RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 51, NUMBER 1 JANUARY 1995

Twin-hole dark solitons

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(Received 8 August 1994)

It is generally believed that interaction between dark solitons is repulsive, and two solitons cannot form a bound state. Recently, it has been pointed out that parametric interactions in a diffractive quadratic nonlinear medium can support one-dimensional *twin-hole* spatial dark solitons [K. Hayata and M. Koshiba, Phys. Rev. A **50**, 675 (1994)]. We demonstrate numerically and analytically that this twin-hole dark soliton is *modulationally unstable*. Instead, we show that, in the other physical context, wave mixing between the first and second harmonics due to cascaded second-order processes in a dispersive quadratic media can support *stable* twin- (or multi-) hole dark solitons. We explain the physical mechanism that allows these solitons to exist and analyze their stability numerically and analytically.

PACS number(s): 42.50.Rh, 02.30.Hq, 42.60.Jf, 42.65.Jx

It is well known that *optical dark solitons* can exist in a Kerr-like (cubic or $\chi^{(3)}$) nonlinear medium such as local dips (or holes) on a modulationally stable continuous wave (cw) background. These solitons have already been observed experimentally as intensity-modulated pulses in optical fibers (*temporal* dark solitons), dark strips in the transverse cross section of laser beams passing through a nonlinear medium, and also as self-guided "holes" in the beams propagating in waveguides (*spatial* dark solitons) (see, e.g., the review paper [1] and references therein). In recent years significant attention has been paid to the physics and properties of dark solitons. For example, spatial solitons are considered as perfect self-induced waveguides, which allow guiding and steering of light beams [2], i.e., manipulation of light by light itself, the basic goal of future optical devices.

It is generally believed that interaction between dark solitons is *repulsive*, and two dark solitons cannot form a bound state (see, e.g., [1]). However, if such a bound state still exists, it must have two holes in the intensity profile, so that it can be naturally called *a twin-hole dark soliton*. Recently, Hayata and Koshiba [3] have pointed out that parametric wave interactions in a quadratic nonlinear medium can support one-dimensional twin-hole *spatial* dark solitons. In particular, they have found an *exact* twin-hole dark soliton solution of the generalized second-harmonic generation equations, which also include the diffraction effect. The purpose of the present paper is twofold. First, we demonstrate numerically and analytically that the solitons discussed by Hayata and Koshiba are *modulationally unstable* and thus it might be difficult to observe these solitons experimentally. Second, looking for the existence of stable dark solitons in another physically important case, we show that stable *temporal* two- (and multi-) hole dark solitons can exist due to wave mixing between the first and second harmonics in a quadratic nonlinear medium. We also point out the physical mechanism that allows these solitons to exist.

Considering interaction of the first ($\omega_1=\omega$) and second ($\omega_2=2\omega$) harmonics in a dielectric medium with $\chi^{(2)}$ nonlinear susceptibility, we assume their amplitudes E_1 and E_2 to be slowly varying and derive from Maxwell's equations the system of two equations coupled through components $\chi^{(2)}_{ijk}$ of the nonlinear susceptibility tensor,

$$i\frac{\partial E_1}{\partial z}+\gamma_1\frac{\partial^2 E_1}{\partial\xi^2}+\chi_1 E_1^* E_2 e^{-i\delta kz}=0,$$

$$i\frac{\partial E_2}{\partial z}+\gamma_2\frac{\partial^2 E_2}{\partial\xi^2}+\chi_2 E_1^2 e^{i\delta kz}=0, \quad (1)$$

where $\chi_1=(4\pi\omega^2/k_1c^2)\chi^{(2)}(\omega;2\omega,-\omega)$ and $\chi_2=(8\pi\omega^2/k_2c^2)\chi^{(2)}(2\omega;\omega,\omega)$, z is the propagation distance, and δk is a phase mismatch between the first and second harmonics. The system (1) generalizes the standard equations of the theory of the second harmonic generation [4]. It describes two different physical situations: In the first, *spatial* case, ξ stands for the transverse spatial coordinate, and $\gamma_{1,2}=1/2k_{1,2}$, so that Eqs. (1) take into account the effect of diffraction. In the second, *temporal* case, ξ stands for retarded time and $\gamma_j\equiv-\frac{1}{2}\partial^2k_j/\partial\omega_j^2$ ($j=1,2$) are the dispersion coefficients.

