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Multiparticle correlations in quaternionic quantum systems

S. P. Brumby* and G. C. Joshi†
School of Physics, University of Melbourne, Research Centre for High Energy Physics, Parkville, Victoria 3052, Australia

R. Anderson‡
Dibner Institute for the History of Science and Technology, 38 Memorial Drive, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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We investigate the outcome of measurements on correlated, few-body quantum systems described by a quaternionic quantum mechanics that allows for regions of quaternionic curvature. We find that a multiparticle interferometry experiment using a correlated system of four nonrelativistic, spin-half particles has the potential to detect the presence of quaternionic curvature. Two-body systems, however, are shown to give predictions identical to those of standard quantum mechanics when relative angles are used in the construction of the operators corresponding to measurements of particle spin components.

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I. INTRODUCTION

Quaternionic quantum mechanics (QQM) was raised as a possibility by Birkhoff and von Neumann [1] in 1936, and has been elaborated since then by a variety of authors using different theoretical approaches [2–10]. In particular, Finkelstein *et al.* [2] have presented a generalization of standard complex quantum mechanics (CQM) which uses the mathematics of QQM to introduce space-time dependence into the quantum dynamics. They do this by choosing to replace the imaginary unit i of standard CQM with what in effect is a nonintegrable, almost complex structure on the space-time manifold [11], which is representable by the introduction of a field of pure imaginary unit quaternions throughout space-time. In order to relate the complex algebras from point to point, use is made of concepts from differential geometry; gauge connections, covariant derivatives, and curvature have their quaternionic analogues in Q connections, Q covariant derivatives, and Q curvature. In the following, we refer to this general theory as GQQM.

To date, however, the absence of clear experimental evidence has meant that researchers have constructed models with a concern to ensure that the quaternionic aspects are hidden in situations where CQM is successful. Fully quaternionic interactions are permitted, but the higher quaternionic components of the wave functions in these models exponentially decay in the absence of quaternionic-dissipative potentials [7]. Further, it is customary to work in what Finkelstein *et al.* [2] have classified as the Q-flat limit of their theory, thereby neglecting much of the geometrical content of the full theory.

Our contention is that within GQQM the quaternionic nature of the states can manifest itself in collective, nonlocal effects. In particular, there emerges a possibility of testing the prediction of GQQM that the i of CQM becomes a field of quaternions on space-time, by the simultaneous measurement at remote points of the intrinsic spin of particles in entangled states.

To treat many-body systems in their nonmutually interacting state (i.e., to construct Fock spaces of particles), we use the tensor product of quaternionic Hilbert modules developed by Razon and Horwitz [10]. This permits the definition of a scalar product of quaternionic multiparticle wave functions, by abandoning linearity in each factor of the tensor product in favor of a quotient-group structure (we present a summary in the Appendix).

In the next section, we examine the predictions of GQQM for the archetypal experimental test (Bohm) [12] of the Einstein-Podolsky-Rosen (EPR) program [13]. This was also the situation Bell [14] used in his famous demonstration that CQM could not be completed in accord with the EPR paper's assumptions. This case raises a number of foundational questions about how physical information is encoded in the mathematical operators chosen to represent particular measurements.

We then consider a similar experiment on a correlated four-electron state proposed by Greenberger *et al.* [15]. It is in this situation that we expect manifestations of quaternionic behavior.

Given the enormous importance of the results engendered by Bell's work for exploring the foundations of CQM, we view the results of our paper as a preliminary contribution to locating their significance for exploring the foundations of QQM.

*Electronic address: spb@tauon.ph.unimelb.EDU.AU
†Electronic address: joshi@tauon.ph.unimelb.EDU.AU
‡Electronic address: anderso@bcuxs2.bc.edu
Permanent address: Department of Philosophy, Boston College, Chestnut Hill, MA 02167.

II. TWO BODY CORRELATIONS

Bohm's *gedankenexperiment* consists of simultaneous spin component measurements on a system of two spin-half particles prepared in an entangled state

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}\big(|+\rangle_1 \otimes |-\rangle_2 - \;|-\rangle_1 \otimes |+\rangle_2\big). \tag{1}$$

The preparation procedure, which is visualized as the decay out of a two-body S state, results in a state which is rotationally invariant. That is, the complete entangled state is invariant under the group of rotations $\{g\otimes g \mid g \in \mathrm{SU}(2)\}$, where SU(2) is the covering group of O_3. This is equivalent to stating that the measurement of internal spins of both particles along the same spatial axis, is independent of the orientation of that axis. (Note that this use of the quaternions, related to the spin-half nature of fundamental fermions, is located wholly within the domain of CQM.) It is this invariance which essentially means we cannot ascribe any physical reality to definite spin components prior to their measurement.

