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Intrinsic and operational observables in quantum mechanics

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The concept of intrinsic and operational observables in quantum mechanics is introduced. In any realistic description of a quantum measurement that includes a macroscopic detecting device, it is possible to construct from the statistics of the recorded raw data a set of operational quantities that correspond to the intrinsic quantum-mechanical observable. This general approach is illustrated by the example of an operational measurement of the position and the momentum of a particle, as well as by an analysis of the operational detection of the phase of an optical field. For the latter we identify the intrinsic phase operator and report its explicit form.

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The quantum measurement theory provides for a conceptual framework in which one can understand the features of the quantum world in terms of measurable or observable quantities. Since the birth of quantum physics, the theory of measurement has proved to be controversial, both in its physical and philosophical aspects. These controversies have generated long lasting debates about the relation of the quantum formalism to the quantities that are actually measured by macroscopic devices used in real experiments [1].

It is the purpose of this Rapid Communication to present a general, down-to-earth approach, connecting in a natural way the standard formalism of quantum mechanics with the statistical raw data recorded in an experiment. In this approach an operational link is established and discussed between the quantum observables and the macroscopic devices used to detect and measure quantum phenomena. We argue that, for each measurement, it is possible to construct from the statistics of the recorded raw data a set of operational quantities that correspond to the quantum-mechanical observables in a certain way. Here, the "raw data" do not refer to the unprocessed laboratory records but rather to the "positive-operator-valued-measure" or POVM that is the mathematical representation of the statistical information gathered. In one way of looking at quantum measurements [2] the emphasis is on such POVMs. For us, however, the underlying intrinsic observable is the heart of the matter.

We illustrate our approach to operational measurements using two different examples. The first example deals with a model measurement of the position and the momentum of a particle, and the second example is devoted to a real homodyne detection of the phase of optical signals.

We start with a general description of our approach. For a quantum system described by a density operator $\hat{\rho}$, statistical properties of an arbitrary observable $\hat{A}$ can be evaluated with the aid of the moment-generating function

$$Z(\lambda)=\mathrm{Tr}\{\hat{\rho}\exp(\lambda\hat{A})\} \tag{1}$$

in accordance with

$$\langle\hat{A}^n\rangle=\mathrm{Tr}\{\hat{\rho}\hat{A}^n\}=\frac{d^n}{d\lambda^n}Z(\lambda)\bigg|_{\lambda=0}. \tag{2}$$

Thus the generating function $Z(\lambda)$ contains all the relevant statistical information about the system in state $\hat{\rho}$, but it makes no reference to the apparatus employed in an actual measurement of the observable $\hat{A}$ and its moments. To begin with, $Z(\lambda)$ is a purely theoretical quantity; it is what would be measured in an ideal noise-free measurement.

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There are, however, numerous examples in the literature of measurements that require realistic detecting devices. To name just a few, we mention the quantum-mechanical models of the "pointer" introduced by von Neumann [3] and Arthurs and Kelly [4], their extension and refinement by Lamb [5], the operational approach to the Heisenberg microscope [6], the quantum Zeno effect [7], the operational phase-space in quantum mechanics [8], or the role of the apparatus in the decoherence theory [9].

A realistic experiment necessarily involves additional degrees of freedom that eventually enable the experimenter to convert the laboratory records into a probability density, or rather a *propensity* density $\Pr(a)$ of a classical variable a [10]. For this purpose an analysis of the experimental setup is required, best perhaps in the spirit of Lamb's operationalism [5]. The propensity thus found determines classical averages as exemplified by

$$\overline{a^n}=\int da\, a^n \Pr(a)\,. \tag{3}$$

In the typical situations that we have in mind, the net effect of the measuring device can be described by an a-dependent *filter* $\mathcal{F}$, represented by a positive operator $\hat{\mathcal{F}}(a)$, such that

$$\Pr(a)=k\,\mathrm{Tr}\{\hat{\rho}\hat{\mathcal{F}}(a)\}, \tag{4}$$

where the coefficient k is chosen in such a way that $\int da\,\Pr(a)=1$. In view of this linear relation, the requirement that

$$\overline{a^n}=\langle \hat{A}^{(n)}_{\mathcal{F}}\rangle \tag{5}$$

hold for all $\hat{\rho}$ specifies a unique set of operators $\hat{A}^{(n)}_{\mathcal{F}}$,

$$\hat{A}^{(n)}_{\mathcal{F}}=k\int da\, a^n \hat{\mathcal{F}}(a)\,, \tag{6}$$

for the given filter $\mathcal{F}$.

