Bell's inequality for an entanglement of nonorthogonal states

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Given an entanglement of two systems involving nonorthogonal states, we find the Schmidt decomposition for the state. The relation between the Schmidt representation and an ideal measurement of the degree of entanglement of the states is discussed, and a Bell inequality is shown to be violated. The maximal violation of the Bell inequality provides a measurement of the degree of entanglement. The entangled coherent states are provided as a concrete example of the Bell inequality for entangled nonorthogonal states.

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A state $|\Psi\rangle$ is entangled provided that there exists the Schmidt decomposition for the state

$$|\Psi\rangle = \sum_i a_i |f_i\rangle^A |g_i\rangle^B, \tag{1}$$

where A and B refer to different systems

$$^A\langle f_i|f_j\rangle^A = \delta_{ij} = ^B\langle g_i|g_j\rangle^B \tag{2}$$

and $a_i \neq 0$ for at least two distinct values of i. Such a state cannot be written in any representation as a product state and, therefore, violates some Bell inequality [1]. A standard example of entanglement is given by [2–4]

$$|\Psi\rangle = \mu|\alpha\rangle^A|\beta\rangle^B + \nu|\gamma\rangle^A|\delta\rangle^B, \tag{3}$$

where $|\alpha\rangle^A$ and $|\gamma\rangle^A$ are orthogonal states of system 1 and similarly for $|\beta\rangle^B$ and $|\delta\rangle^B$ for system 2. However, entanglements of nonorthogonal states are important as well, particularly in the context of entangled coherent states [5,6]. An entanglement involving nonorthogonal states, which is expressed in the form (3), would have the property that the overlaps $^A\langle\gamma|\alpha\rangle^A$ and $^B\langle\beta|\delta\rangle^B$ are nonzero. The entanglement becomes apparent by writing the state in the Schmidt form, and here we introduce an explicit algorithm for obtaining the Schmidt form for entangled nonorthogonal states in general and for an example of an entangled coherent state in particular.

Suppose that the overlaps $^A\langle\alpha|\gamma\rangle^A$ and $^B\langle\delta|\beta\rangle^B$ are not zero. However, the two nonorthogonal states $|\alpha\rangle^A$ and $|\gamma\rangle^A$ are assumed to be linearly independent and span a two-dimensional subspace of the Hilbert space. Thus two orthonormal Hilbert space vectors, which are representable as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4}$$

can be introduced that span the same subspace and a similar procedure follows for system B. These states can be chosen such that

$$|\alpha\rangle^A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^A, \qquad |\gamma\rangle^A = \begin{pmatrix} \mathcal{N}^A \\ \langle\alpha|\gamma\rangle \end{pmatrix}^A,$$
$$|\delta\rangle^B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^B, \qquad |\beta\rangle^B = \begin{pmatrix} \mathcal{N}^B \\ \langle\delta|\beta\rangle \end{pmatrix}^B, \tag{5}$$

for

$$\mathcal{N}^A = \sqrt{1 - |^A\langle\alpha|\gamma\rangle^A|^2}\,, \qquad \mathcal{N}^B = \sqrt{1 - |^B\langle\delta|\beta\rangle^B|^2}, \tag{6}$$

in the new basis (4). In this basis the state (3) can be written as

$$\psi^{AB} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}^A \otimes \begin{pmatrix} \mathcal{N}^B \\ \langle\delta|\beta\rangle \end{pmatrix}^B + \nu \begin{pmatrix} \mathcal{N}^A \\ \langle\alpha|\gamma\rangle \end{pmatrix}^A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^B = \begin{pmatrix} 0 \\ \nu\mathcal{N}^A \\ \mu\mathcal{N}^B \\ \mathcal{M} \end{pmatrix}, \tag{7}$$

where

$$\mathcal{M} \equiv \mu\langle\delta|\beta\rangle + \nu\langle\alpha|\gamma\rangle \tag{8}$$

and the superscripts A and B are ignored for cases where ambiguity does not arise. Also normalization of Eq. (7) requires that

$$\left|\mu\mathcal{N}^B\right|^2 + \left|\nu\mathcal{N}^A\right|^2 + |\mathcal{M}|^2 = 1. \tag{9}$$

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The state (7) is a pure state whose density matrix is $\rho^{AB} = \psi^{AB}\psi^{AB\dagger}$. The reduced density matrices for each system A and B are

$$\rho^A = \mathrm{Tr}_B\rho^{AB} = \begin{pmatrix} |\nu\mathcal{N}^A|^2 & \nu\mathcal{N}^A\mathcal{M}^* \\ \nu^*\mathcal{N}^A\mathcal{M} & |\mu\mathcal{N}^B|^2 + |\mathcal{M}|^2 \end{pmatrix},$$
$$\rho^B = \mathrm{Tr}_A\rho^{AB} = \begin{pmatrix} |\mu\mathcal{N}^B|^2 & \mu\mathcal{N}^B\mathcal{M}^* \\ \mu^*\mathcal{N}^B\mathcal{M} & |\nu\mathcal{N}^A|^2 + |\mathcal{M}|^2 \end{pmatrix}, \quad (10)$$

respectively, and the determinants

$$\det\rho^A = |\mu\nu\mathcal{N}^A\mathcal{N}^B|^2 = \det\rho^B \quad (11)$$

are equal. The two eigenvalues of ρ^A, given by

$$\lambda_\pm = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4\det\rho^A} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4|\mu\nu\mathcal{N}^A\mathcal{N}^B|^2}, \quad (12)$$

are identical to the two eigenvalues of ρ^B. The corresponding eigenvectors of ρ^A are $|\pm\rangle^A$ and the corresponding eigenvectors of ρ^B are designated as $|\pm\rangle^B$. The general theory of the Schmidt decomposition [7,8] implies that the state (7) can be expressed in the Schmidt form

$$|\psi\rangle^{AB} = c_-|-\rangle^A|-\rangle^B + c_+|+\rangle^A|+\rangle^B, \quad (13)$$

with $|c_\pm|^2 = \lambda_\pm, |c_+|^2 + |c_-|^2 = 1$.

