Geometric spin and charge

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A mathematical representation of measurements on geometric excitations is given for the geometric unified theory. Application to excitations generated by rotations results in quantum numbers $n\hbar/2$ for a geometric angular-momentum current. Excitations generated by the other compact subgroup result in quantum numbers qe for the geometric electric charge. The method is compatible with quantum mechanics but additionally gives the quantum of electric charge.

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I. INTRODUCTION

It is possible to give a geometrical representation to the postulates of quantum mechanics. In most cases this is only a geometrical superstructure that may not contain new physical ideas. On the contrary, it may be possible to start from a geometrical physical theory, and try to obtain its quantum implications. In this manner new physical phenomena may arise. This is the object of this paper. Similar aims originally guided Dreschler in discussing extended objects as functions on homogeneous de Sitter fiber bundles within a geometric theory of hadrons [1,2].

A geometrical relativistic unified theory of gravitation and electromagnetism unexpectedly leads to equations of relativistic quantum mechanics [3]. The theory considers physical matter as representations of the structure group of the geometrical theory. From this idea it follows that there are certain discrete numbers associated with the states of matter. It was shown that these numbers may be interpreted as quanta of spin, charge, and magnetic flux, providing a plausible explanation to the fractional quantum Hall effect (FQHE) [4]. Furthermore, the theory also leads to a geometrical model for the process of field quantization [5], implying the existence of fermionic and bosonic operators and their rules of quantization. It should be clear that a process of physical measurement should display these discrete numbers as experimental quanta, leading to a particle (atomic) description of matter. It is necessary, therefore, to discuss measurements within the geometrical theory.

We may wonder whether these results are accidental or the consequence of a deep fundamental relation of the geometrical structure of the unified field with the standard quantum structure. Here we pursue the second alternative. In particular, we consider the following questions: Can we define, within our theory, a geometric operation representing the process of physical measurement with results compatible with well known facts of experimental physics? Can we define geometric angular momentum and charge? First we shall review the geometrical theory.

In order to understand the ideas and calculations involved, some knowledge of the geometry of principal fiber bundles, universal Clifford algebras, and their correlated spinor bundles is required. These subjects were covered in previous publications, and due to their extensiveness we find it inconvenient to repeat them here except for a few necessary facts.

Recently, Drechsler incorporated electromagnetism in his theory by using a modified Weyl geometry in the construction of the homogeneous de Sitter bundles and relating the full curvature to quantized matter currents [6,7]. Our approach has been different. Originally our aim was to unify gravitation and electromagnetism by means of the connection. To avoid contradictions we had to introduce groups acting on the Clifford algebra of space and time, forcing a geometrical structure that implies quantum aspects.

In geometric terms, our unified interaction is represented by a connection on a principal bundle, and matter by sections in the principal bundle. Because of Infeld's and Van der Waerden's work [8–10], we used a group that contains SL(2,ℂ) rather than SO(3.1), the related Lorentz group. A gravitation theory based on SL(2,ℂ) was discussed by Carmeli [11]. To obtain a generalization the simplest choice, apparently, was the group U(1)×SL(2,ℂ), which is the group that preserves the metric associated with a tetrad induced from a spinor base. By using the integrability conditions of the field equations of the theory, it was shown that the predicted motion of charges is incorrect, and that it is false to assume that any structure group that has SL(2,ℂ)×U(1) as a subgroup gives a unified theory without contradicting the Lorentz equation of motion. The correct classical motion is a fundamental requirement of a unified theory.

We require the use of the group of automorphisms of the universal Clifford algebra associated to flat four-dimensional space time or equivalently its highest-dimensional simple subgroup SL(4,ℝ). This implies an extension of relativity. The connection, which represents the interaction, not only unifies gravitation with electromagnetism including the correct motion but actually gives a gravitational theory that differs in principle with Einstein's theory and resembles Yang's [12] theory. This may be seen from the field equation of the theory, which relates the derivatives of the Ehresmann curvature to a current source J:

$$D^*\Omega = kJ\ . \tag{1.1}$$

Because of the geometrical structure of the theory the source current must be a geometrical object compatible with the field equation and the geometry. The field equation implies integrability conditions in terms of J. Together with the geometric structure of J, these conditions imply a generalized Dirac equation which, therefore, is not required to be separately postulated as is normally done in nonunified theories. Actually, the nonlinear field equation for the connection and the simplest geometric structure of the current are sufficient to predict this generalized Dirac equation. The structure of J, of course, is given in terms of geometric objects acted upon by the connection:

$$kJ^{u} = k\tilde{e}(\iota^{\alpha} u^{\mu}_{\alpha})e\ , \tag{1.2}$$

in terms of the frame e, an orthonormal set of the algebra ι, the correlation on the spinor spaces $\sim$, and a space-time tetrad u.

