\tilde{\partial} + V(r)\right]f(r)=E_q f(r) , \tag{34}$$

which looks like an s-wave equation, but now with a typical $2l+N$ dependence concerning the coefficients in front of the radial derivatives. However, proceeding further and putting $f(r)=r^{-l}g(r)$, it can easily be verified that Eq. (34) can be rewritten equivalently as

$$\left[-q^{N-1}\tilde{\partial}^2-\frac{\mu}{q+1}([[N-1]]_q-q^{-l}\delta_q)\frac{1}{r}\tilde{\partial} + \frac{\mu^2}{(q+1)^2}\frac{q^{-l}}{r^2}[[l]]_q[[l+N-2]]_q + V(r)\right]g(r)=E_q g(r) , \tag{35}$$

which represents properly the $SO_q(N)$ solution to the q deformation of Eq. (2), thereby explaining the onset of the q defect δ_q. Notwithstanding, Eq. (16) remains a valuable candidate for the general description of the (not at all unique) q deformation of Eq. (2), which amounts to modifying reasonably the original $SO_q(N)$ invariance in conjunction with Eq. (15). In addition, one finds that Eq. (34) yields the reduced radial equation

$$\left[-\frac{q}{(q+1)^2}(q^{-l-N/2+1}+q^{l+N/2-1})^2\tilde{\partial}^2 - \frac{\mu^2 q^2}{(q+1)^4}[[-2l-N+1]]_q[[2l+N-3]]_q\frac{1}{r^2} + V(r)\right]u(r)=E_q u(r) , \tag{36}$$

via $f(r)=r^a u(r)$, provided that $q^a=(q^{-2l-N+2}+1)/(q+1)$. Such a reduction proceeds, as before, irrespective of N but with a q-dependent power exponent a, such that $a\rightarrow(1-N)/2-l$ for $q\rightarrow 1$, as one might expect.

Relevant scalar products for the q oscillator wave functions have been analyzed in both one [29] and N [12,14,30] space dimensions. We have to recognize, however, that to date the estimation of matrix elements is still a quite involved task. Moreover, the concrete form of the related q spherical functions (see, for instance, Ref. [14]), remains open for further studies.

Performing comparisons with related $SU_q(2)$ energy results [27,28], one realizes immediately that the noncommutative geometry leads specifically to a rescaling of the ground-state energy. This proceeds by virtue of the factor $\mu/(q+1)$ exhibited by Eqs. (5)–(7). A somewhat drastic simplification is to postulate Eq. (16) as a q analog of Eq. (2), but now by identifying r with the radial coordinate of the usual Euclidean space. Strictly speaking, this alternative leads to some consistency problems, because the usual coordinate is generated by the noncommutative one via $q\rightarrow 1$. Nevertheless, such a reinterpretation is interesting as it brings out the effect of changing the metric.

Comparisons concerning the q-deformed centrifugal barrier exhibited by Eq. (16) are in order. First, Eq. (16) leads us to say, by virtue of a direct analogy, that the q analog of the eigenvalue $l(l+N-2)$ of the second $SO(N)$ Casimir operator is

$$\lambda_l=\frac{\mu^2}{(q+1)^2}[[l]]_q[[l+N-2]]_q . \tag{37}$$

However, one also has the identity

$$[[l]]_q=-q^l[[-l]]_q , \tag{38}$$

which shows, in accord with Eq. (35), that the alternative choice

$$\lambda_l^*=-\frac{\mu^2 q^2}{(q+1)^2}[[-l]]_q[[l+N-2]]_q , \tag{39}$$

which is invariant under $q\rightarrow q^{-1}$ up to the factor q^{N-2}, should also be considered. Comparing these candidates with the eigenvalue Ω_l of the $SO_q(N)$ Casimir operator discussed before [14], one sees immediately that

$$\frac{\mu^2}{(q+1)^2}\Omega_l=\lambda_l^*=q^{2-l}\lambda_l , \tag{40}$$

which shows that $\lambda_2^*=\lambda_2$, but, in general, $\sum_l\lambda_l^*\neq\sum_l\lambda_l$. Moreover, replacing q by q^2, one realizes that a similar agreement remains valid with respect to Eq. (6.15) in Ref. [30].

There is still an interconnection which seems to have been ignored so far. Indeed, Eq. (6) shows that the number of space dimensions is itself subject to the deformation

$$N\rightarrow N_q=\frac{\mu}{q+1}[[N]]_q, \tag{41}$$

which competes with Eq. (25). Then an appealing idea is to parametrize certain modifications and/or uncertainties in the number of space dimensions with the help of Eq. (41). The q parameter established in this manner is then able to serve as an input for Eq. (25) or Eq. (28).