We are interested in stationary solutions of the system (1), and look for the fields in the form

$$E_1=w\frac{\beta}{\sqrt{2\sigma\chi_1\chi_2}}e^{i\beta z}, \qquad E_2=v\,\frac{\beta}{\chi_1}\,e^{i(2\beta+\delta k)z}, \quad (2)$$

where β is a nonlinearity-induced shift of the propagation constant and $\sigma\equiv|\gamma_1|/|\gamma_2|$. Equations for w and v are

$$i\frac{\partial w}{\partial\zeta}+r\frac{\partial^2 w}{\partial\tau^2}-w+w^*v=0,$$

$$i\sigma\frac{\partial v}{\partial\zeta}+s\frac{\partial^2 v}{\partial\tau^2}-\alpha v+\frac{1}{2}w^2=0, \quad (3)$$

where $\zeta=\beta z$, $\tau=(|\beta|/|\gamma_1|)^{1/2}\xi$, $r=\mathrm{sgn}(\beta\gamma_1)$, $s=\mathrm{sgn}(\beta\gamma_2)$, and $\alpha\equiv(2\beta+\delta k)\sigma/\beta$.

Following Hayata and Koshiba [3], we consider the case of spatial solitons when $\mathrm{sgn}(\gamma_1)=\mathrm{sgn}(\gamma_2)=+1$, and look for stationary (independent of ζ) localized solutions of Eqs. (3), which are described by the system of the ordinary differential equations for real $w(\tau)$ and $v(\tau)$,

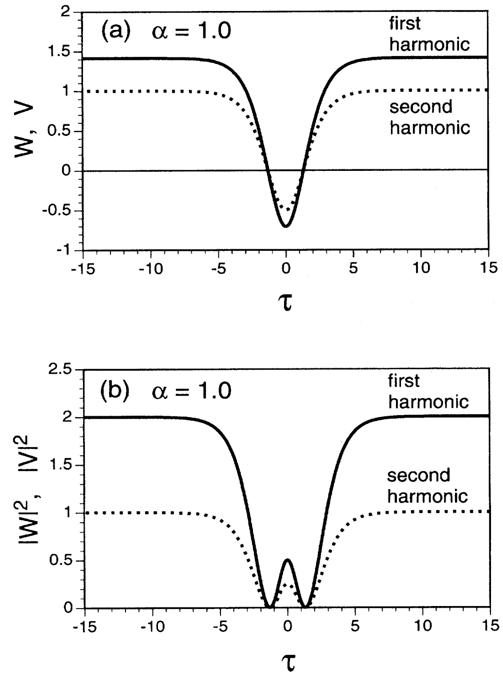

FIG. 1. Twin-hole dark soliton (5) found in Ref. [3]. (a) Profiles of the first (w) and second (v) harmonics, and (b) the corresponding intensities, $|w|^2$ and $|v|^2$.

$$r\,\frac{d^2w}{d\tau^2}-w+wv=0,\qquad s\,\frac{d^2v}{d\tau^2}-\alpha v+\frac{1}{2}\,w^2=0. \tag{4}$$

Note that for the case $r=s=+1$ we have found recently [5] one family of bright solitons of Eqs. (3). This family includes, as a particular case, the exact solution found earlier [6,7].

According to Hayata and Koshiba [3], for $\mathrm{sgn}(\gamma_1)=\mathrm{sgn}(\gamma_2)=+1$ but $\beta<0$, i.e., when $r=s=-1$ in our notation, the system (3) supports spatial dark solitons of a special profile. This soliton solution exists only for one fixed value of α ($\alpha=1$) and has the form (see Fig. 1)

$$w=\pm\left[\sqrt{2}-\frac{3\sqrt{2}}{2\cosh^2(\tau/2)}\right],\quad v=\left[1-\frac{3}{2\cosh^2(\tau/2)}\right]. \tag{5}$$

As can be seen from Fig. 1(b), the intensity of the soliton (5) has *two holes* for both harmonics. Another important feature of the soliton (5) is the absence of the phase difference between the background waves from the right and left. These properties are in sharp contrast with the conventional dark solitons of $\chi^{(3)}$ nonlinearities [1].

