Unitary transformations for the time-dependent quantum oscillator

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An approach to study the time-dependent quantum oscillator with the Schrödinger operator $\mathbf{S}(\Omega(t))\equiv\frac{1}{2}[\mathbf{p}^2+\Omega^2(t)\mathbf{x}^2]-i(\partial/\partial t)$ is presented. A family of unitary operators $\{\mathbf{W}_{lz}(t)\}$ is found such that $\mathbf{S}(\Omega(t))=\mathbf{W}_{1z}(t)\mathbf{S}(\omega)\mathbf{W}^{\dagger}_{1z}(t)$, where $z=z(t)$ is a certain function and ω is a positive number. Wave vectors, the boson creation and destruction operators, the evolution operator, and invariants for the time-dependent oscillator are obtained by means of $\{\mathbf{W}_{1z}(t)\}$. The approach is applied to calculate transition probabilities, Berry phases, and to study coherent states of the time-dependent oscillator.

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I. INTRODUCTION

Oscillator models are widely used in quantum optics and atomic, molecular, and solid-state physics. Here an approach to study the time-dependent harmonic oscillator is developed. A family of unitary operators $\{\mathbf{W}_{1z}(t)\}$ acting on an abstract Hilbert space $\mathcal{R}$ is found that transforms the Schrödinger operator

$$S_0(t)\equiv H_0-i\frac{\partial}{\partial t} \tag{1}$$

of a time-independent oscillator (the reference system), where

$$\mathbf{H}_0\equiv\tfrac{1}{2}(\mathbf{p}^2+\omega^2\mathbf{x}^2)\ , \tag{2}$$

$\mathbf{x}$ and $\mathbf{p}$ are the coordinate and momentum operators on $\mathcal{R}$, $\omega>0$, into the Schrödinger operator

$$\mathbf{S}_1(t)\equiv\mathbf{H}_1(t)-i\frac{\partial}{\partial t} \tag{3}$$

of the parametrically excited oscillator, where

$$\mathbf{H}_1(t)\equiv\tfrac{1}{2}[\mathbf{p}^2+\Omega^2(t)\mathbf{x}^2]\ , \tag{4}$$

$\Omega(t)\geq 0$, namely,

$$\mathbf{S}_1(t)=\mathbf{W}_{1z}(t)\mathbf{S}_0(t)\mathbf{W}^{\dagger}_{1z}(t)\ . \tag{5}$$

In this paper we use $\hbar=m=1$; the case $m=m(t)$ can be reduced to the case $m=1$ [1]. The family $\{\mathbf{W}_{1z}(t)\}$ depends upon a family of complex functions $\{z(t)\}$ [see (17) and (18) below]. The use of $\{\mathbf{W}_{1z}(t)\}$ provides a uniform treatment of a number of problems related to the parametrically excited harmonic oscillator. It gives insight into the problems previously discussed in the literature, such as finding a general form of solutions of the Schrödinger equation (Sec. V), finding a general form of the density operator (Sec. VI) and its P representation (Sec. VI), calculation of matrix elements of operators (Sec. V), calculation of transition probabilities for the oscillator subjected to perturbation finite in time (Sec. VIII), and properties of exact time-dependent invariants (Sec. II). It also provides a number of new results: a family of expressions for the evolution operator (Sec. IV); properties of a wider set of families of generalized coherent states of the time-dependent oscillator, including a description in terms of the expectation values $\langle\mathbf{x}\rangle$ and $\langle\mathbf{p}\rangle$, the resolution of unity, and the diagonal representation of operators (Sec. VI); the Floquet form of the evolution operator and Berry phases of cyclic states in the cases of evolution finite in time and evolution under a T-periodic parametric excitation (Sec. VII); a general expression for the amplitude of a transition from a number state into a quasienergy state under a periodic perturbation and an example of perturbation for which the amplitude of transition equals δ_{nm} (Sec. VIII).

The time-dependent oscillator has been studied in a number of papers. Explicit expressions for the Green's function and for the transition probabilities were obtained by Husimi [2]. A thorough study of the oscillator was undertaken by Popov and Perelomov [1,3,4]. They have obtained an explicit form of a general solution of the Schrödinger equation in the x representation, an analytical expression for coherent states, the P representation for the density operator, and an explicit form of the Wigner function for the case

$$\Omega(t)\xrightarrow[t\to-\infty]{}\omega_-\ . \tag{6}$$

Popov and Perelomov have also obtained the quasienergy states of the oscillator under a periodic perturbation and transition probabilities for the oscillator subjected to a perturbation finite in time. The importance of exact invariants for description of time-dependent quantum systems was pointed out by Lewis and Riesenfeld [5], who considered a certain class of exact quadratic invariants to obtain solutions of the Schrödinger equation of the oscillator. Linear invariants and the invariant displacement operator were used by Malkin, Man'ko and Trifonov [6] to obtain coherent states of the oscillator. An explicit form of the propagator was obtained by Khandekar and Lawande [7] and Farina de Sousa [8]. Generalized canonical transformations for the time-dependent oscillator were considered by Leach; see [9] and references therein. Note that all these papers deal with an analyti-

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cal description of the time-dependent harmonic oscillator in the x representation, which has proven to be rather complicated in form. Lie algebraic techniques were applied to study coherent states by Perelomov [10,11], to study Noether invariants of the oscillator by Profilo and Soliana [12], and to study the evolution operator by Magnus [13], Wei and Norman [14], Cheng and Fung [15], and Lo [16–21]. The expressions for the evolution operator obtained by the Lie algebraic method are not always valid for all moments of time [13,14]. Canonical transformations for the quantum oscillator with the frequency satisfying (6) were considered by Blaizot and Ripka [22]. The evolution of the density operator and the Wigner function for the particular case where $\Omega(t)$ is a linear function on $[0,T]$ was considered by Agarwal and Kumar [23]. In all the papers listed above the problem in question is reduced to a problem of solving ordinary differential equations for scalar functions. The above papers differ by the choice of parametrization, namely, by the equation, by the particular solution of the equation, and by the frequency of the reference system ω [$\omega=1$, $\Omega(0)$, or $\lim_{t\to-\infty}\Omega(t)$]. The final form of the results depends dramatically upon the chosen parametrization.

The approach presented here, based upon the family of unitary operators $\{\mathbf{W}_{1z}(t)\}$, has several distinctive features. (i) It uses the abstract Hilbert space $\mathcal{R}$ rather than its realization by the x representation; most of the results are obtained by operator techniques. (ii) It uses the invariant form of the destruction, creation, and displacement operators, which provides a natural transition from the description of the reference system to the description of the time-dependent oscillator. (iii) The time dependence of $\Omega(t)$ is arbitrary. (iv) The frequency of the reference system ω is not necessarily unity or the initial value of $\Omega(t)$. The frequency $\Omega(t)$ may have no limit and, what is more important, the initial value of $\Omega(t)$ may have nothing to do with the characteristic frequencies of the system, as in the case of a T-periodic parametric excitation. (v) It is flexible in the choice of $z(t)$. An appropriate parametrization produces the results in a compact and physically clear form. (vi) The results obtained by this method are valid for all moments of time.