The entangled character of the state is also invariant under this group, which implies that under linear, continuous evolution in free space, the state remains entangled and does not collapse into a trivial product of wave functions at distinct points. In addition, this invariance means that when we measure spin components along different spatial directions, it is only the relative angular displacements of the analyzing directions that are physically significant.

In CQM, this state can be multiplied by an additional arbitrary phase factor which does not change the physical system (i.e., we actually deal with an equivalence class of states, of which the above is merely representative, the class relation being multiplication by a phase factor). This enables us to choose the initial state Eq. (1) to be real, and in particular, to choose each factor to be real.

Now at each point of space-time GQQM singles out a complex slice of the operator algebra on the quaternionic Hilbert space $\mathcal{H}_{\boldsymbol{H}}$. All of the local physical observables are restricted to this slice, and so the theory specifies a complete and distinct CQM at each point of space-time, related to each other through the Q connection. Hence, GQQM retains the property of local phase invariance of states. In this case, the choice of a real initial state corresponds to nulling the phase factors associated with each single particle wave function separately.

The O_3 invariance of the quaternionic system, in the sense of spin measurements along the same spatial axis being independent of the orientation of that axis, is assured by the local equivalence between CQM and GQQM. Hence the rotational invariance of the state still holds within GQQM with the theory therefore still implying that individual particle spins fail to have any physical reality prior to their experimental resolution.

The Hermitian operator corresponding to the measurement, at position $\mathbf{x}$, of internal spin of a single particle, along a spatial unit vector

$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{2}$$

where the angles are defined with respect to some laboratory frame of reference, is

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta e^{-\phi\eta} \\ \sin\theta e^{\phi\eta} & -\cos\theta \end{pmatrix}. \tag{3}$$

Here, η is the quaternionic generalization of the imaginary unit of CQM which, expanded with respect to a fixed basis of imaginary units, is

$$\eta = \eta(\mathbf{x}) = \sum_{r=1}^{3} h_r(\mathbf{x}) i_r, \tag{4}$$

where $\forall\ \mathbf{x},\ \mathbf{h}(\mathbf{x}) \in \boldsymbol{R}^3,\ \|\mathbf{h}\| = 1$. That is, GQQM is a theory in which the observables have an additional space dependence due to the phenomenon of Q curvature. We now explore the significance of this feature for experiments on this two-body correlated system (we examine a four-body system in the following section).

(We remark in passing that we only deal with a static Q curvature; the processes by which the Q curvature is generated and evolves have still to be conjectured. Nevertheless, we expect that a mathematical exposition of this problem will involve quaternionic gauge fields, the fundamental theory of which has been investigated by Adler [8].)

The expectation value of the product of a component of each particles' spin is given by

$$E_\Psi(\mathbf{n},\mathbf{n}') = \langle\Psi|\ \mathbf{n}\cdot\boldsymbol{\sigma}_{(1)} \otimes \mathbf{n}'\cdot\boldsymbol{\sigma}_{(2)}|\Psi\rangle, \tag{5}$$

which is equivalent to calculating

$$E_\Psi(\mathbf{n},\mathbf{n}') \equiv \tfrac{1}{2}(P_{++} + P_{--} - P_{+-} - P_{-+}). \tag{6}$$

Here, the subscripts of P match the labels of the left most elements in the tensor products of bras and kets, e.g.,

$$P_{++} = \langle +| \otimes \langle -|\ (\mathbf{n}\cdot\boldsymbol{\sigma}_{(1)} \otimes \mathbf{n}'\cdot\boldsymbol{\sigma}_{(2)})\ |+\rangle \otimes |-\rangle, \tag{7}$$

and we retain the usual definition of tensor products of operators action on tensor products of states. That is, $\forall|\psi\rangle,\ |\phi\rangle \in \mathcal{H}_{\boldsymbol{H}}$, the Hilbert space of quaternionic single particle states, and $\forall A,\ B \in \boldsymbol{B}(\mathcal{H}_{\boldsymbol{H}})$, the algebra of linear operators on $\mathcal{H}_{\boldsymbol{H}}$, we define

$$(A\otimes B)\ |\psi\rangle \otimes |\phi\rangle = (A|\psi\rangle) \otimes (B|\phi\rangle). \tag{8}$$