The solution (5) exists only at $\alpha=1$. One can suppose that twin-hole dark solitons still exist for every value of the parameter α forming a family, which includes the exact solution (5) as a particular case at $\alpha=1$. However, as has been shown by our analysis, the hypothesis of the existence of a continuous family of dark solitons for the system (3) at $r=s=-1$ is not correct. Indeed, Eqs. (4) can be treated as the classical mechanics equations of motion for a particle on the plane (w,v), where w and v are two coordinates of the particle. Then, any localized solution of Eqs. (4) is a separatrix trajectory that separates different regimes of the particle motion. The trajectories, which correspond to dark solitons, must start and finish at nonzero extremum points of the potential $U(w,v)$ of the corresponding mechanical problem. For the case $r=s=-1$ this potential has the form $U(w,v)=\frac{1}{2}(w^2+\alpha v^2-w^2v)$, so that the nonzero extremum points are

$$w_0=\pm\sqrt{2\alpha},\qquad v_0=1. \tag{6}$$

Choosing the initial conditions in the vicinity of one of the points (6) we can search numerically for all trajectories on the plane (w,v) that finally come back to one of these points. Such separatrix trajectories correspond to dark solitons of the system (3). However, the extremum points (6) are the points of the saddle type and, therefore, at any fixed value of α we have only one trajectory to check in our search for a separatrix. This absence of an internal shooting parameter leads to the result that for the system (3) at $r=s=-1$ *localized solutions exist only for some particular fixed values of the parameter* α (e.g., $\alpha\approx0.2173$, 0.4005, 3.9434, 7.3103, and so on). Thus, instead of a continuous family, there is only a discrete set of dark-soliton solutions. The simplest one among these solutions is the solution (5), which may be considered as the dark soliton of *the lowest order*. It is important to note that all these solutions are very different from each other. Each of them exists only at one particular value of α and can be treated, at least qualitatively, as a two- (or multi-) soliton bound state, although we have not found any example of a single dark soliton. This situation resembles the case of the nonlinear Schrödinger (NLS) equation with an additional third-order dispersion term where a single bright soliton always radiates but a pair of such solitons can exist as a quasistationary weakly unstable localized state at certain values of some parameter due to the suppression of radiation through the wave interference in the asymptotic regions [8].

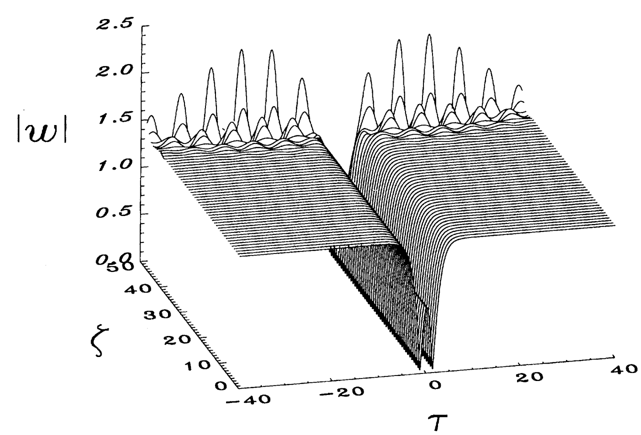

FIG. 2. Instability of the dark soliton (5) shown in Fig. 1. Only the evolution of the first harmonic w is shown ($\sigma=2.0$). Corresponding plots of Ref. [3] are limited by shorter propagation distances and do not display this instability.

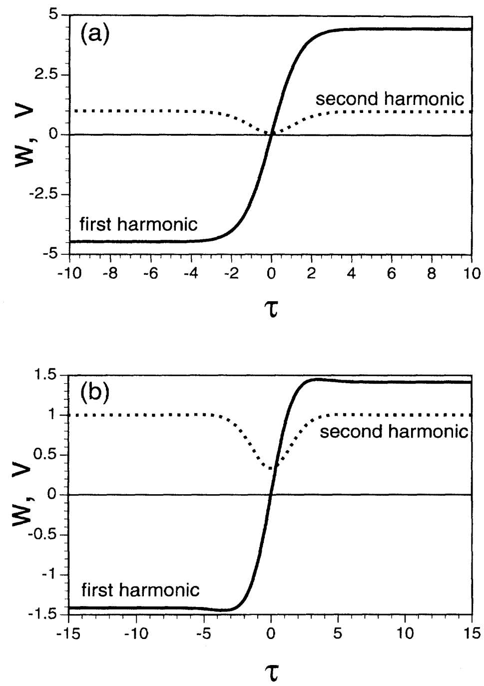

FIG. 3. Characteristic profiles of the single dark-soliton solutions of Eq. (3) at $r=-1$ and $s=+1$: (a) $\alpha=10.0$ and (b) $\alpha=1.0$. Note the nonzero amplitude minimum of the second harmonic in both cases and nonmonotonic tails in case (b). In contrast to the solution (5), solitons of this family exist for *every value* of $\alpha>0$.