II. INVARIANTS OF A TIME-DEPENDENT QUANTUM SYSTEM

In this section we obtain some properties of exact invariants which are essential for the present approach. Let $\mathbf{H}(t)$ be the Hamiltonian of the system and $S(t)\equiv\mathbf{H}(t)-i(\partial/\partial t)$ its Schrödinger operator. An operator $\mathbf{A}(t)$ is called an exact invariant of the system if

$$[\,\mathbf{A}(t),\mathbf{S}(t)]=0\ . \tag{7}$$

It follows from (7) that action of an invariant on a solution of the Schrödinger equation of the system produces another solution. Since the inner product of any two exact solutions does not depend on time [24], all the matrix elements of the invariant between the states of the system are time independent. The general solution of Eq. (7) in the set of operator-valued functions of time acting on $\mathcal{R}$ is $\mathbf{A}(t)=\mathbf{U}(t,0)\,\mathbf{A}_0\mathbf{U}^\dagger(t,0)$, where $\mathbf{U}(t,0)$ is the evolution operator of the system and $\mathbf{A}_0$ is an operator on $\mathcal{R}$. Let $|\varphi_0\rangle\in\mathcal{R}$ be an eigenvector of $\mathbf{A}_0$ associated with an eigenvalue λ; then $|\varphi(t)\rangle=\mathbf{U}(t,0)|\varphi_0\rangle$ is an eigenvector of $\mathbf{A}(t)$ associated with the eigenvalue λ. Thus the eigenvectors of the invariants that are operator-valued functions acting on $\mathcal{R}$ can be chosen in such a way that they satisfy the Schrödinger equation of the system. This is a generalization of the result obtained by Lewis and Riesenfeld [5] for Hermitian invariants which have a discrete spectrum of nondegenerate eigenvalues. It is easy to see that the uncertainty of the invariant in an arbitrary state of the system,

$$(\Delta\,\mathbf{A}(t))^2\equiv\langle[\,\mathbf{A}(t)]^2\rangle-\langle\,\mathbf{A}(t)\rangle^2\ , \tag{8}$$

does not depend on time.

In the following we shall use the invariant destruction and creation operators $\mathbf{a}_0(t),\mathbf{a}_0^\dagger(t)$ [6],

$$\mathbf{a}_0(t)=e^{i\omega t}\mathbf{a}(\omega),\quad \mathbf{a}_0^\dagger(t)=e^{-i\omega t}a^\dagger(\omega)\ , \tag{9}$$

$$[\mathbf{a}_0(t),\mathbf{a}_0^\dagger(t)]=1\ , \tag{10}$$

where

$$\begin{aligned}\mathbf{a}(\omega)&=(2\omega)^{-1/2}(\omega\mathbf{x}+i\mathbf{p})\ ,\\ \mathbf{a}^\dagger(\omega)&=(2\omega)^{-1/2}(\omega\mathbf{x}-i\mathbf{p})\ .\end{aligned} \tag{11}$$

The operators (9) are invariants of the reference system linear in $\mathbf{x}$ and $\mathbf{p}$. We shall also need the following quadratic invariants:

$$\mathbf{K}_+=\tfrac{1}{2}\mathbf{a}_0^{\dagger 2},\quad \mathbf{K}_-=\tfrac{1}{2}\mathbf{a}_0^2,\quad \mathbf{K}_0=\tfrac{1}{4}(\mathbf{a}_0^\dagger\mathbf{a}_0+\mathbf{a}_0\mathbf{a}_0^\dagger)\ , \tag{12}$$

$$[\mathbf{K}_0,\mathbf{K}_\pm]=\pm\mathbf{K}_\pm,\quad [\mathbf{K}_-,\mathbf{K}_+]=2\mathbf{K}_0\ . \tag{13}$$

Here the time argument in $\mathbf{a}_0$, $\mathbf{a}_0^\dagger$, $\mathbf{K}_0$, $\mathbf{K}_-$, and $\mathbf{K}_+$ is omitted.

Let $\mathbf{A}(t)$ be an invariant of the reference system; then

$$\tilde{\mathbf{A}}(t)=\mathbf{W}_{1z}(t)\,\mathbf{A}(t)\mathbf{W}_{1z}^\dagger(t) \tag{14}$$

is an invariant of the time-dependent oscillator (4). The invariant $\tilde{\mathbf{A}}(t)$ has the same spectrum as $\mathbf{A}(t)$ and its eigenvectors are obtained by action of $\mathbf{W}_{1z}(t)$ on the eigenvectors of $\mathbf{A}(t)$.

III. EXPLICIT FORMS OF $\mathbf{W}_{1z}(t)$

It can be verified by substitution that

$$\mathbf{W}_{1z}(t)=\frac{1}{\sqrt{r}}e^{i\mu\mathbf{x}^2}e^{-i\vartheta\mathbf{x}\mathbf{p}}e^{i(t-\gamma/\omega)\mathbf{H}_0}\ , \tag{15}$$

where

$$\begin{aligned}r(t)&=|z(t)|,\quad \mu(t)=\frac{\dot r(t)}{2r(t)}\ ,\\ \vartheta(t)&=\ln r(t),\quad \gamma(t)=\arg z(t)\ ,\end{aligned} \tag{16}$$

and $z(t)$ satisfies the classical equation

$$\ddot z(t)+\Omega^2(t)z(t)=0 \tag{17}$$

and the condition

$$\dot{z}(t)z^*(t)-\dot{z}^*(t)z(t)=2i\omega \ . \tag{18}$$

Another expression for $\mathbf{W}_{1z}(t)$ is given by

$$\mathbf{W}_{1z}(t)=e^{\xi\mathbf{K}_+-\xi^*\mathbf{K}_-}e^{-2i\varphi_u\mathbf{K}_0} , \tag{19}$$

where

$$\xi=|\xi|e^{i\varphi}, \quad |u|=\cosh|\xi|, \quad \varphi=\varphi_v-\varphi_u , \tag{20}$$

$$u\equiv|u|e^{i\varphi_u}=\frac{e^{-i\omega t}}{2}\left[z(t)-\frac{i\dot{z}(t)}{\omega}\right] , \quad v\equiv|v|e^{i\varphi_v}=\frac{e^{i\omega t}}{2}\left[z(t)+\frac{i\dot{z}(t)}{\omega}\right] . \tag{21}$$

Formula (19) can be verified by substitution into (5), expressing the Hamiltonians $\mathbf{H}_0$ and $\mathbf{H}_1(t)$ in $\mathbf{K}_0,\mathbf{K}_-,\mathbf{K}_+$ and using (13). Using (14) we obtain linear invariants

$$\mathbf{a}_1(z,t)=(2\omega)^{-1/2}[-i\dot{z}(t)\mathbf{x}+iz(t)\mathbf{p}] , \quad \mathbf{a}_1^\dagger(z,t)=(2\omega)^{-1/2}[i\dot{z}^*(t)\mathbf{x}-iz^*(t)\mathbf{p}] \tag{22}$$

and quadratic invariants

$$\mathbf{K}_{1+}=\tfrac{1}{2}a_1^{\dagger 2}, \quad \mathbf{K}_{1-}=\tfrac{1}{2}\mathbf{a}_1^2, \quad \mathbf{K}_{10}=\tfrac{1}{4}(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{a}_1\mathbf{a}_1^\dagger) \tag{23}$$

of the time-dependent oscillator (4). The linear invariants (22) are equivalent to the invariants obtained by Malkin, Man'ko, and Trifonov [6], who used a different parametrization. Using the identity $\mathbf{W}_{1z}(t)=\mathbf{W}_{1z}(t)\mathbf{W}_{1z}(t)\mathbf{W}_{1z}^\dagger(t)$, the operator $\mathbf{W}_{1z}(t)$ can also be expressed in terms of $\mathbf{K}_{1+}$, $\mathbf{K}_{1-}$, and $\mathbf{K}_{10}$:

$$\mathbf{W}_{1z}(t)=e^{\xi K_{1+}-\xi^* K_{1-}}e^{-2i\varphi_u\mathbf{K}_{10}} . \tag{24}$$

Thus we have obtained three families of expressions for $\{\mathbf{W}_{1z}(t)\}$. The families (19) and (24) are useful for treatment of coherent states of the parametrically excited oscillator and of the forced oscillator, respectively, while (15) is especially convenient for the x representation.