Inasmuch as the experimenter is guided by classical intuition when designing the apparatus, we shall take for granted that $\hat{A}=\hat{A}^{(1)}_{\mathcal{F}}$ holds and that the quantum expectation value $\langle\hat{A}^n\rangle$ agrees with the classical average $\overline{a^n}$ in the correspondence limit. In other words, a good measurement is characterized by the property that the classical limits [11] of $\hat{A}^n$ and $\hat{A}^{(n)}_{\mathcal{F}}$ are the same.

We shall employ the following terminology. We call $\hat{A}$ an intrinsic quantum observable (IQO), whereas each $\hat{A}^{(n)}_{\mathcal{F}}$ is an operational quantum observable (OQO). Thus, from the point of view that we wish to advance, the measuring device $\mathcal{F}$ effectively replaces the powers of IQOs by a set of OQOs. Rather than determining the generating function $Z(\lambda)$ of Eq. (1), which refers to the IQO of interest, the experimental results are compactly summarized in the filter-dependent generating function

$$Z_{\mathcal{F}}(\lambda)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathrm{tr}\{\hat{\rho}\hat{A}^{(n)}_{\mathcal{F}}\}=\int da\,\exp(\lambda a)\Pr(a)\,. \tag{7}$$

The comparison with

$$Z(\lambda)=\int dA\,\langle A|\hat{\rho}|A\rangle\exp(\lambda A) \tag{8}$$

shows that the probability distribution that is associated with the spectral decomposition of $\hat{A}$ is effectively replaced by the propensity $da\,\Pr(a)$, which refers to the filter $\mathcal{F}$ of the measuring device. Note that the quantity $k\,da\,\hat{\mathcal{F}}(a)$ is the POVM of the experiment in question. From our point of view, this POVM is not interesting in itself; the filter function is merely necessary for the identification of the OQOs, but the IQOs remain the objects of primary interest.

There is then the obvious question: What is the relation between the OQOs and the powers of the IQO? Two cases must be distinguished. First, we have the standard situation in which the IQO is known, so that one just needs to identify the OQOs corresponding to the filter of the actual measurement. The noise introduced in the course of determining the propensity density $\Pr(a)$ can then be accounted for explicitly. In this way, $Z(\lambda)$ can possibly be expressed in terms of $Z_{\mathcal{F}}(\lambda)$ whereafter the propensity has served its purpose. We shall illustrate this standard case at a model of position and momentum measurements with respect to a reference pointer in thermal equilibrium.

In the second case one deals with the unusual situation that the quantum properties of the IQO are largely unknown, although the IQO has a well known classical analog. The guidance provided by this classical analog suggests one or more measurement schemes, each of which specifies a set of OQOs. While it is clear that the looked-for IQO cannot be identified uniquely in such an operational approach, the choice $\hat{A}=\hat{A}^{(1)}_{\mathcal{F}}$ is certainly the most natural one for the IQO associated with the OQOs of one experimental setup. Once this IQO is identified, its $Z(\lambda)$ is available in principle and can possibly be related to the generating function $Z_{\mathcal{F}}(\lambda)$ that is determined experimentally. This second case is exemplified by the recent measurements of the phase properties of optical fields by Noh, Fougères, and Mandel (NFM) [12]. Here the filter $\mathcal{F}$ accounts for the beam splitters, mirrors, and photon counters used in the homodyne detection. We shall treat this example and identify the intrinsic phase operator that corresponds most naturally to the OQOs defined by the NFM apparatus.

As a rule, the algebraic properties of the $\hat{A}^{(n)}_{\mathcal{F}}$ operators are quite different from those of the powers of $\hat{A}$. In particular, a factorization is typically impossible, so that, for instance, $\hat{A}^{(2)}_{\mathcal{F}}$ does not equal $(\hat{A}^{(1)}_{\mathcal{F}})^2$. The operational spread $\delta a=(\overline{a^2}-\bar{a}^2)^{1/2}=(\langle\hat{A}^{(2)}_{\mathcal{F}}\rangle-\langle\hat{A}^{(1)}_{\mathcal{F}}\rangle^2)^{1/2}$ is then different from the quantum uncertainty $\Delta\hat{A}=(\langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2)^{1/2}$. Indeed, since the operational spread δa may refer to expectation values of two different operators, its physical significance could be rather murky, in contrast to the quantum uncertainty $\Delta\hat{A}$ with its familiar physical meaning. Further, it is clear that the Heisenberg uncertainty relation obeyed by the product $\Delta\hat{A}\,\Delta\hat{B}$ for two IQOs need not be equally valid for the product $\delta a\,\delta b$ of the corresponding operational spreads.