We have investigated the exact solution given by Eq. (5) by the split-step beam-propagation method (see, e.g., [9]) to analyze numerically its stability. The example of the propagation dynamics is presented in Fig. 2. It is clearly seen that, after some propagation distance, the cw background that supports the twin-hole dark soliton becomes *unstable*. The distance at which the instability manifests itself growing from a numerical noise is bigger than that used by Hayata and Koshiba in their paper [3], where this instability has not been found. Nevertheless, if we start from an initial condition in the form of *a slightly perturbed* solution (5), the instability appears to be seen immediately. The instability observed here does not affect much the localized part of the solution. It is, in fact, the *modulational instability* of the cw background. Analysis of the linear stability of cw solutions (6) against modulations $\sim\exp(iq\zeta+ip\tau)$ reveals *two branches* for the dispersion relation,

$$q_{1,2}^2(\rho^2)=[\Delta_1\pm\sqrt{\Delta_1^2-\Delta_2^2}]/\sigma, \tag{7}$$

where $\Delta_1=2\alpha\sigma+\frac{1}{2}(s\rho^2+\alpha)+\frac{1}{2}r\rho^2\sigma^2(2+r\rho^2)$ and $\Delta_2^2=4\alpha^2+r\rho^2(2+r\rho^2)(s\rho^2+\alpha)^2-4\alpha(s\rho^2+\alpha)(1+r\rho^2)$. These two spectrum branches resemble "optical" and "acoustic" modes of a diatomic lattice. It is important that instabilities can appear due to both "acoustic" (as for a single NLS equation) and "optical" branches (parametric modulational instability). The analysis of the expression (7) displays modulational instability of all dark solitons of the system (3) at $r=s=-1$, since in this case the cw background is not stable for any combination of the parameters σ and α, and thus the necessary condition for the existence of stable dark solitons is not satisfied. In addition to the analysis of the cw stability we have also investigated the evolution of perturbation eigenmodes for the discrete set of the dark-soliton solutions found numerically. To do this, we linearize Eq. (3) around the solution of interest and solve the resulting linearized equation numerically to find exponentially growing modes. In agreement with the cw analytical results, we have found that all considered dark-soliton solutions of Eqs. (3) at $r=s=-1$ have *exponentially growing modes* and thus are *unstable*. For example, the increment λ of the fastest growing instability mode of the solution (5) lies in the interval $0.65>\lambda>0.45$ for $0.5<\sigma<4.0$.

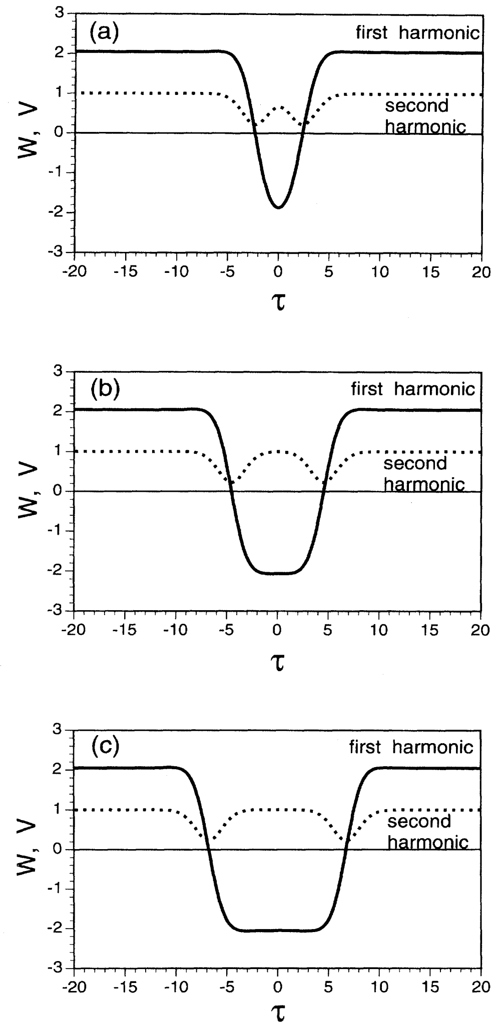

FIG. 4. Bound states of a pair of dark solitons with nonmonotonic tails at $\alpha=2.1$. These bound states are stable twin-hole dark solitons. The plots (a) to (c) show three different relative distances between a pair of single dark solitons (which correspond to three different local extrema of their effective interaction potential; thus only certain relative distances between single solitons are possible).