IV. EVOLUTION OPERATOR

The evolution operator $\mathbf{U}(t,0)$ of a quantum system is defined as

$$i\frac{\partial}{\partial t}\mathbf{U}(t,0)=\mathbf{H}(t)\mathbf{U}(t,0), \quad \mathbf{U}(0,0)=\mathbf{I} , \tag{25}$$

where $\mathbf{I}$ is the identity transformation. In this section we will use $\{\mathbf{W}_{1z}(t)\}$ to obtain expressions for the evolution operator of the parametrically excited oscillator $\mathbf{U}_1(t,0)$. It is easy to see that

$$\mathbf{U}_1(t,0)=\mathbf{W}_{1z}(t)\mathbf{U}_0(t,0)\mathbf{W}_{1z}^\dagger(0) =\mathbf{W}_{1z}(t)\mathbf{W}_{1z}^\dagger(0)e^{-2i\omega t\mathbf{K}_{10}(0)} , \tag{26}$$

where $\mathbf{U}_0(t,0)=e^{-iH_0t}\equiv e^{-2i\omega t\mathbf{K}_0}$ is the evolution operator of the reference system. For $z_0(t)$, such that $z_0(0)=1$ and $\dot{z}_0(0)=i\omega$, we have $\mathbf{W}_{1z_0}(0)=\mathbf{I}$ and

$$\mathbf{U}_1(t,0)=\mathbf{W}_{1z_0}(t)e^{-i\mathbf{H}_0t} . \tag{27}$$

Substitution of (15) and (19) into (27) gives

$$\mathbf{U}_1(t,0)=\frac{1}{\sqrt{r_0}}e^{i\mu_0\mathbf{x}^2}e^{-i\vartheta_0\mathbf{x}\mathbf{p}}e^{-2i\gamma_0\mathbf{K}_0} , \tag{28}$$

$$\mathbf{U}_1(t,0)=e^{\xi_0\mathbf{K}_+-\xi_0^*\mathbf{K}_-}e^{-2i\varphi_{u0}\mathbf{K}_0} , \tag{29}$$

where the time-dependent parameters correspond to $z(t)=z_0(t)$. Note that the expression similar to (29), which can be found in Lo [16–21] (see, for example, Eqs. 72–74 of [18]), is incorrect because it fails to satisfy the necessary condition

$$\mathbf{U}_1(t,0)=\mathbf{U}_0(t,0) \quad \text{if } \Omega(t)=\omega .$$

V. SOLUTIONS OF THE SCHRÖDINGER EQUATION

It follows from (5) that if $|\Psi_0(t)\rangle$ is a state of the reference system

$$\mathbf{S}_0(t)|\Psi_0(t)\rangle=\mathbf{0} , \tag{30}$$

then

$$|\Psi_1(t)\rangle=\mathbf{W}_{1z}(t)|\Psi_0(t)\rangle \tag{31}$$

is a state of the time-dependent oscillator

$$\mathbf{S}_1(t)|\Psi_1(t)\rangle=\mathbf{0} . \tag{32}$$

The action of $\mathbf{W}_{1z}(t)$ on $\{|n_0(\omega,t)\rangle\}$,

$$|n_0(\omega,t)\rangle=e^{-iE_nt}|n(\omega)\rangle, \quad E_n=(n+\tfrac{1}{2})\omega , \tag{33}$$

$$\langle x|n(\omega)\rangle=\left[\frac{1}{2^n n!}\left[\frac{\omega}{\pi}\right]^{1/2}\right]^{1/2} \times\exp\left\{-\frac{\omega x^2}{2}\right\}H_n(\sqrt{\omega}x) , \tag{34}$$

where H_n is a Hermite polynomial, gives the eigenvectors of $\mathbf{K}_{10}$

$$|n_1(t)\rangle=\mathbf{W}_{1z}(t)|n_0(\omega,t)\rangle . \tag{35}$$

It is important to use states (33) satisfying the Schrödinger equation (30) because the expansion coefficients of any exact solution of (30) in the basis (33) do not depend on time. The invariants (9) are lowering and raising operators for the states (33). The states $\{|n_1(t)\rangle\}$ have the same properties: they satisfy the corresponding Schrödinger equation (32), any solution of (32) has time-independent coefficients in $\{|n_1(t)\rangle\}$, and the invariants (22) are lowering and raising operators for $\{|n_1(t)\rangle\}$. Using (31) one can reduce matrix elements of operators between the states $\{|n_1(t)\rangle\}$ to matrix elements between the states $\{|n_0(\omega,t)\rangle\}$. For example,

$$\langle n_1(t)|\mathbf{H}_1(t)|n_1'(t)\rangle =\frac{E_n}{2\omega^2}[|\dot{z}|^2+\Omega^2(t)|z|^2]\delta_{n,n'} +\frac{\sqrt{n(n-1)}}{4\omega}[\dot{z}^2+\Omega^2(t)z^2]\delta_{n,n'+2} +\frac{\sqrt{n'(n'-1)}}{4\omega}[\dot{z}^{*2}+\Omega^2(t)z^{*2}]\delta_{n,n'-2} . \tag{36}$$

The uncertainties $\Delta\mathbf{x}$ and $\Delta\mathbf{p}$ in the state $|n_1(t)\rangle$ are given by

$$(\Delta\mathbf{x})^2=\frac{|z|^2}{\omega}(n+\tfrac{1}{2}),\quad (\Delta\mathbf{p})^2=\frac{|\dot{z}|^2}{\omega}(n+\tfrac{1}{2}) . \qquad (37)$$

The x representation of $\{|n_1(t)\rangle\}$ can be obtained from (35) by taking into account that

$$e^{i\mu\mathbf{x}^2}\Psi(x)=e^{i\mu x^2}\Psi(x),\quad e^{-i\vartheta\mathbf{x}\mathbf{p}}\Psi(x)=\Psi(x/r) . \qquad (38)$$

We have

$$\langle x|n_1(t)\rangle=\langle x|\mathbf{W}_{1z}(t)|n_0(\omega,t)\rangle$$
$$=\frac{1}{\sqrt{r}}e^{i\mu x^2}\left\langle\frac{x}{r}\left|n_0\left[\omega,\frac{\gamma(t)}{\omega}\right]\right.\right\rangle . \qquad (39)$$

The right-hand side of (39) can be reduced to the expression for the eigenfunctions of the quadratic invariant $\mathbf{a}_1^\dagger\mathbf{a}_1$ obtained in [6] in a different form. The advantage of (39) is that it provides a general relation between the sets of wave functions $\{\Psi_1(x,t)\equiv\langle x|\Psi_1(t)\rangle\}$ and $\{\Psi_0(x,t)\equiv\langle x|\Psi_0(t)\rangle\}$. The expansion coefficients of $\Psi_0(x,t)$ in $\{\langle x|n_0(t)\rangle\}$ are time independent; therefore

$$\Psi_1(x,t)=\frac{1}{\sqrt{r}}e^{i\mu x^2}\Psi_0\left[\frac{x}{r},\frac{\gamma(t)}{\omega}\right] . \qquad (40)$$

The family of relations (40) is determined by the family $\{z(t)\}$ satisfying (17) and (18). A particular case of (40) was obtained in [3] for

$$\Omega(t)\underset{t\to-\infty}{\longrightarrow}\omega_-$$

and a particular solution

$$z(t)\underset{t\to-\infty}{\longrightarrow}e^{i\omega_-t} ,$$

$\omega=\omega_-$. If $\Omega(t)=0$, then (40) gives a family of relations between the wave functions of a free particle and the wave functions of the oscillator. A particular solution $z_0(t)=i\omega t+1$ gives the relation between the wave functions having the same initial conditions. The fact that in order to obtain a wave function of a free particle for $0\le\omega t\le\infty$ we need to know the corresponding wave function of the oscillator only for $0\le\omega t\le\pi/2$ can be useful for computations.