As an illustration of the general scheme we now turn to operational measurements of the position and momentum of a particle in one dimension. In particular, we consider a device that determines the overlap of the density operator $\hat{\rho}$ of the system with the density operator of a reference oscillator.

This reference oscillator is supposed to be in a state of thermal equilibrium with a temperature corresponding to $\bar{n}$ oscillator quanta. The oscillator also supplies natural units for distances and momenta. Therefore, we shall take as the IQOs the dimensionless position and momentum operators $\hat{Q}$ and $\hat{P}$ that refer to these oscillator units. Now, in order to probe the system, the reference oscillator is displaced both in position and in momentum by the amounts q and p, respectively. With these classical variables, the filter function is

$$\hat{\mathscr{F}}(q,p)=\exp(ip\hat{Q}-iq\hat{P})\,\hat{\mathscr{F}}(0,0)\,\exp(iq\hat{P}-ip\hat{Q})\,, \tag{9}$$

where

$$\hat{\mathscr{F}}(0,0)=\frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{(\hat{Q}^2+\hat{P}^2-1)/2} \tag{10}$$

is the density operator of the reference oscillator when it is at rest and located at the origin.

The propensity $\Pr(q,p)=k\langle\hat{\mathscr{F}}(q,p)\rangle$ is normalized according to $\int dq\,dp\,\Pr(q,p)=1$. The generating function for the OQOs, for which

$$Z_{\mathscr{F}}(\lambda,\mu)=\int dq\,dp\,\exp(i\lambda q-i\mu p)\Pr(q,p) \tag{11}$$

is a convenient choice here, is then given by

$$Z_{\mathscr{F}}(\lambda,\mu)=\langle\exp(i\lambda\hat{Q}-i\mu\hat{P})\rangle\exp[-\tfrac{1}{4}(2\bar{n}+1)(\lambda^2+\mu^2)]\,. \tag{12}$$

The first factor can be regarded as a generating function $Z(\lambda,\mu)$ for expectation values of the intrinsic observables $\hat{Q}$ and $\hat{P}$, and the second factor accounts for the noise that is unavoidably introduced during the measurement.

With the generating function $Z_{\mathscr{F}}(\lambda,\mu)$ at hand we can proceed to identify the operational observables. Upon expanding $Z_{\mathscr{F}}(\lambda,\mu)$ in powers of λ and μ, the OQOs can be read off in accordance with (5). For example, for those OQOs that correspond to powers of q only, this produces

$$\hat{Q}^{(n)}_{\mathscr{F}}=\left(\frac{1}{2i}\sqrt{2\bar{n}+1}\right)^n H_n(i\hat{Q}/\sqrt{2\bar{n}+1})\,, \tag{13}$$

where H_n is the nth Hermite polynomial. An analogous equation holds for $\hat{P}^{(n)}_{\mathscr{F}}$. These relations can be inverted in order to express the powers of $\hat{Q}$ and $\hat{P}$ in terms of the OQOs whose expectation values are measured directly, as exemplified by

$$\hat{Q}=\hat{Q}^{(1)}_{\mathscr{F}},\quad \hat{Q}^2=\hat{Q}^{(2)}_{\mathscr{F}}-(\bar{n}+\tfrac{1}{2})\,, \tag{14}$$

and so forth, and likewise for $\hat{P}^n$. An immediate consequence is the analog of Heisenberg's uncertainty relation for the operational spreads, viz [13]

$$\delta q\,\delta p\geqslant\bar{n}+1\,, \tag{15}$$

where the equal sign holds only if $\Delta\hat{Q}=\Delta\hat{P}=1/\sqrt{2}$. Owing to the noise of the measuring device, the lower limit in (15) is at least twice as large as that for the product of the intrinsic uncertainties, $\Delta\hat{Q}\,\Delta\hat{P}\geqslant\frac{1}{2}$.

As an illustration of the second case we now turn to the operational phase difference of two monochromatic electromagnetic waves determined by measuring its sine and cosine simultaneously in a fittingly designed interferometer. Such a device has been used in the recent NFM experiments [12] for a measurement of the quantum phase properties of a low-intensity laser, relative to a high intensity classical field (local oscillator). The experimental data are summarized in the so-called "phase distribution," which is nothing but the propensity density $\Pr(\varphi)$ for the classical phase variable φ that NFM associate operationally with the phase properties of the probe field.