Thus, the twin-hole dark soliton found by Hayata and

Koshiba has been proved to be modulationally unstable. Therefore, the question remains: *Does a stable twin-hole dark soliton exist?* This is a more general question than that for the particular model (3), because, to our knowledge, up to the present time no one has reported stable twin-hole dark solitons in other fields of nonlinear science. However, as we show below, the positive answer does exist for the model (3) at $r=-1$ and $s=+1$. Now the system (3) describes *a temporal* pulse propagation in a dispersive $\chi^{(2)}$ medium. It can be derived for the case of the opposite signs of the second-order dispersions γ_j ($j=1,2$) in the system (1).

To find stationary dark-soliton solutions, we analyze again Eqs. (4) for real $w(\tau)$ and $v(\tau)$ by means of the shooting technique but this time taking $r=-1$ and $s=+1$. In the case $\alpha>0$ we have found numerically a continuous family of dark one-soliton solutions. Two characteristic representatives of this family are shown in Figs. 3(a) and 3(b) for $\alpha=10.0$ (monotonic tails) and $\alpha=1.0$ (nonmonotonic tails), respectively. Dark solitons of this family exist for every value of α ($\alpha>0$). However, the modulational instability analysis [Eqs. (7)] predicts that at $r=-1$ and $s=+1$ the cw background, which supports these dark solitons, can be stable only if $\alpha>2$ and $\sigma>\sigma_{cr}$ ($\sigma_{cr}\approx 1.689$). Additionally, the dark solitons with nonmonotonic tails shown in Fig. 3(b) exist only if $\alpha<8$ (this can be shown by an analysis similar to that presented in a recent paper [10]), so that the stable dark solitons with nonmonotonic tails are expected to exist in the interval $2<\alpha<8$. In the analytical adiabatic approach [11] where dark solitons are considered as effective particles interacting through exponentially decaying forces, such nonmonotonic tails produce local minima in the effective interaction potential of weakly overlapping solitons, and, therefore, *a dark soliton with nonmonotonic tails can trap another one*. The physical picture of the soliton bound states is well understood for bright solitons [12], but after corresponding generalizations it can be used for dark solitons as well. In our problem this qualitative picture suggests the existence of infinitely many *families* of bound states of two (or more) dark solitons. These solitons have two (or more) intensity minima, so that they are twin- (or multi-) hole dark solitons. Our numerical analysis shows that these solitons really exist for every value of α ($0<\alpha<8$). Some of the simplest examples of the two-soliton bound states are shown in Fig. 4 at $\alpha=2.1$. In spite of the fact that the solution for the first harmonic shown in Fig. 4(a) looks similar to that found by Hayata and Koshiba, it differs greatly from the result (5), especially by the structure of the second harmonic [cf. Figs. 1(a) and 4(a)]. We have found several twin- (and multi-) hole dark-soliton families. For every value of α, solitons of the various families can be classified by the number of single dark solitons forming these bound states, and also by the relative distance between the neighboring dark solitons [see Figs. 4(a)–4(c)]. Our numerical analysis shows that at least some of these families consist of stable solitons (in the interval $2<\alpha<8$), e.g., the twin-hole dark solitons shown in Figs. 4(a) and 4(b) are stable for $\sigma>1.7$. However, the bound energy strongly depends on the relative distance between solitons, i.e., on the order of the minimum of the effective interaction potential.

In conclusion, we have demonstrated numerically and analytically that the exact solution found by Hayata and Koshiba [3], which describes a twin-hole dark soliton, is modulationally unstable. Instead, we have pointed out the other physical situation when many families of twin- (and multi-) hole dark solitons can remain stable. The physical reason for the existence of these stable bound states of dark solitons is in their nonmonotonic tails. Bound states of two (or more) dark solitons are formed due to mutual trapping of the neighboring solitons in the positions corresponding to the local minima of the effective interaction potential.

The work was supported by the Australian Photonics Cooperative Research Centre.

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