VI. COHERENT STATES

Coherent states were first introduced by Schrödinger [25] to describe nonspreading wave packets of an oscillator with a constant frequency and were later employed by Glauber [26] for a quantal description of laser light beams. The properties of the coherent states in both the cases are most similar to those of classical states. At present the concept of coherence is widely used in various domains of physics and mathematics and the notion of coherent states is generalized and applied to families of quantum-mechanical states quite different from the Glauber states [10,11,26,27]. In this section we use $\{\mathbf{W}_{1z}(t)\}$ to obtain a description of coherent states of the time-dependent oscillator parallel to the description of coherent states of the reference system. Most of the results are obtained by operator techniques. The expectation values of the coordinate, momentum, and energy in the coherent states are obtained in terms of the corresponding real solutions of the classical equation (17). The description is illustrated by the example of coherent states of the free particle.

In terms of the invariant displacement operator

$$\mathbf{D}(\mathbf{a}_0(t),\alpha)\equiv\exp\{\alpha\mathbf{a}_0^\dagger(t)-\alpha^*\mathbf{a}_0(t)\} \qquad (41)$$

and the evolving basis states (33) a coherent state of the reference system with a starting vector $|n_0(\omega,t)\rangle$ and amplitude α is given by

$$|\alpha,n_0\rangle\equiv\mathbf{D}(\mathbf{a}_0(t),\alpha)|n_0(\omega,t)\rangle . \qquad (42)$$

In this representation a state $|\alpha,n_0\rangle$ satisfies the Schrödinger equation (30) and its amplitude α is constant. The states (42) with $n=0$ are just Schrödinger's states [25]; they are often called the standard [11] or the canonical [26] coherent states. The states with $n>0$ were considered by Roy and Singh [27] in terms of $\mathbf{U}(\alpha(t))\equiv\exp\{\alpha(t)\mathbf{a}^\dagger-\alpha^*(t)\mathbf{a}\}$ and the time-independent basis $\{|n(\omega)\rangle\}$ under the name of generalized coherent states. It is shown in [27] that many of the properties of the states $|\alpha,n_0\rangle$ with $n>0$ are similar to these of the standard coherent states. For example, for any given n the set $\{|\alpha,n_0\rangle,\alpha\in\mathbb{C}\}$, where $\mathbb{C}$ is the complex plane, is overcomplete.

Consider now

$$W_{1z}(t)|\alpha,n_0\rangle=\mathbf{D}(a_1(t),\alpha)|n_1(t)\rangle\equiv|\alpha,n_1\rangle , \qquad (43)$$

where

$$\mathbf{W}_{1z}(t)\mathbf{D}(\mathbf{a}_0(t),\alpha)\mathbf{W}_{1z}^\dagger(t)=e^{\alpha\mathbf{a}_1^\dagger(t)-\alpha^*\mathbf{a}_1(t)}\equiv\mathbf{D}(\mathbf{a}_1(t),\alpha) \qquad (44)$$

is the displacement operator corresponding to $\mathbf{a}_1(t),\mathbf{a}_1^\dagger(t)$ [6]. States of this type with $n=0$, often called squeezed states, were intensively studied in a number of references [1,4,6,10,11,28,29].

Since $\mathbf{D}(\mathbf{a}_1(t),\alpha)$ is a unitary operator the set $\{|\alpha,n_1\rangle,$ n is an integer$\}$ is a complete orthonormal set for any given α. A state $|\alpha,n_1\rangle$ is an eigenvector of the Hermitian invariant $(\mathbf{a}_1^\dagger(t)-\alpha^*)(\mathbf{a}_1(t)-\alpha)$ associated with the eigenvalue n:

$$(\mathbf{a}_1^\dagger(t)-\alpha^*)(\mathbf{a}_1(t)-\alpha)|\alpha,n_1\rangle$$
$$=\mathbf{D}(\mathbf{a}_1(t),\alpha)\mathbf{a}_1^\dagger(t)\mathbf{a}_1(t)|n_1(t)\rangle=n|\alpha,n_1\rangle . \qquad (45)$$

The expansion of $|\alpha,n_1\rangle$ in the basis $\{|m_1(t)\rangle\}$,

$$|\alpha,n_1\rangle=\exp\left[-\frac{|\alpha|^2}{2}\right]\times\sum_{m=0}^{\infty}\left[\frac{n!}{m!}\right]^{1/2}\alpha^{m-n}L_n^{m-n}(|\alpha|^2)|m_1(t)\rangle , \qquad (46)$$

where $L_n^k(x)$ are Laguerre polynomials [30,31], can be obtained by $\mathbf{W}_{1z}(t)$ acting on the corresponding expansion for $|\alpha,n_0\rangle$ given in [27]. It is known that the state $|\alpha,0_1\rangle$ is an eigenstate of the invariant $\mathbf{a}_1(t)$ [6]. This property does not hold for the states $|\alpha,n_1\rangle$ with $n>0$; nevertheless

$$\langle\alpha,n_1|\mathbf{a}_1(t)|\alpha,n_1\rangle=\alpha,\quad \langle\alpha,n_1|\mathbf{a}_1^\dagger(t)|\alpha,n_1\rangle=\alpha^* \ . \tag{47}$$

Using (43) we obtain the expectation values of $\mathbf{x}$ and $\mathbf{p}$ in the state $|\alpha,n_1\rangle$

$$\langle\mathbf{x}\rangle\equiv\langle\alpha,n_1|\mathbf{x}|\alpha,n_1\rangle=\left[\frac{2}{\omega}\right]^{1/2}\mathrm{Re}(z^*(t)\alpha)\equiv\chi_\alpha(t) \ , \tag{48}$$

$$\langle\mathbf{p}\rangle\equiv\langle\alpha,n_1|\mathbf{p}|\alpha,n_1\rangle=\left[\frac{2}{\omega}\right]^{1/2}\mathrm{Re}(\dot z^*(t)\alpha)\equiv\dot\chi_\alpha(t) \ , \tag{49}$$

which, as expected, satisfy the classical equations of motion of the force-free oscillator. The function $\chi_\alpha(t)$ is the real solution of the equation (17) corresponding to the state $|\alpha,n_1\rangle$; this solution is defined uniquely by the initial values of $\langle\mathbf{x}\rangle$ and $\langle\mathbf{p}\rangle$. Using the expression for $\mathbf{D}(\mathbf{a}_1(t),\alpha)$ in terms of $\chi_\alpha=\chi_\alpha(t)$,

$$\mathbf{D}(\mathbf{a}_1(t),\alpha)=\exp(i\mathbf{x}\dot\chi_\alpha-i\mathbf{p}\chi_\alpha) \ , \tag{50}$$

we get

$$\langle x|\alpha,n_1\rangle=\exp\left[-i\frac{\chi_\alpha\dot\chi_\alpha}{2}+ix\dot\chi_\alpha\right]\langle x-\chi_\alpha|n_1(t)\rangle \ , \tag{51}$$

which is similar to the corresponding relation for the reference system. Note that the displacement operator conserves the uncertainties of the coordinate and momentum operators (see [11], p. 16); therefore $(\Delta\mathbf{x})^2$ and $(\Delta\mathbf{p})^2$ for the state $|\alpha,n_1\rangle$ are given by (37). The expectation value of the energy in the state $|\alpha,n_1\rangle$ can be found using (43). After some rearrangements we get

$$\langle\alpha,n_1|\mathbf{H}_1(t)|\alpha,n_1\rangle=\langle n_1(t)|\mathbf{H}_1(t)|n_1(t)\rangle+\tfrac{1}{2}[\dot\chi_\alpha^2(t)+\Omega^2(t)\chi_\alpha^2(t)] \ . \tag{52}$$