Applying the q-deformed $1/N$ energy formulas (29)–(31), we have also the opportunity to check energy corrections for arbitrary spherically symmetrical potentials. Of course, the accuracy of $1/N$ formulas can be enhanced either by resorting to higher orders [24,31,32] or by performing suitable modifications [33–35]. So far, such spectroscopic applications are able to be supported, e.g., by the successful $\mathrm{SU}_q(2)$ description of vibrational spectra of diatomic molecules [36,37], by the rotational nuclear spectra of deformed nuclei [38,39], or by the U(3) symmetry of the nuclear shell model [40]. The application of Eq. (16) to the spectroscopy of the q-deformed Klein-Gordon Coulomb system has also been made [41]. Either way, one has by now classes of physical systems which are parametrized by the q number, which may be of interest in the description of further interconnections between certain near but nonidentical results.

The invariance of the q Casimir eigenvalue λ_l^* under the substitution $q \to q^{-1}$ indicates the existence of an additional symmetry, which otherwise would be obscured by the particular $q=1$ choice. We can then say that the virtue of the $q \neq 1$ approach is the actual possibility to incorporate from the very beginning unexpected nonlinear [42] and squeezing [43–46] effects, for which the underlying interactions are automatically produced by the q deformations. Further generalizations can also be generated by the symmetries characterizing exactly solvable models in statistical mechanics [47] and especially Bethe-ansatz solutions to spin chains [48], factorized S matrices [49], the Hubbard [50] and Perk-Schultz [51] models, or exact solutions to q-boson-hopping models [52]. Tentative descriptions of collective excitations in terms of q bosons should also be mentioned [53,54]. Coming back to the present quantum-mechanical q deformations, it should be stressed that the meaning of the q parameter can also be analyzed with the help of WKB [55] and/or $1/N$ [56] equivalent potentials. Relevant perturbations can then be produced systematically by pertinent ε expansions ($q=\exp\varepsilon$).

An alternative to the interpolation formula (14) in Ref. [26] can also be done by using q^2 instead of q in the $\mathrm{SU}_q(2)$ deformation of the Coulomb system ($\nu=1$). Accordingly, Eq. (14) in Ref. [26] remains valid as it stands, provided that $\omega=\Omega^{-1/2}$ is replaced by $\omega=2/\Omega$. Then the q-deformed counterpart of d_0 is

$$d_0 \to \tilde{d}_0 = \frac{2}{\Omega}\frac{q-q^{-1}}{q^{2/\Omega}-q^{-2/\Omega}}[d_0]_q \ , \tag{42}$$

where $\Omega=2-\nu$ for power-law potentials. This result agrees, up to a d_0-independent factor, with Eq. (26) established above.

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[1] L. D. Fadeev, N. Yu. Reshetikhin, and L. A. Takhtajan, in *Algebraic Analysis,* edited by M. Kashiwara and T. Kawai (Academic, Boston, 1989), Vol. 1, p. 129.
[2] E. K. Sklyanin, Functional Anal. Appl. **16**, 262 (1982).
[3] V. G. Drinfeld, in *Quantum Groups,* Proceedings of the International Congress of Mathematicians, Berkeley, 1986, edited by M. Gleason (American Mathematical Society, Providence, 1987), p. 798.
[4] M. Jimbo, Lett. Math. Phys. **11**, 247 (1986).
[5] *Symmetries in Science VI,* edited by B. Gruber (Plenum, New York, 1993).
[6] *Quantum Symmetries,* Proceedings of the 1991 Clausthal Workshop on Mathematical Physics, edited by H. D. Doebner and V. K. Dobrev (World Scientific, Singapore, 1993).
[7] A. Connes, Publ. Math. Inst. Hautes Études Scientifiques **62**, 41 (1986).
[8] Yu. I. Manin, Ann. Inst. Fourier (Grenoble) **37**, 191 (1987).
[9] S. L. Woronowicz, Commun. Math. Phys. **122**, 125 (1989).
[10] J. Wess and B. Zumino, Nucl. Phys. Proc. Suppl. B **18**, 302 (1990).
[11] U. C. Watamura, M. Schlieker, and S. Watamura, Z. Phys. C **49**, 439 (1991). Note that Eqs. (IV.9) and (IV.11) in this reference are actually identical.
[12] G. Fiore, Int. J. Mod. Phys. A **7**, 7597 (1992). In this paper the q Hamiltonian reads $[-q^N\Delta_q+V(r)]/2$ instead of $-\Delta_q+V(r)$.
[13] O. Ogievetsky and B. Zumino, Lett. Math. Phys. **25**, 121 (1992).
[14] A. Hebecker and W. Weich, Lett. Math. Phys. **26**, 245 (1992).
[15] T. Brzezinski, H. Dabrowski, and J. Remblienski, J. Math. Phys. **33**, 19 (1992).
[16] L. D. Mlodinow and N. Papanicolaou, Ann. Phys. (N.Y.) **128**, 314 (1980).
[17] C. C. Gerry and J. B. Togeas, J. Phys. A **19**, 3797 (1986).
[18] X.-C. Song and L. Liao, J. Phys. A **25**, 623 (1992).
[19] M. Kibler, in *Generalized Symmetries in Physics,* Proceedings of the 1993 Clausthal Symposium, edited by H. D. Doebner, V. Dobrev, and U. Ushveridze (World Scientific, Singapore, 1994).
[20] F. R. Tangherlini, Nuovo Cimento **27**, 636 (1963).
[21] D. Bergmann and Y. Frishman, J. Math. Phys. **6**, 1855 (1965).
[22] The quantum number of the angular momentum and the quantum deformation parameter have been denoted by l and $q>0$, as usual.
[23] See also the Appendix in Ref. [13] and Eq. (4.11) in Ref. [18].
[24] E. Papp, Phys. Rev. A **38**, 2158 (1988). Note that d_0 is produced to first $1/N$ order by Eqs. (2) and (3) in this reference via $\varepsilon^{(1)}(x_0)=0$.
[25] U. C. Watamura and S. Watamura, Int. J. Mod. Phys. A **9**, 3989 (1994). Here $\mathcal{H}_q=-q^N\Delta_q+V$, whereas $f(x^2)$ and D_{x^2} stand for $f(r)$ and $(1/\mu r)\partial$.
[26] E. Papp, Phys. Lett. A **189**, 341 (1994).
[27] E. G. Floratos, J. Phys. A **24**, 4739 (1991).