The three compact generators κ_0, κ_5, and $\kappa_1\kappa_2\kappa_3$ are equivalent as electromagnetic generators within the theory because there are automorphisms that transform any one of them into any other. It follows that the set ι^{α} that enters in the current is defined up to an algebra automorphism. This allows us to take the ι^0 element as any of the three electromagnetic generators without changing the physical content of the theory.

II. MEASUREMENT OF GEOMETRY CURRENTS

It is possible to study the properties of fluctuations or excitations of the geometric elements of the unified theory. Furthermore, if, as suggested before [4], a particle may be represented as an excitation of the geometry, its physical properties may be determined by its associated fluctuations. These fluctuations may be characterized mathematically by a variational problem. From a variational principle, if the equations of motion hold, it is possible to define the generator of the variation. There is a canonical geometric current associated with this generator which geometrically represents the excitation, and should be considered the subject of a physical measurement (observable).

The Lagrangian density, in general, has units of energy per volume and the action has units of energy time. In the natural units defined by the connection ($c=1,\hbar=1,e=1$), the action is dimensionless. In the standard relativistic units ($c=1$), the constant $\hbar$ arises as a factor in the action W.

It is well known that the variation of an action integral along a transformation of the variable y with parameter λ is

$$\delta W = \int_R \frac{\delta L}{\delta y}\delta y\, d^4x + \int_{\partial R} \delta Q^{\mu} d\sigma_{\mu}\ , \tag{2.1}$$

where the canonical current $\mathcal{J}$, and the conjugate momentum Π are

$$\delta Q^{\mu} = \mathcal{J}^{\mu}\delta\lambda = \left[\Pi^{\mu}\frac{dy}{d\lambda} + L\frac{dx^{\mu}}{d\lambda}\right]\delta\lambda\ , \tag{2.2}$$

$$\Pi^{\mu} = \frac{\partial L}{\partial y_{,\mu}}\ . \tag{2.3}$$

The current $\mathcal{J}$ is a geometrical object field that represents an observable property of a physical excitation, e.g., angular-momentum density. In general, a measurement is not a point process but rather it is an interaction with an apparatus over a space-time region. In the geometric theory, the results of a physical measurement should be a number depending on a variational current $\mathcal{J}$ about a background section e, over a region R around a characteristic point m on the base space M, with some instrumental averaging procedure over the region. Then we shall make the hypothesis that a measurement on a geometrical excitation is represented mathematically by a functional of the observable geometric current defined by the associated variation,

$$\mathcal{M}(\mathcal{J}) = \int_{\partial R} \mathcal{M}\mathcal{J}^{\mu} d\sigma_{\mu}\ . \tag{2.4}$$

In some cases, excitations may be approximated as point excitations with no extended structure. In order to show the relation of our geometric theory to other theories, without using any knowledge about the structure of excitations, we shall define a geometric measurement of a point excitation property by a process of shrinking the region $R(m)$ of the current to the point m. With this procedure, the local sections representing the excitation shrink to singular sections at m. We may express this mathematically by

$$\lim_{R(m)\to m} \mathcal{M}(\mathcal{J}) = \delta_m(\mathcal{J})\ , \tag{2.5}$$

where the functional δ_m is the Dirac functional

$$\delta_m(\mathcal{J}) = \mathcal{J}(m)\ . \tag{2.6}$$

This process shrinks the current to a timelike world line. We may visualize the boundary ∂R of region R as an infinitesimal cylindrical pillbox pierced by the current at the bottom and top spacelike surfaces Σ. As the pillbox is shrunken to point m, the functionals of the current $\delta_m(\mathcal{J})$ at the top and bottom surfaces are equal, if the current is continuous. The functional $\delta_m(\mathcal{J})$ on either of the spacelike surfaces Σ is the geometric measurement

$$\delta_m(\mathcal{J}) = \int_{\Sigma} \delta^3_m \mathcal{J}^{\mu} d\sigma_{\mu} = \int \delta^3(x-m)\mathcal{J}^{\mu}(x)u_{\mu}(x)d^3x \tag{2.7}$$

$$= \mathcal{J}^{\mu}u_{\mu}(m)\ , \tag{2.8}$$

where u is the timelike velocity, orthogonal to Σ, of the space-time observer.

From the Lagrangian given for the theory [1], the geometric current of an arbitrary matter excitation has the general form

$$\mathcal{J}^{\mu} = \tilde{e}\iota^{\mu}Xe \tag{2.9}$$

in terms of the frame e, an orthonormal set of the algebra ι, the correlation on the spinor spaces $\sim$, and the group generator X of the variation.

It should be noted that the frame e is associated with the set of states of an irreducible representation of the geometric field. It does not represent a single physical state, but a collection of physical states. For any operator Λ in the algebra, as the column vectors of the frame e, we may select the eigenvectors ϕ corresponding to Λ. Then we may write

$$\Lambda e=\Lambda(\phi_1,\phi_2,\ldots)=(\lambda^1\phi_1,\lambda^2\phi_2,\ldots)=e\lambda\ , \tag{2.10}$$

where λ is the diagonal matrix formed by the eigenvalues λ_i.