It is interesting that $\langle\alpha,n_1|\mathbf{H}_1(t)|\alpha,n_1\rangle$ is a sum of a "quantum" term and a "classical" term. The first term is an expectation value of the energy in the starting basis state $|n_1(t)\rangle$ given by (36). The second term is the classical energy of an oscillator with the coordinate $\chi_\alpha(t)$ and the momentum $\dot\chi_\alpha(t)$. Equations (48)–(52) give a description of coherent states in terms of the corresponding solutions of the classical equation (17).

Thus the expectation values of the operators $\mathbf{a}_1^\dagger(t),\mathbf{a}_1(t),\mathbf{x},\mathbf{p}$ and the classical part of the energy in the state $|\alpha,n_1\rangle$ depend only on α and $z(t)$, but do not depend on the number of the starting vector, n. Therefore these quantities for the state $|\alpha,n_1\rangle$ are equal to the corresponding quantities for the coherent state $|\alpha,0_1\rangle$. The uncertainty magnitude and the quantum part of the energy in the state $|\alpha,n_1\rangle$ depend on n and are equal to the corresponding quantities for the starting basis state $|n_1(t)\rangle$.

For any given n the set $\{|\alpha,n_1\rangle,\alpha\in\mathbb{C}\}$ is given by the unitary transformation of $\{|\alpha,n_0\rangle,\alpha\in\mathbb{C}\}$; therefore,

$$\frac{1}{\pi}\int|\alpha,n_1\rangle\langle\alpha,n_1|d^2\alpha=\mathbf{I} \ , \tag{53}$$

where $\alpha\equiv\alpha_1+i\alpha_2$, $d^2\alpha\equiv d\alpha_1 d\alpha_2$, and

$$\mathcal{K}_n(\alpha,\beta)\equiv\langle\alpha,n_1|\beta,n_1\rangle=L_n(|\alpha-\beta|^2)\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)+\beta\alpha^*] \ . \tag{54}$$

Equations (53) and (54) establish overcompleteness of $\{|\alpha,n_1\rangle,\alpha\in\mathbb{C}\}$. For $n=0$ these equations are obtained in [6]. For arbitrary solution of (32), $|\Psi_1(t)\rangle$, we have the set of integral equations

$$\Psi_1(\alpha,n)=\int\mathcal{K}_n(\alpha,\beta)\Psi_1(\beta,n)d^2\alpha \ , \tag{55}$$

where

$$\Psi_1(\alpha,n)\equiv\langle\alpha,n_1|\Psi_1(t)\rangle \ . \tag{56}$$

Note that the reproducing kernel $\mathcal{K}_n(\alpha,\beta)$ and the wave function in the coherent states representation, $\Psi_1(\alpha,n)$, do not depend on time.

Let $\mathbf{B}$ be a bounded operator or a polynomial in $\mathbf{a}_1(t)$ and $\mathbf{a}_1^\dagger(t)$. Then $\mathbf{B}$ is uniquely determined by its diagonal matrix elements

$$B_1(\alpha)\equiv\langle\alpha,0_1|\mathbf{B}|\alpha,0_1\rangle \ . \tag{57}$$

The diagonal or P representation of $\mathbf{B}$ is given by

$$\mathbf{B}=\frac{1}{\pi}\int b_1(\alpha)|\alpha,0_1\rangle\langle\alpha,0_1|d^2\alpha \ , \tag{58}$$

where $b_1(\alpha)$ is defined by

$$B_1(\alpha)=\frac{1}{\pi}\int b_1(\beta)e^{-|\alpha-\beta|^2}d^2\beta \ . \tag{59}$$

The P representation of $\mathbf{B}$ (57)–(59) can be obtained by introducing $\mathbf{A}=\mathbf{W}_{1z}^\dagger(t)\mathbf{B}\mathbf{W}_{1z}(t)$. The operator $\mathbf{A}$ is a bounded operator or a polynomial in $\mathbf{a}_0(t)$ and $\mathbf{a}_0^\dagger(t)$; therefore it admit a diagonal representation in standard coherent states $\{|\alpha,0_0\rangle,\alpha\in\mathbb{C}\}$ [26]. We have

$$A_0(\alpha)\equiv\langle\alpha,0_0|\,\mathbf{A}|\alpha,0_0\rangle=B_1(\alpha),\quad a_0(\alpha)=b_1(\alpha) \ . \tag{60}$$

In particular, it follows from (5) that the density operator for the time-dependent oscillator is given by

$$\rho_1(t)=\mathbf{W}_{1z}(t)\rho_0\mathbf{W}_{1z}^\dagger(t) \ . \tag{61}$$

ρ_0 is the density operator of the reference system. Therefore ρ_1 admits a diagonal representation in $\{|\alpha,0_1\rangle,\alpha\in\mathbb{C}\}$ and its weight function $P(\alpha)$ coincides with the weight function of ρ_0 in $\{|\alpha,0_0\rangle,\alpha\in\mathbb{C}\}$. The diagonal representation of the density operator in a particular set of coherent states which at $t=0$ coincide with the standard coherent states of the reference system was considered in [1]. The expectation value of $\mathbf{B}$ in an arbitrary state with the density operator $\rho_1(t)$ is given by

$$\langle \mathbf{B} \rangle \equiv \mathrm{Tr}\{\rho_1(t)\mathbf{B}\} = \int P(\alpha) B_1(\alpha) d^2\alpha$$
$$= \int P(\alpha) A_0(\alpha) d^2\alpha \ . \tag{62}$$

Let us consider a free particle $\Omega(t)=0$. It is well known that its Hamiltonian $\mathbf{H}_1=\frac{1}{2}\mathbf{p}^2$ has no eigenvectors which belong to $\mathcal{R}$. Using the approach of the present paper we can obtain complete sets of classical-like states of the free particle which belong to $\mathcal{R}$. Let

$$z_{fp}(t)=i\omega t+1 \ . \tag{63}$$

The states $\{|n_1(t)\rangle \equiv |n_{fp}(t)\rangle, \ |n_{fp}(0)\rangle = |n(\omega)\rangle\}$ are eigenvectors of the quadratic invariant of a free particle

$$\tilde{\mathbf{H}}_0(t)=\tfrac{1}{2}[\mathbf{p}^2+\omega^2(\mathbf{x}-\mathbf{p}t)^2] \ , \tag{64}$$

$$\tilde{\mathbf{H}}_0(t)|n_{fp}(t)\rangle=(n+\tfrac{1}{2})\omega|n_{fp}(t)\rangle \ . \tag{65}$$