[28] I. E. Pris' and E. A. Tolkachev, Yad. Fiz. **56**, 66 (1993) [Phys. At. Nucl. **56**, 748 (1993)]. See the results which come from Eq. (15) in this reference.
[29] J. A. Minahan, Mod. Phys. Lett. A **5**l, 2625 (1990).
[30] G. Fiore, Int. J. Mod. Phys. A **8**, 4679 (1993).
[31] T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Rev. D **29**, 1669 (1984).
[32] V. S. Popov and A. V. Sergeev, Zh. Eksp. Teor. Fiz. **105**, 568 (1994) [JETP **78**, 303 (1994)].
[33] A. Chatterjee, Phys. Rep. **186**, 249 (1990).
[34] M. Bag, M. M. Panja, R. Dutt, and Y. P. Varshni, Phys. Rev. A **46**, 6059 (1992).
[35] S. Kais and D. R. Herschbach, J. Chem. Phys. **98**, 3990 (1993).
[36] Z. Chang and H. Yan, Phys. Rev. A **44**, 7405 (1991).
[37] D. Bonatsos and C. Daskaloyannis, Phys. Rev. A **46**, 75 (1992); **50**, 890 (1994).
[38] P. P. Raychev, R. P. Roussev, and Yu. F. Smirnov, J. Phys. G **16**, L137 (1990).
[39] F. Pan, Phys. Rev. C **50**, 1876 (1994).
[40] A. Del Sol Mesa, G. Loyola, M. Moshinsky, and V. Velázquez, J. Phys. A **26**, 1147 (1993).
[41] E. Papp, Phys. Lett. B **339**, 83 (1994).
[42] V. I. Man'ko, G. Marmo, S. Solimeno, and F. Zaccaria, Int. J. Mod. Phys. A **8**, 3577 (1993).
[43] M. Chaichian, D. Ellinas, and P. Kulish, Phys. Rev. Lett. **65**, 980 (1990).
[44] E. Celeghini, M. Rasetti, and G. Vitiello, Phys. Rev. Lett. **66**, 2056 (1991).
[45] J. Katriel and A. Solomon, J. Phys. A **24**, 2093 (1991).
[46] M. Artoni, J. Zang, and J. L. Birman, Phys. Rev. A **47**, 2555 (1993).
[47] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).
[48] C. N. Yang and C. P. Yang, Phys. Rev. **150**, 321 (1966).
[49] A. B. Zamolodchikov and A. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).
[50] D. B. Uglov and V. E. Korepin, Phys. Lett. A **190**, 238 (1994).
[51] M. Couture, J. Phys. A **25**, 1953 (1992).
[52] N. M. Bogoliubov, R. K. Bullough, and G. D. Pang, Phys. Rev. B **47**, 11 495 (1993).
[53] D. Bonatsos, J. Phys. A **25**, L101 (1992).
[54] J. L. Birman, Solid State Commun. **84**, 259 (1992).
[55] D Bonatsos, C. Daskaloyannis, and K. Kokkotas, Phys. Rev. A **45**, R6153 (1991); J. Phys. A **24**, L795 (1991).
[56] E. Papp, TU-Clausthal Report No. ASI-TPA/14/94 (unpublished).