The two-system entangled state (13) violates a Bell inequality. More specifically, we choose Hermitian operators $\hat\Theta$ for each system A, B such that the eigenvalues are ± 1. The general form for such an operator is

$$\hat\Theta = \cos\lambda\,[|+\rangle\langle+| - |-\rangle\langle-|] + \sin\lambda\left[e^{i\varphi}|+\rangle\langle-| + e^{-i\varphi}|-\rangle\langle+|\right]. \quad (14)$$

Each two-state system is being treated as a spin-1/2 system and the operator Θ then corresponds to the component of a "spin" operator along the axis determined by the angles λ and φ.

The Bell operator is defined as [9]

$$\hat{\mathcal{B}} = \hat\Theta^A\hat\Theta^B + \hat\Theta^A\hat\Theta'^B + \hat\Theta'^A\hat\Theta^B - \hat\Theta'^A\hat\Theta'^B. \quad (15)$$

For the choices

$$\lambda^A = 0, \qquad \lambda'^A = \pi/2,$$
$$\lambda^B = -\lambda'^B = \cos^{-1}\left[1 + |2c_+c_-|^2\right]^{-1/2}, \quad (16)$$
$$\varphi^A + \varphi^B = \varphi'^A + \varphi'^B = \varphi_+ - \varphi_-,$$

where $\varphi_\pm$ are the phases of $c_\pm$, the expectation value of the Bell operator for the state (13) is

$$\mathcal{B} \equiv \langle\Psi|\hat{\mathcal{B}}|\Psi\rangle = 2\sqrt{1 + |2c_+c_-|^2} > 2; \quad (17)$$

the degree of violation depends on the values of $c_\pm$, but a violation always occurs.

For the case that (3) is an entangled coherent state, the expectation value of the Bell operator can be determined. The example of the entangled coherent state [5]

$$|\alpha;\beta\rangle = 2^{-1/2}\left[|\alpha\rangle^A|\beta\rangle^B + i|i\beta\rangle^A|-i\alpha\rangle^B\right] \quad (18)$$

is used where $\{|\alpha\rangle, |\beta\rangle\}$ are coherent states. For simplicity $\bar n = |\alpha|^2$ is introduced and we set $\beta = 0$. The eigenvalues of the reduced density matrices ρ^A and ρ^B are

$$\lambda_\pm = \tfrac{1}{2} \pm \tfrac{1}{2}e^{-\bar n/2}\sqrt{2 - e^{-\bar n}}. \quad (19)$$

The Schmidt basis states can be expressed as

$$|\pm\rangle^A = \frac{e^{i\pi/4}}{\sqrt{2 - e^{-\bar n} \pm e^{-\bar n/2}\sqrt{2 - e^{-\bar n}}}}\Big[|0\rangle^A - \left(e^{-\bar n/2} - 2^{-1/2}e^{-i\pi/4}\left[e^{-\bar n/2} \pm \sqrt{2 - e^{-\bar n}}\right]\right) \times|\alpha\rangle^A\Big],$$
$$|\pm\rangle^B = \frac{e^{-i\pi/4}}{\sqrt{2 - e^{-\bar n} \pm e^{-\bar n/2}\sqrt{2 - e^{-\bar n}}}}\Big[|0\rangle^B - \left(e^{-\bar n/2} - 2^{-1/2}e^{i\pi/4}\left[e^{-\bar n/2} \pm \sqrt{2 - e^{-\bar n}}\right]\right) \times|-i\alpha\rangle^B\Big] \quad (20)$$

and the coefficients $c_\pm$ can be determined. The coefficients depend explicitly on $\bar n$ and this dependence indicates that the appropriate measurement for observing a violation of a Bell inequality itself depends on $\bar n$.

The expectation value of the Bell operator for the entangled coherent state $|\alpha;0\rangle$ is given by

$$\mathcal{B} = 2\sqrt{1 + (1 - e^{-\bar n})^2}, \quad (21)$$

which is strictly greater than 2 for all $\bar n > 0$. [At $\bar n = 0$, entanglement no longer holds as the state (18) can be expressed as a product state and we see that $\mathcal{B} \to 2$ as $\bar n \to 0+$.] The value of $\mathcal{B}$ increases monotonically with increasing $\bar n$ and asymptotically approaches $2\sqrt{2}$ very rapidly as the overlap between $|\alpha\rangle$ and $|0\rangle$ becomes negligible. In contrast to the use of the quadrature phase Bell inequality [10] for the entangled coherent state, which is violated only for sufficiently large overlap of $|\alpha\rangle$ and $|0\rangle$ [5], the ideal Bell inequality treated here produces an increased violation as the overlap is decreased. The quadrature phase Bell inequality becomes less useful as the photon number becomes more macroscopic whereas the ideal Bell inequality provides a good measure of the degree of entanglement.

In summary we have determined the Schmidt decomposition for the entangled nonorthogonal state (3) and this representation corresponds to an abstract spin-1/2 system for each of the systems A and B. The violation of Bell's inequality then follows by using the standard analysis [1,9] and we show in particular that Bell's inequality is violated in principle for entangled coherent states. Other entangled nonorthogonal states can be considered using this technique. For example, a Schrödinger cat state [11] which has been developed for SU(2) coherent states [12] could be extended to an entanglement of nonorthogonal SU(2) coherent states and the analysis above applies to this case as well.

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