With the detectors at $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, without loss of generality we set $\eta(\mathbf{x}_1) = i_1$ and drop the position label from $\eta(\mathbf{x}_2)$.

$$P_{+-} = \sum_{(\rho)\in\mathcal{I}^2} \{[(\langle +| \otimes \langle -|\,]\,[(\sin\theta e^{\phi i_1}|+\rangle)_{\rho_1} \otimes (\sin\theta' e^{-\phi'\eta}|-\rangle)_{\rho_2}]\}(1\otimes 1\otimes 1\cdot 1_{D(\boldsymbol{H})}, i_{\rho_1}\otimes i_{\rho_2}\otimes 1\cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}}. \tag{9}$$

For the reasons given previously, we consider the basis spin states $|\pm\rangle$ to be real. Then we can identify them with their formally real 0th components (in the sense of [10])

$$(|\pm\rangle)_\sigma = \tfrac{1}{4}\sum_{\varpi\in\mathcal{I}} i_\varpi(|\pm\rangle i_\sigma^*)i_\varpi^* = |\pm\rangle\delta_{\sigma 0}. \tag{10}$$

Hence

$$P_{+-} = \sin\theta\sin\theta'\sum_{\rho\in\mathcal{I}}(e^{-\phi'\eta})_\rho \times[\cos\phi\,(1_{D(\boldsymbol{H})},\,1\otimes i_\rho\otimes 1\cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}} + \sin\phi\,(1_{D(\boldsymbol{H})},\,i_1\otimes i_\rho\otimes 1\cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}}]. \tag{11}$$

Applying the recursive definition to the inner product of tensor producted quaternionic algebras (given in the Appendix), we have

$$(1_{D(\boldsymbol{H})}, 1\otimes i_\rho\otimes 1\cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}} = \tfrac{1}{3}i_\rho + \tfrac{2}{3}\delta_{\rho 0}, \tag{12}$$

$$(1_{D(\boldsymbol{H})}, i_1\otimes i_\rho\otimes 1\cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}} = \tfrac{1}{3}\delta_{\rho 0}i_1 - \tfrac{1}{3}\delta_{\rho 1}. \tag{13}$$

Hence

$$P_{+-} = \sin\theta\sin\theta'[\cos\phi\cos\phi' + \tfrac{1}{3}\sin\phi\sin\phi' h_1 + \tfrac{1}{3}\sin\phi\cos\phi' i_1 - \tfrac{1}{3}\cos\phi\sin\phi'\eta].$$

Similarly,

$$\begin{aligned} P_{-+} &= \langle -|\otimes\langle +|\,(\mathbf{n}\cdot\boldsymbol{\sigma}_{(1)}\otimes\mathbf{n}'\cdot\boldsymbol{\sigma}_{(2)})\,|+\rangle\otimes|-\rangle, \\ &= \sin\theta\sin\theta'(\langle -|\otimes\langle +|,\, e^{-\phi i_1}|+\rangle\otimes e^{\phi'\eta}|-\rangle)_{\boldsymbol{H}} \\ &= P_{+-}(\phi\to -\phi\,;\,\phi'\to -\phi'), \end{aligned}$$

so that in combination, we obtain the real result

$$P_{+-} + P_{-+} = 2\sin\theta\sin\theta'(\cos\phi\cos\phi' + \tfrac{1}{3}\sin\phi\sin\phi' h_1). \tag{14}$$

Now

$$P_{++} = \langle +|\otimes\langle -|\,(\mathbf{n}\cdot\boldsymbol{\sigma}_{(1)}\otimes\mathbf{n}'\cdot\boldsymbol{\sigma}_{(2)})\,|+\rangle\otimes|-\rangle, \tag{15}$$

$$P_{--} = P_{++} = -\cos\theta\cos\theta'. \tag{16}$$

Therefore, we have the (real) expectation value,

$$E_\Psi(\mathbf{n},\mathbf{n}') = -\cos\theta\cos\theta' - \sin\theta\sin\theta'(\cos\phi\cos\phi' + \tfrac{1}{3}h_1\sin\phi\sin\phi'). \tag{17}$$

The final term in this sum represents the manifestation of QQM in this nonrelativistic, correlated quantum system. It is a specialization of the Euclidean inner product $\mathbf{h}(\mathbf{x}_1)\cdot\mathbf{h}(\mathbf{x}_2)$, arising from the natural inner product structure imposed upon the quaternionic algebra.