The corresponding invariant destruction and creation operators are

$$\mathbf{a}_{fp}(t)=(2\omega)^{-1/2}[\omega(\mathbf{x}-\mathbf{p}t)+i\mathbf{p}] \ ,$$
$$\mathbf{a}^{\dagger}_{fp}(t)=(2\omega)^{-1/2}[\omega(\mathbf{x}-\mathbf{p}t)-i\mathbf{p}] \ . \tag{66}$$

The Hermitian invariant $\mathbf{x}-\mathbf{p}t$ can be interpreted as the operator of the initial coordinate of the particle. The wave packets of the states $|n_{fp}(t)\rangle$ are spreading in the x representation (39) and keeping their shapes in the p representation. We have

$$(\Delta \mathbf{x})^2=\frac{\omega^2t^2+1}{\omega}(n+\tfrac{1}{2}), \quad (\Delta \mathbf{p})^2=\omega(n+\tfrac{1}{2}) \ . \tag{67}$$

The parameter ω can be chosen to obtain desirable initial squeezing of the states $|n_{fp}(t)\rangle$. The set $|n_{fp}(t)\rangle$ has equally spaced expectation values of the energies

$$\langle n_{fp}(t)|\mathbf{H}_1|n_{fp}(t)\rangle=\frac{(2n+1)\omega}{4} \ . \tag{68}$$

The real solution of the classical equation, $\chi_{fp\alpha}$ corresponding to the coherent state $|\alpha,n_{fp}\rangle$, is given by

$$\chi_{fp\alpha}=\left[\frac{2}{\omega}\right]^{1/2}\alpha_1+\sqrt{2\omega}\alpha_2 t \ . \tag{69}$$

Thus the amplitude of the coherent state α defines expectation values of the momentum and coordinate in this state at $t=0$. The expectation value of the energy is given by

$$\langle \alpha,n_{fp}|\mathbf{H}_1|\alpha,n_{fp}\rangle=\frac{(2n+1)\omega}{4}+\omega\alpha_2^2 \ , \tag{70}$$

where the first term can be omitted if we use a set $\{|\alpha,n_{fp}\rangle,\alpha\in\mathbb{C}\}$ corresponding to fixed n. The families of orthonormal sets $\{|n_{fp}(t)\}$ and overcomplete sets of coherent states $\{|\alpha,n_{fp}\rangle,\alpha\in\mathbb{C}\}$ can be useful in the quantum-mechanical description of the free particle.

Thus utilization of invariant forms of creation, destruction, and displacement operators together with the choice of a particular class of solutions of the classical equation (17) satisfying (18), $\{z(t)\}$, proved to be most suitable for the description of coherent states of the time-dependent oscillator. It enables the presentation of properties of these states in a compact and physically clear form.

VII. THE CASE OF A PERIODIC FREQUENCY AND THE CASE OF EVOLUTION FINITE IN TIME

In this section we obtain the Floquet [32] form of the evolution operator and calculate Berry phases [33] for cyclic states of the time-dependent oscillator. Let $\Omega(t)$ be piecewise continuous and

$$\Omega(t+T)=\Omega(t), \quad T=\frac{2\pi}{\omega'} \ , \tag{71}$$

where ω' is a positive real number. In this case, according to Floquet theory [34,35], there exist solutions of the classical equation (17) in the form [3,4]

$$z_\lambda(t)=z_p(t)e^{i\lambda t} \ ,$$
$$z^*_\lambda(t)=z^*_p(t)e^{-i\lambda t} \ , \tag{72}$$

where $z_p(t+T)=z_p(t)$, and λ is constant. In the stability domain of (17) λ is real and without loss of generality can be assumed to be positive. In this section we shall consider only T-periodic functions $\Omega(t)$, which belong to the stability domain of Eq. (17). In this case the Schrödinger equation (32) has a set of quasienergy solutions [3,4] with the quasienergies

$$\varepsilon_n=\lambda(n+\tfrac{1}{2}) \ . \tag{73}$$

Wave functions of the quasienergy states of the oscillator in the x representation are obtained in [4]. Note that the quasienergy states can be expressed in terms of $\mathbf{W}_{1z_\lambda}(t)$ corresponding to the cyclic solution of the classical equation (72),

$$|n_1(z_\lambda,t)\rangle=\mathbf{W}_{1z_\lambda}(t)|n(\omega,t)\rangle$$
$$=e^{-i\lambda(n+1/2)t}|n_{1p}(z_\lambda,t)\rangle \ , \tag{74}$$

where $|n_{1p}(z_\lambda,t)\rangle$ is a T-periodic vector. The destruction and creation operators (22) for the quasienergy states are of the form

$$\mathbf{a}_1(z_\lambda,t)=e^{i\lambda t}\mathbf{a}_{1p}(t) \ ,$$
$$\mathbf{a}^\dagger_1(z_\lambda,t)=e^{-i\lambda t}\mathbf{a}^\dagger_{1p}(t) \ , \tag{75}$$

where $\mathbf{a}_{1p}(t+T)=\mathbf{a}_{1p}(t)$.

It is known that the evolution operator of a quantum-mechanical system with a self-adjoint T-periodic Hamiltonian can be presented in the Floquet form [32]

$$\mathbf{U}(t,0)=\mathbf{P}(t)e^{-i\mathbf{G}t} \ , \tag{76}$$

where $\mathbf{G}$ is a time-independent self-adjoint operator and $\mathbf{P}(0)=\mathbf{P}(kT)=\mathbf{I}$ for k is an integer. The Floquet form of the evolution operator for the harmonic oscillator with T-periodic frequency is given by (26) with $z_\lambda(t)$ in place of $z(t)$ and $\omega=\lambda$. The transformation $\mathbf{W}_{1z_\lambda}(t)$ corresponding to $\omega=\lambda$ is T-periodic. We have

$$\mathbf{P}_1(t)=\mathbf{W}_{1z_\lambda}(t)\mathbf{W}^\dagger_{1z_\lambda}(0), \quad \mathbf{G}_1=2\lambda\mathbf{K}_{10}(z_\lambda,0) \ . \tag{77}$$

An expression for the evolution operator at the moments of time $t_k = kT$,

$$\mathbf{U}_1(kT,0) = e^{-2i\lambda \mathbf{K}_{10}(z_\lambda,0)kT} = [\mathbf{U}_1(T,0)]^k \ , \tag{78}$$

where k is an integer, is similar to the expression for the evolution operator of the reference system. It depends only on λ, $z_\lambda(0)$, and $\dot{z}_\lambda(0)$ and gives the long-term behavior of the oscillator under a periodic perturbation.

It follows from (74) that $\Delta \mathbf{x}$ and $\Delta \mathbf{p}$ in the basis states $|n_1(z_\lambda,t)\rangle$ and in the coherent states $|\alpha_0,n_1\rangle$ are T-periodic functions of time. The expectation values of the energy in the basis states $|n_1(z_\lambda,t)\rangle$ are also T-periodic.