By construction, this propensity is periodic, $\Pr(\varphi)=\Pr(\varphi+2\pi)$, and we normalize it such that

$$\int_{(2\pi)}d\varphi\,\Pr(\varphi)=1 \tag{16}$$

holds, where the integration covers any φ interval of length 2π. The classical average of a periodic function $g(\varphi)=g(\varphi+2\pi)$ is then given by

$$\overline{g(\varphi)}=\int_{(2\pi)}d\varphi\,g(\varphi)\Pr(\varphi)\,. \tag{17}$$

This number equals the quantum expectation value $\langle\hat{G}_{\mathscr{F}}\rangle$ of the corresponding operational operator $\hat{G}_{\mathscr{F}}(\hat{b}^\dagger,\hat{b})$, which is a function of $\hat{b}^\dagger$ and $\hat{b}$, the creation and annihilation operators for photons in the probe field. It is obvious that any $\hat{G}_{\mathscr{F}}$ of this kind is an OQO of the NFM experiment with the filter $\mathscr{F}$ denoting the homodyne detection scheme used.

In the terminology of Ref. [14], these OQOs are operators of the phase — *phasors*. In analogy to (5), the phasor basis $\hat{E}^{(n)}_{\mathscr{F}}$ is thus identified by the defining property

$$\overline{\exp(in\varphi)}=\langle\hat{E}^{(n)}_{\mathscr{F}}\rangle \tag{18}$$

for $n=0,\pm1,\pm2,\ldots$. The reality of the propensity density $\Pr(\varphi)$ implies that $\hat{E}^{(-n)}_{\mathscr{F}}$ is the adjoint of $\hat{E}^{(n)}_{\mathscr{F}}$, and $\hat{E}^{(0)}_{\mathscr{F}}=1$ is an immediate consequence of the normalization (16).

The members of the phasor basis are the basic OQOs because all other ones are weighted sums of these fundamental OQOs. Indeed, a Fourier decomposition,

$$\hat{G}_{\mathscr{F}}=\sum_{n=-\infty}^{\infty}\hat{E}^{(n)}_{\mathscr{F}}\int_{(2\pi)}\frac{d\varphi}{2\pi}\exp(-in\varphi)g(\varphi)\,, \tag{19}$$

establishes the quantum counterpart $\hat{G}_{\mathscr{F}}$ to any periodic function $g(\varphi)$. This relation enables one to map classical trigonometry onto the corresponding quantum trigonometry associated with the NFM experiment. As an example we have for the cosine and the (cosine)^2 functions these operational definitions:

$$\hat{C}^{(1)}_{\mathscr{F}}=\tfrac{1}{2}(\hat{E}^{(1)}_{\mathscr{F}}+\hat{E}^{(-1)}_{\mathscr{F}})\,,$$
$$\hat{C}^{(2)}_{\mathscr{F}}=\tfrac{1}{4}(\hat{E}^{(2)}_{\mathscr{F}}+2\hat{E}^{(0)}_{\mathscr{F}}+\hat{E}^{(-2)}_{\mathscr{F}})\,. \tag{20}$$

In fact, using relation (19) one can infer the entire quantum trigonometry from the operational phasors. Note that, due to the operational character of these cosine operators, they differ considerably from the Susskind-Glogower operators [15], which are intrinsic in character.

The NFM experiment has been analyzed in two different, and largely independent, ways. One analysis [16,17] found that the propensity density $\Pr(\varphi)$ is given by

$$\Pr(\varphi)=\frac{1}{2\pi}\int_0^\infty dI\,\langle\beta|\hat{\rho}|\beta\rangle\,, \tag{21}$$

where $\hat{\rho}$ is the density operator of the photon state of the probe field and $|\beta\rangle$ is a normalized eigenstate of $\hat{b}$. Here, $\beta=\sqrt{I}\exp(i\varphi)$ relates the eigenvalue β to the phase variable φ and the intensity I. In the jargon of quantum optics [18], $\Pr(\varphi)$ is the radially integrated Q function of $\hat{\rho}$, and $|\beta\rangle$ is a coherent state or Glauber state.

The other analysis [19] has identified the NFM phasors in normally ordered form, compactly presented as [20]

$$\hat{E}^{(n)}_{\mathscr{F}}=\frac{(n/2)!}{n!}:M(n/2,n+1,-\hat{b}^\dagger\hat{b}):\hat{b}^n \tag{22}$$

for $n=0,1,2,\ldots$, where M denotes the confluent hypergeometric function, and the pair of colons indicates normal ordering of the operators $\hat{b}^\dagger$ and $\hat{b}$; that is, all $\hat{b}^\dagger$'s to the left of all $\hat{b}$'s. The connection between (21) and (22) is implicitly contained in a 1974 paper by Paul [21]. A particularly convenient form of the basic phasors is [20,21]

$$\hat{E}^{(n)}_{\mathscr{F}}=\frac{(\hat{b}^\dagger\hat{b}+n/2)!}{(\hat{b}^\dagger\hat{b}+n)!}\hat{b}^n \quad\text{for } n=0,1,2,\ldots\,; \tag{23}$$

it is perhaps best suited for the construction of the OQOs associated with a classical observable $g(\varphi)$.