As stated before, rotational invariance of the initial entangled state Eq. (1) requires the use of relative angles for the phases ϕ, ϕ', i.e., $\phi\to\phi_{\rm rel} = \phi-\phi'$, $\phi' = 0$, so $\mathbf{n}' = (\sin\theta',\, 0,\, \cos\theta')$ and

$$\mathbf{n}\cdot\mathbf{n}' = \cos\phi_{\rm rel}\sin\theta\sin\theta' + \cos\theta\cos\theta'.$$

Hence, we retrieve the CQM result,

$$E_\Psi(\mathbf{n},\mathbf{n}') = -\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\phi_{\rm rel} = -\mathbf{n}\cdot\mathbf{n}'. \tag{18}$$

The fact that GQQM hides itself at this level of two-body systems is an interesting result, especially as it tends to imply that GQQM might not manifest itself in models neglecting $\geq$ three-body interactions. This would suggest that the present lack of experimental evidence for GQQM may be due to the subtlety of the theory.

III. FOUR-BODY CORRELATIONS

We now consider a four-particle correlated system described by Greenberger *et al.* [15].

$$|\Psi_{\rm GHSZ}\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle_1\otimes|+\rangle_2\otimes|-\rangle_3\otimes|-\rangle_4 - |-\rangle_1\otimes|-\rangle_2\otimes|+\rangle_3\otimes|+\rangle_4). \tag{19}$$

We carry out simultaneous (or at least, spacelike separated) measurements of a component of spin of each particle, and consider the product of these values. The quantum mechanical expectation value is, thus,

$$E_{\rm GHSZ}(\{\mathbf{n}_{(j)}\}) = \langle\Psi_{\rm GHSZ}|\bigotimes_{j=1}^{4}\mathbf{n}_{(j)}\cdot\boldsymbol{\sigma}_{(j)}|\Psi_{\rm GHSZ}\rangle, \tag{20}$$

$$P_{+-} = \langle +|\otimes\langle +|\otimes\langle -|\otimes\langle -|\,\Big(\bigotimes_{j=1}^{4}\mathbf{n}_{(j)}\cdot\boldsymbol{\sigma}_{(j)}\Big)\,|-\rangle\otimes|-\rangle\otimes|+\rangle\otimes|+\rangle, \tag{21}$$

$$P_{+-} = \Big(\prod_{j=1}^{4}\sin\theta_j\Big)\sum_{(\rho)\in\mathcal{I}^4}(e^{\phi_1\eta(\mathbf{x}_1)})_{\rho_1}(e^{\phi_2\eta(\mathbf{x}_2)})_{\rho_2}(e^{-\phi_3\eta(\mathbf{x}_3)})_{\rho_3}(e^{-\phi_4\eta(\mathbf{x}_4)})_{\rho_4}(1_{D(\boldsymbol{H})},\,\mathrm{i}_{(\rho)}\otimes 1\cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}}, \tag{22}$$

where $\{\theta_j,\phi_j\}$ are the polar and azimuthal angles specifying the direction $\mathbf{n}_j$ along which a measurement is made on the jth particle.

We now simplify this expression by considering the special configuration $\forall j,\ \theta_j = \pi/2$ and without loss of generality take $\eta_3 = i_1$. Further, we use relative angles: $\forall j,\ \phi_j\to\phi_j-\phi_4$.

Hence,

$$P_{+-} = \sum_{(\rho)\in\mathcal{I}^2} (e^{\phi_1\eta(\mathbf{x}_1)})_{\rho_1} (e^{\phi_2\eta(\mathbf{x}_2)})_{\rho_2} \big[\cos\phi_3\,(1_{D(\mathbf{H})},\ \mathrm{i}_{(\rho)} \otimes 1 \otimes 1 \otimes 1 \cdot 1_{D(\mathbf{H})})_{\mathbf{H}} - \sin\phi_3\,(1_{D(\mathbf{H})},\ \mathrm{i}_{(\rho)} \otimes i_1 \otimes 1 \otimes 1 \cdot 1_{D(\mathbf{H})})_{\mathbf{H}}\big]. \tag{23}$$