If $\lambda \neq l\omega'$, where l is an integer, only the basis states $|n_1(z_\lambda,t)\rangle$ are cyclic. The nonadiabatic Berry phase [33] of a basis state can be easily obtained from (36) and (73)

$$\beta(|n_1(z_\lambda,t)\rangle) = \frac{1}{\lambda}(n+\tfrac{1}{2})\int_0^T \Omega^2(t)|z_\lambda|^2 dt - \lambda T(n+\tfrac{1}{2}) \ . \tag{79}$$

On the other hand, Berry phases of the states $|n_1(z_\lambda,t)\rangle$ can be calculated using the general relation between the quasienergy ε and the Berry phase of a cyclic state $|\Psi_\varepsilon(t)\rangle$, continuous with respect to the frequency ω' [36],

$$\beta(|\Psi_\varepsilon(t)\rangle) = -2\pi \frac{\partial \varepsilon(\omega', \{\Omega^{(k)}\})}{\partial \omega'} \ , \tag{80}$$

where $\{\Omega^{(k)}\}$ are Fourier coefficients of $\Omega(t)$ considered as independent variables. We have

$$\beta(|n_1(z_\lambda,t)\rangle) = -2\pi \frac{\partial \lambda(\omega', \{\Omega^{(k)}\})}{\partial \omega'}(n+\tfrac{1}{2}) \ . \tag{81}$$

If $\lambda = l\omega'$ for some l, then the function $z_\lambda(t)$ is T-periodic and all the states of the oscillator are cyclic with the same phase $\phi = l\pi$. Berry phases of the states which are cyclic only for isolated values of the frequency ω' can be calculated using (36). Taking into account (52) we obtain the Berry phase of a coherent state $|\alpha,n_1\rangle$,

$$\beta(|\alpha,n\rangle) = \beta(|n_1(z_\lambda,t)\rangle) + \tfrac{1}{2}\int_0^T [\dot{\chi}_\alpha^2(t) + \Omega^2(t)\chi_\alpha^2(t)]dt \ , \quad \mathrm{mod}(2\pi) \ ,$$

where $\chi_\alpha(t)$ is a solution of the classical equation (17) corresponding to the coherent state $|\alpha,n_1\rangle$ and given by (48).

Suppose we are interested in an evolution of the oscillator with a piecewise continuous frequency in a finite time interval $[0,T]$ (system I). The results obtained for systems with T-periodic frequencies can be applied to this case by considering another system (system II), such that [36]

$$\begin{aligned} &\Omega_{\mathrm{II}}(t) = \Omega_{\mathrm{I}}(t) \quad \text{for } t \in [0,T) \ , \\ &\Omega_{\mathrm{II}}(t+T) = \Omega_{\mathrm{II}}(t) \quad \text{for any } t \ , \end{aligned} \tag{82}$$

where $\Omega_{\mathrm{I}}(t)$ is the frequency of system I and $\Omega_{\mathrm{II}}(t)$ is the frequency of system II [36]. The function $\Omega_{\mathrm{II}}(t)$ is T-periodic and piecewise continuous. The description of system I coincides with the description of system II on the time interval $[0,T]$. For example, the quasienergy states of system II taken in the time interval $[0,T]$ form a complete basis set of cyclic states for system I.

Finally, let us note that an expression for the Berry phase of a cyclic coherent state of the harmonic oscillator with time-dependent frequency was obtained previously in [37] in terms of a loop integral over a Cartan form. However, the question of the existence of cyclic states, their explicit form, and their relation to solutions of the classical equation was not considered in [37].

VIII. MATRIX ELEMENTS OF $\mathbf{W}_{1z}(t)$ AND TRANSITION AMPLITUDES

The calculation of transition amplitudes is important for applications. It is shown in this section that by appropriate choices of $z(t)$, transition amplitudes for the oscillator can be reduced to matrix elements of $\mathbf{W}_{1z}(t)$ between the basis states (33) corresponding to the same frequencies. The expression for the matrix elements in terms of the corresponding $z(t)$, which follows from (83), is rather simple. There is an analogy between the classical equation (17) for $z(t)$ and a certain one-dimensional Schrödinger equation [1] which can be used to study the relation between the shape of $\Omega(t)$ and the transition probabilities. This analogy is illustrated by the case of a transition from a number state into a quasienergy state.

It can be proved by induction with respect to n that

$$\mathbf{W}_{1z}(t)|n_0(\omega,t)\rangle = |n_1(t)\rangle = \sum_{m=0}^{\infty} \frac{1+(-1)^{m+n}}{2}\left[\frac{n!}{m!u}\right]^{1/2} e^{(i/2)[(\varphi_v+\pi)(m-n)-\varphi_u(m+n)]} P_{(m+n)/2}^{(m-n)/2}\left[\frac{1}{|u|}\right] |m_0(\omega,t)\rangle \ , \tag{83}$$

where $P_{(m+n)/2}^{(m-n)/2}(x)$ are associated Legendre functions [30] and u and φ are given by (20) and (21). Suppose the perturbation is turned on for $t_1 \le t \le t_2$,

$$\Omega(t) = \begin{cases} \omega_- & \text{for } t < t_1 \\ \omega_+ & \text{for } t > t_2 \ . \end{cases} \tag{84}$$

Then for $t < t_1$ and $t > t_2$ there exist complete sets of solutions $\{|n_0(\omega_-,t)\rangle\}$ and $\{|m_0(\omega_+,t)\rangle\}$, respectively. Suppose the system was in the state $|n_0(\omega_-,t)\rangle$ at $t \le t_1$. The amplitude of probability A_{nm} to find the system in the state $|m_0(\omega_+,t)\rangle$ at $t \ge t_2$ is given by

$$A_{nm} = \langle \Psi_2(t)|\Psi_1(t)\rangle \ , \tag{85}$$

where $|\Psi_1(t)\rangle, |\Psi_2(t)\rangle$ satisfy the Schrödinger equation with $\Omega(t)$ given by (84) for all t and

$$|\Psi_1(t)\rangle=|n_0(\omega_-,t)\rangle \quad \text{for } t\le t_1 , \tag{86}$$

$$|\Psi_2(t)\rangle=|m_0(\omega_+,t)\rangle \quad \text{for } t\ge t_2 . \tag{87}$$

First let $\omega=\omega_-$. One can verify that

$$|\Psi_2(t)\rangle=\mathbf{W}_{1z_2}(t)|m_0(\omega_-,t)\rangle , \tag{88}$$

where $z_2(t)$ is a solution of (17) with $\Omega(t)$ given by (84), such that

$$z_2(t)=\begin{cases}\sigma_1 e^{i\omega_- t}+\sigma_2 e^{-i\omega_- t} & \text{for } t\le t_1\\ \left[\dfrac{\omega_-}{\omega_+}\right]^{1/2} e^{i\omega_+ t} & \text{for } t\ge t_2 ,\end{cases} \tag{89}$$

where the coefficients σ_1,σ_2 are determined by $\Omega(t)$ and the condition (18),

$$|\sigma_1|^2-|\sigma_2|^2=1 , \tag{90}$$

and

$$u(z_2)=\sigma_1, \quad v(z_2)=\sigma_2 \quad \text{for } t\le t_1 . \tag{91}$$

The inner product of any two exact solutions of the Schrödinger equation does not depend on time; therefore it is enough to calculate (85) for any given moment of time. We have

$$\begin{aligned}A_{nm}&=\langle\Psi_2(t)|\Psi_1(t)\rangle|_{t\le t_1}\\ &=\langle m_0(\omega_-,t)|\mathbf{W}^\dagger_{1z_2}(t)|n_0(\omega_-,t)\rangle|_{t\le t_1} .\end{aligned} \tag{92}$$

On the other hand,

$$|\Psi_1(t)\rangle=\mathbf{W}_{1z_1}(t)|n_0(\omega_+,t)\rangle , \tag{93}$$

where $z_1(t)$ is a particular solution of the equation (17) satisfying the condition

$$z_1(t)=\left[\frac{\omega_+}{\omega_-}\right]^{1/2} e^{i\omega_- t} \quad \text{for } t\le t_1 . \tag{94}$$