The general procedure for finding the relations between the operationally defined phasors and the intrinsic phase operator $\hat{\Phi}$ is not applicable to the NFM experiment, simply because $\hat{\Phi}$ is unknown. It can even be argued [14] that a unique phase operator does not exist at all. There is a plethora of acceptable definitions that are all equally good on general grounds. Nevertheless, the NFM experiment can be analyzed, of course, and the phasor basis (23) has been identified as the OQOs.

From this basis one can construct an operational phase operator $\hat{\Phi}_{\mathscr{F}}$. We use relation (19) to calculate the weight factors of the phasors; these are just the Fourier components of a periodic function that is equal to the classical phase variable φ in an interval $\varphi_0<\varphi<\varphi_0+2\pi$ [14]. The result is

$$\hat{\Phi}_{\mathscr{F}}=(\varphi_0+\pi)\hat{E}^{(0)}_{\mathscr{F}}+\sum_{n=1}^{\infty}\frac{i}{n}\left(e^{-in\varphi_0}\hat{E}^{(n)}_{\mathscr{F}}-e^{in\varphi_0}\hat{E}^{(-n)}_{\mathscr{F}}\right). \tag{24}$$

It is that Hermitian phase operator which is most naturally associated with the NFM phase propensity, inasmuch as

$$\langle\hat{\Phi}_{\mathscr{F}}\rangle=\int_{\varphi_0}^{\varphi_0+2\pi}d\varphi\,\varphi\Pr(\varphi) \tag{25}$$

equates the quantum expectation value of $\hat{\Phi}_{\mathscr{F}}$ to the classical average of the phase variable φ. The specific choice made for the value of the constant φ_0 is without physical significance, of course, so that the NFM experiment does not lead to one single phase operator but rather to a family of closely related operators labeled by the classical parameter φ_0. The spectrum of $\hat{\Phi}_{\mathscr{F}}$ consists of all φ values in said range. The eigenstates of $\hat{\Phi}_{\mathscr{F}}$, however, are unknown as yet and remain to be found.

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[1] *Quantum Theory of Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, 1983).
[2] See, for example, P. Busch, P. J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement* (Springer-Verlag, Berlin, 1991).
[3] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1932).
[4] E. Arthurs and J. Kelly, Bell Syst. Tech. J. **44**, 725 (1965).
[5] W. E. Lamb, Jr., Phys. Today **22** (4), 23 (1969); Ann. N.Y. Acad. Sci. **480**, 407 (1986).
[6] P. Storey, M. Collett, and D. Walls, Phys. Rev. Lett. **68**, 472 (1992).
[7] M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A **41**, 2295 (1990); V. Frerichs and A. Schenzle, *ibid.* **44**, 1962 (1991).
[8] K. Wódkiewicz, Phys. Rev. Lett. **52**, 1064 (1984); Phys. Lett. A **115**, 304 (1986).
[9] W. H. Zurek, Phys. Today **44** (10), 36 (1991).
[10] About *propensity* or *potentiality* or *tendency*, see, for example, K. R. Popper, *Quantum Theory and the Schism in Physics* (Hutchinson, London, 1982); and W. Heisenberg, *Physics and Philosophy* (Harper & Row, New York, 1958).
[11] We use the term "classical limit" with the clear technical meaning of Ref. [14].
[12] J. W. Noh, A. Fougères, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991); Phys. Rev. A **45**, 424 (1992); **46**, 2840 (1992); Phys. Rev. Lett. **71**, 2579 (1993).
[13] K. Wódkiewicz, Phys. Lett. A **124**, 207 (1987).
[14] J. Bergou and B.-G. Englert, Ann. Phys. (NY) **209**, 479 (1991).
[15] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[16] M. Freyberger and W. Schleich, Phys. Rev. A **47**, R30 (1993).
[17] U. Leonhardt and H. Paul, Phys. Rev. A **47**, R2460 (1993).
[18] See, for example, D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).
[19] P. Riegler and K. Wódkiewicz, Phys. Rev. A **49**, 1387 (1994).
[20] B.-G. Englert, K. Wódkiewicz, and P. Riegler (unpublished).
[21] H. Paul, Fortschr. Phys. **22**, 657 (1974).