Using the recursive definition for the inner product of quaternion algebras, we obtain from Eqs. (19)–(23),

$$\begin{aligned} E_{\mathrm{GHSZ}}(\{\mathbf{n}_{(j)}\}) &= -\tfrac{1}{2}(P_{+-} + P_{-+}) \\ &= -\cos\phi_1\cos\phi_2\cos\phi_3 - \tfrac{1}{3}h_1(\mathbf{x}_1)\sin\phi_1\cos\phi_2\sin\phi_3 - \tfrac{1}{3}h_1(\mathbf{x}_2)\cos\phi_1\sin\phi_2\sin\phi_3 \\ &\quad + \tfrac{1}{3}\mathbf{h}(\mathbf{x}_1)\cdot\mathbf{h}(\mathbf{x}_2)\sin\phi_1\sin\phi_2\cos\phi_3. \end{aligned} \tag{24}$$

Hence, we find that GQQM effects are manifest in experiments on this four-body correlated system. Moreover, we expect GQQM to give different predictions from CQM for all $N \geq 2$ body systems. (On this point, we mention that Razon and Horwitz [10] have suggested that as $N \to \infty$, with their definition of a multiparticle inner product, we recover the quantized field commutator relations of complex quantum field theory.)

The CQM expectation value is

$$E^0_{\mathrm{GHSZ}}(\{\mathbf{n}_{(j)}\}) = -\cos(\phi_1 + \phi_2 - \phi_3)\ . \tag{25}$$

Comparing this with Eq. (24), we see that the CQM limit is not reached by simply nulling the Q curvature, i.e.,

$$\mathbf{h}(\mathbf{x}) \cdot \mathbf{h}(\mathbf{x}') \to 1, \quad \forall\ \mathbf{x}, \mathbf{x}' \tag{26}$$

The recursive definition of the scalar product of Razon and Horwitz [10] has introduced numerical factors which depend on the number of subsystems, but the occurrence of $\mathbf{h}(\mathbf{x}) \cdot \mathbf{h}(\mathbf{x}')$ coefficients to the relative-orientation dependent terms in the expectation value might be characteristic of GQQM, whatever our choice of scalar product (the same terms arise in the simplest approaches to this calculation [16]). If QQM does extend CQM, then it must be hoped that there are definitions of tensor and inner products which reach CQM via some natural limiting process, such as nulling of the Q curvature in GQQM. Alternatively, it may be that QQM theories dealing with composite objects cannot support tensor products and we must look for alternative mathematical tools of description (e.g., composition of quaternionic Hilbert spaces could be viewed as a lattice theoretic problem [6]). For the moment, we are content to regard Eq. (24) as being a useful example of the type of prediction made by a class of almost complex extensions of quantum mechanics, in the sense that it demonstrates how QQM effects can be uncovered in correlated systems.

Note that if instead, we considered an experiment which made use of plane polarized photons, in the manner described by [15], then because photons possess a single quantum of angular momentum, the operator corresponding to a polarization filter has real components [17] and GQQM has no opportunity to manifest itself. This situation is changed by the use of circularly polarized photons, in which case the quantum mechanical prescription for calculating the expectation values necessarily uses complex numbers, giving the quaternionic entangled state a probability distribution sensitive to a changing η field (hence, not predicted by CQM).

IV. CONCLUSION

We see that it is a special result of particle number which hides GQQM in two-body situations. For more complicated systems, GQQM gives expectation values that differ formally from those of CQM. The importance of particle number in this situation is analogous to that pertaining to correlations in CQM. As Greenberger *et al.* [15] have shown, Bell type inequalities which permit local realistic hidden variable theories to agree with quantum mechanical predictions in certain regions of parameter space (though not in all regions) only occur for two-particle entangled states. For more complicated systems, however, CQM predicts correlations for entangled systems which are unable to be described by local hidden variable theories.

Our result in Eq. (24) indicates that experimental tests of four-particle correlations have either the potential to reveal quaternionic components or to set limits on their values. Our prediction of quaternionic terms in multiparticle correlation experiments provides a further motivation to the reasons given by Greenberger *et al.* to undertake such experiments.