This solution corresponds to $\omega=\omega_+$. Substitution of (93) into (85) gives

$$\begin{aligned}A_{nm}&=\langle\Psi_2(t)|\Psi_1(t)\rangle|_{t\ge t_2}\\ &=\langle m_0(\omega_+,t)|\mathbf{W}_{1z_1}(t)|n_0(\omega_+,t)\rangle|_{t\ge t_2} .\end{aligned} \tag{95}$$

Thus we have obtained two equivalent expressions for the amplitude of the transition probability A_{nm} corresponding to two solutions (89) and (94) of Eq. (17). In the particular case $t_1\to-\infty$ and $t_2\to+\infty$ the expression (95) is equivalent to the result obtained by Malkin, Man'ko, and Trifonov [6]. Note that the amplitude A_{nm} is given by just one coefficient of the expansion (83) because it is reduced to matrix elements of $\mathbf{W}_{1z}(t)$ between number states corresponding to the *same* frequency. The amplitude A_{nm} can be (up to a phase factor) reduced to a matrix element of the evolution operator $\mathbf{U}_1(t,0)$ between the states $|n(\omega_-)\rangle$ and $|m(\omega_+)\rangle$. The explicit form of this matrix element is rather complicated because $\langle m(\omega_+)|n(\omega_-)\rangle\neq\delta_{nm}$ if $\omega_+\neq\omega_-$.

Consider now

$$\begin{aligned}&\Omega(t)=\omega_- \quad \text{for } t<t_1 ,\\ &\Omega(t+T)=\Omega(t) \quad \text{for } t\ge t_1 .\end{aligned} \tag{96}$$

In the stability domain of Eq. (17) it has solutions of the type (72) and the oscillator has a complete set of quasienergy states $\{|m_1(t)\rangle\}$ for $t\ge t_1$. The amplitude of the transition of the system from the basis state $|n_0(\omega_-,t)\rangle$ into the quasienergy state $|m_1(t)\rangle$, B_{nm}, is given by

$$B_{nm}=\langle m_0(\omega_-,t)|\mathbf{W}^\dagger_{1z_3}(t)|n_0(\omega_-,t)\rangle|_{t\le t_1} , \tag{97}$$

where $\omega=\omega_-$,

$$z_3(t)=\begin{cases}\sigma_1 e^{i\omega_- t}+\sigma_2 e^{-i\omega_- t} & \text{for } t\le t_1\\ z_p(t)e^{i\lambda t}, z_p(t+T)=z_p(t) & \text{for } t\ge t_1 .\end{cases} \tag{98}$$

The parameters σ_1,σ_2 are determined by the initial conditions for the cyclic solution $z_p(t)e^{i\lambda t}$ and satisfy (90). It is interesting that $z_3(t)$ coincides with the wave function of the one-dimensional Schrödinger equation if we replace t by x. The condition (96) corresponds to a spatially periodic potential barrier. The solution $z_3(x)$ describes propagation of a Bloch-type wave through the semi-infinite periodic structure (96). The transition probability $|B_{nm}|^2$ depends only on the reflection coefficient

$$R=\frac{\sigma_2}{\sigma_1}, \quad \frac{1}{|u|}=\sqrt{1-|R|^2} . \tag{99}$$

If $\sigma_1=1$, then $\sigma_2=0$, $R=0$, and $B_{nm}=\delta_{nm}$. This corresponds to the case where a number state of the oscillator becomes a quasienergy state at $t\ge t_1$. Let us consider a particular realization of this case. It is not difficult to verify that a solution

$$z_4(t)=\begin{cases}e^{i\omega_- t} & \text{for } t\le 0\\ e^{ik\lambda T}(c_1 e^{i\Omega_1 t_k}+c_2 e^{-i\Omega_1 t_k}) & \text{for } 0\le t_k\le T_1\\ e^{ik\lambda T}e^{ip\pi}e^{i\omega_-(t_k-T_1)} & \text{for } T_1\le t_k<T ,\end{cases} \tag{100}$$

where p and k are integers, $0\le t_k\equiv t-kT<T$, and

$$c_1=\frac{\Omega_1+\omega_-}{2\Omega_1}, \quad c_2=\frac{\Omega_1-\omega_-}{2\Omega_1}, \quad \lambda=\frac{p\pi+\omega_-(T-T_1)}{T} \tag{101}$$

satisfy (17) and (18) with $\omega=\omega_-$ and

$$\Omega(t)=\begin{cases}\omega_- & \text{for } t<0\\ \Omega_1=p\pi/T_1 & \text{for } 0\le t_k<T_1\\ \omega_- & \text{for } T_1\le t_k<T .\end{cases} \tag{102}$$

For $0\le t$ the frequency (102) is T-periodic and

$$z_4(t+T)=z_4(t)e^{i\lambda T} . \qquad (103)$$

The basis states of the oscillator corresponding to the solution (100) coincide with the number states of the reference system with $\omega=\omega_-$ for $t\leq 0$ and are quasienergy states for $0\leq t$. The quasienergies of the states are given by (73) together with (101). If $\lambda=l\omega'\equiv 2\pi l/T$, which is equivalent to $\omega_-(T-T_1)=\pi(2l-p)$, all states of the system are T periodic for $0\leq t$.

IX. CONCLUDING REMARKS

Canonical transformations (transformations that preserve the commutation relations between the canonical variables) are widely used in classical and quantum mechanics. For instance, Leach (see [9] and references therein) has applied a generalized canonical transformation

$$\{q,p,t,H(q,p,t)\}\rightarrow\{Q,P,T,\overline{H}(Q,P)\} , \qquad (104)$$

where q, p, Q, and P are classical canonical variables; $H(q,p,t)$ and $\overline{H}(Q,P)$ are classical Hamiltonian functions of the time-dependent oscillator and of the reference system with $\omega=1$, respectively, to relate solutions of the corresponding Schrödinger equations and to relate matrix elements of observables[1] for the two systems. In the present paper a description of the time-dependent harmonic oscillator parallel to the description of the time-independent reference system has been presented in terms of the family of the unitary operators $\{\mathbf{W}_{1z}(t)\}$. The unitary operator $\mathbf{W}_{1z}(t)$ transforms the Schrödinger operator of the reference system (5) and, consequently, its Schrödinger equation. However, the observables $\mathbf{x}$ and $\mathbf{p}$ and the variable t are the same for both the reference system and the time-dependent oscillator. The consistent quantum-mechanical approach of the present method gives not only the transformation of the states (31) and (40), but also the corresponding transformation of invariants (14) and their eigenvectors, the transformation of the evolution operator (26), and a straightforward method of calculation of matrix elements using (31). Use of the evolving number states (33) and the corresponding invariant lowering and raising operators (9) immediately gives the evolving number states (35), invariant lowering and raising and displacement operators for the time-dependent oscillator (22), and a general form of solution of the Schrödinger equation (40). The parametrization (17) and (18) is well suited for the description of coherent states. The flexibility of the choice of parametrization is very important for applications. One can see from the results of Secs. VII and VIII that it is the appropriate choice of the parametrization which provides a description of quasienergy states, a calculation of Berry phases, and a reduction of transition probabilities to matrix elements of $\mathbf{W}_{1z}(t)$.

[1]Nondiagonal matrix elements of operators $\mathbf{q}^2$, $\mathbf{p}^2$, and $\mathbf{H}$ obtained in [9] are incorrect because the time-dependent phase factors of the evolving basis states, Eq. (3.5) of [9], are not taken into account.

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