If such an experimental test of GQQM were carried out and no quaternionic correction terms were discovered, and given that the universe we inhabit is observed to be complex to a high degree, then in some clear sense we would have evidence against what we would maintain is the most natural interpretation and implementation of the suggestion of Birkhoff and von Neumann that does not relegate QQM to presently inaccessible sectors of physics (e.g., physics at the subquark level [8]). In this case, until there occurs a situation where QQM provides an explanation analogous in significance to general relativity's explanation of the anomalous precession of mercury, or to the quark model's solution of the hadron classification puzzle, the theory will remain tentative.

A more modest perspective would be to sidestep for the moment questions as to whether or not QQM is the "real" theory of quantum phenomena, and simply to view QQM in a phenomenological sense as a theory whose extra parameters affords a way of better capturing experiment results that have defied analysis by other means. The results of the type we present in Sec. III indicate the manner in which this parametrization can occur. Naturally any success on this score invites a deeper consideration of the viability of QQM.

Depending on the outcomes of these possibilities, QQM may tell us something crucial about the necessity of both q numbers and c numbers in the correct description of physical phenomena as we know them. Quaternionic quantum mechanics exists as a potential alternative theory because of the correspondence we set up between experimentally observable quantities and Hermitian operators (which necessarily possess real eigenvalues). If experiments fail to support QQM, then we have found out something about the physical standing of non-Hermitian operators and unobservable properties of a system.

We await experimental clarification of these concerns.

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APPENDIX: TENSOR PRODUCT OF QUATERNION ALGEBRAS

The algebra of quaternions $\boldsymbol{H}$ can be considered to be the real linear span of a set of four abstract basis elements $1, i_1, i_2, i_3$ with the multiplication rule $\forall r, s \in \{1, 2, 3\}$,

$$1 i_r = i_r 1 = i_r, \quad i_r i_s = -\delta_{rs} + \sum\nolimits_{t=1}^{3} \epsilon_{rst} i_t. \tag{A1}$$

We designate the set of pure imaginary quaternions by $\operatorname{Im} \boldsymbol{H} = \{\eta = \sum_{r=1}^{3} h_r i_r : \mathbf{h} \in \boldsymbol{R}^3, \ \|\mathbf{h}\| = 1\}$.

Let $\mathcal{I}$ be the index set $\{0, 1, 2, 3\}$, and define $(\rho) = (\rho_1, \ldots, \rho_N)$, which lies in the Cartesian N-product $\mathcal{I}^N$.

We denote the tensor product of N quaternionic algebras by $D = \boldsymbol{H} \otimes \cdots \otimes \boldsymbol{H}$, which has the product basis $\{\mathsf{i}_{(\rho)} = i_{\rho_1} \otimes \cdots \otimes i_{\rho_N}\}$ (where $i_0 \equiv 1$), and identity element $1_D = 1 \otimes \cdots \otimes 1$. Multiplication in D is defined in a pairwise sense,

$$\begin{aligned} \mathsf{qr} &= (q_1 \otimes \cdots \otimes q_N)(r_1 \otimes \cdots \otimes r_N) \\ &= (q_1 r_1 \otimes \cdots \otimes q_N r_N). \end{aligned} \tag{A2}$$

Let $[q]_j = 1 \otimes \cdots \otimes q \otimes \cdots \otimes 1 \in D$, with q in the jth place.

The requirement of linearity in each factor of the tensor product is now to be weakened [10]. To this end, we consider instead the quotient space $D(\boldsymbol{H}) = D/A_{\boldsymbol{H}}$, $A_{\boldsymbol{H}} = \bigcap_{\eta \in \operatorname{Im} \boldsymbol{H}} A_\eta$, where A_η is a left ideal of D generated by the set $\{[\eta]_1 - [\eta]_j \ : \ 2 \leq j \leq N\}$. The identity element of this space is $1_{D(\boldsymbol{H})} = 1_D + A_{\boldsymbol{H}}$.

$(\mathsf{q})_{\mathcal{Q}}$ is defined inductively for $\mathsf{q} \in D$ by

$$(q_1)_{\mathcal{Q}} = (q_1)_0 \equiv \operatorname{Re} q_1, \tag{A3}$$

and for $N > 1$,

$$\begin{aligned} (\mathsf{q})_{\mathcal{Q}} = \frac{1}{N+1} [& (q_1 q_N \otimes \cdots \otimes q_{N-1})_{\mathcal{Q}} \\ & + \cdots + (q_1 \otimes \cdots \otimes q_{N-1} q_N)_{\mathcal{Q}} \\ & + 2(q_N)_0 (q_1 \otimes \cdots \otimes q_{N-1})_{\mathcal{Q}}]. \end{aligned} \tag{A4}$$

With these conventions in place, we now define an inner product on the quaternionic quotient space,

$$(\mathsf{q}_1 \cdot 1_{D(\boldsymbol{H})}, \, \mathsf{q}_2 \cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}} = \sum_{j=0}^{3} (\mathsf{q}_1^* [i_j^*]_N \mathsf{q}_2)_{\mathcal{Q}} \, i_j. \tag{A5}$$

For each $\psi \in$ a Hilbert $\boldsymbol{H}$ module $\mathcal{H}_{\boldsymbol{H}}$, the formally real components of ψ are defined at each point $\mathbf{x}$ by

$$(\psi)_{\boldsymbol{H}} = \sum_{\sigma \in \mathcal{I}} [\psi \otimes i_\sigma^*(\mathbf{x})]_{\mathcal{Q}} i_\sigma(\mathbf{x}) \tag{A6}$$

$$= \sum_{\sigma \in \mathcal{I}} \psi_\sigma^{\mathbf{x}} i_\sigma(\mathbf{x}). \tag{A7}$$

We now choose to decompose each tetrad $i_\sigma(\mathbf{x})$ as a rotation from a standard basis i_ρ (the rotation being an element of the group $\mathrm{I}_1 \oplus \mathrm{O}_3$).

$$(\psi)_{\boldsymbol{H}} = \sum_{\rho \in \mathcal{I}} \Big(\sum_{\sigma \in \mathcal{I}} \psi_\sigma^{\mathbf{x}} \mathcal{O}_{\sigma\rho}(\mathbf{x}) \Big) i_\rho = \sum_{\rho \in \mathcal{I}} \psi_\rho i_\rho. \tag{A8}$$

We can now form the quaternion tensor product of N single particle systems, which is itself a left module over the quaternionic algebra:

$$\Psi q = \psi^1 \otimes \cdots \otimes \psi^N \otimes q \in \mathcal{H}_{\boldsymbol{H}}^N \otimes \boldsymbol{H}. \tag{A9}$$

As usual, for an interacting system, the states of the N-body interacting quantum system can be decomposed in the basis formed by the N-tensor product of a basis $\mathcal{B}(\mathcal{H}_{\boldsymbol{H}})$ for $\mathcal{H}_{\boldsymbol{H}}$.

Using the decomposition into formally real components of each single particle wave function,

$$\begin{aligned} \Psi &= \sum_{(\rho) \in \mathcal{I}^N} (\psi_{\rho_1}^1 i_{\rho_1}) \otimes \cdots \otimes (\psi_{\rho_N}^N i_{\rho_N}) \otimes 1 \cdot 1_{D(\boldsymbol{H})} \\ &= \sum_{(\rho) \in \mathcal{I}^N} (\psi_{\rho_1}^1 \otimes \cdots \otimes \psi_{\rho_N}^N) \otimes (\mathsf{i}_{(\rho)} \otimes 1 \cdot 1_{D(\boldsymbol{H})}), \end{aligned} \tag{A10}$$

where we have taken the formally real components and formed them into a correspondingly ordered N tuple of functions.

Thus, $\Psi \in [\mathcal{L}_{\boldsymbol{H}}^2(\boldsymbol{R}^d)]^N \otimes D^{(N+1)}(\boldsymbol{H})$, where $\mathcal{L}_{\boldsymbol{H}}^2(\boldsymbol{R}^d)$ is the set of quaternion valued, Lebesgue square-integrable functions on d-dimensional Euclidean space (for particles in a box, these functions are of finite support).

We can now define a scalar product of N-body wave functions:

$$\begin{aligned} \langle \Psi | \Phi \rangle_{\boldsymbol{H}} = \sum_{(\rho),(\sigma) \in \mathcal{I}^N} & \langle \psi_{\rho_1}^1 \otimes \cdots \otimes \psi_{\rho_N}^N | \phi_{\sigma_1}^1 \otimes \cdots \otimes \phi_{\sigma_N}^N \rangle \\ & \times (\mathsf{i}_{(\rho)} \otimes 1 \cdot 1_{D(\boldsymbol{H})}, \, \mathsf{i}_{(\sigma)} \otimes 1 \cdot 1_{D(\boldsymbol{H})})_{\boldsymbol{H}}. \end{aligned} \tag{A11}$$

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