Accordingly, the results of the measurement given by Eq. (2.8), in coordinates adapted to the four-velocity u, is

$$\delta_m(\mathcal{J})=\mathcal{J}^0=\tilde{e}\iota^0 Xe=\tilde{e}\Lambda e\ , \tag{2.11}$$

which defines an associated operator Λ.

In this expression, we should note that $\tilde{e}$ is the group inverse of e, and the correlated product in the spinor space is a scalar. The product $\tilde{e}e$ gives a unit matrix of scalars, and the measurement values of the current $\mathcal{J}$ coincide with the diagonal matrix formed with its eigenvalues,

$$\delta_m(\mathcal{J})=\tilde{e}e\lambda=\lambda=\begin{bmatrix}\lambda_1 & & \\ & \cdots & \\ & & \lambda_i\end{bmatrix}. \tag{2.12}$$

This result agrees with one of the postulates of quantum mechanics. The mathematical content of Eq. (2.12) is truly independent of a physical interpretation of the frame e. In particular, it does not require, but allows, that e is a probability amplitude.

The result of the measurement essentially equals the value of the current at the representative point m. Equivalently it is the average over a characteristic three-volume V of Σ,

$$\langle\mathcal{J}\rangle=\frac{1}{V}\int_V\frac{\delta Q^\mu}{\delta\lambda}d\sigma_\mu=\frac{1}{V}\int_V\mathcal{J}^\mu d\sigma_\mu\ . \tag{2.13}$$

This averaging, indicated by $\langle\ \rangle$, is similar to the operation of taking the expectation value of an operator in wave mechanics.

The different generators of the group produce excitations whose properties may be investigated by measuring the associated geometric currents. In particular, we are interested here in currents associated with generators of the compact subgroups, which were used to characterize the induced representations.

III. GEOMETRIC SPIN

The concept of spin is related to rotations. In geometric theory the compact even generators form an su(2) subalgebra which is related to the rotation algebra. The group homomorphism between this SU(2) subgroup and rotations is

$$R^a_b=\frac{1}{2}\mathrm{tr}(g^\dagger\sigma_b g\sigma^a)\ , \tag{3.1}$$

where $g\in\mathrm{SU}(2)$ and $R\in\mathrm{SO}(3)$. The isomorphism between this SU(2) and the compact even subgroup of SL(4,$\mathbb{R}$) is the well-known isomorphism between the complex numbers and a subalgebra of the real 2×2 matrices,

$$1\leftrightarrow\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ ,\quad i\leftrightarrow\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\ . \tag{3.2}$$

The isomorphism given by Eq. (3.2) is not accidental, but is part of the conceptual definition of the geometric Clifford algebras as a generalization of the complex numbers and the quaternions. These algebras, and the spinor spaces on which they act, have well-defined complex structures.

If we consider that the generators $\kappa^i\kappa^j$ that belong to su(2) are the rotation generators, the associated geometric current is the angular momentum. For example, the result of the measurement of this current in a preferred direction $\mathcal{J}^3$, using as illustration Eq. (2.13) for the expectation value,

$$\langle\mathcal{J}^3\rangle=\frac{1}{V}\int d\sigma_\mu\left[\Pi^\mu\frac{de}{d\lambda}+L\frac{dx^\mu}{d\lambda}\right]\ , \tag{3.3}$$

where the variational δe is generated by $\kappa^1\kappa^2$. Then, for a flat metric,

$$\langle\mathcal{J}^3\rangle=\frac{1}{V}\int d\sigma_\mu\tilde{e}\iota^\alpha u^\mu_\alpha\frac{de}{d\lambda}=\frac{1}{V}\int d^3x\,\tilde{e}\iota^0\frac{de}{d\lambda}\ . \tag{3.4}$$

Since ι^0 commutes with $\kappa^1\kappa^2$ and $\iota^0\iota^0$ is -1, we may use Eq. (3.2) to identify

$$\iota^0\rightarrow iI\ ,\quad \kappa^1\kappa^2\rightarrow i\sigma^3\ , \tag{3.5}$$

and express the variation generated by $\kappa^1\kappa^2$ as the differential of Eq. (3.1),

$$\delta R^a_b=\frac{i\lambda}{2}\mathrm{tr}(\sigma_b\sigma^3\sigma^a-\sigma^3\sigma_b\sigma^a)\ . \tag{3.6}$$

The only nonzero elements are

$$\delta R^1_2=-\delta R^2_1=2\lambda=\theta\ , \tag{3.7}$$

which represents a rotation by angle θ in the 1-2 plane. This rotation induces a change in the functions on three-space, giving a total variation for e of

$$\delta e=\left[\frac{i}{2}\sigma^3 e+(x\partial_y-y\partial_x)e\right]\delta\lambda\ , \tag{3.8}$$

which leads to

$$\langle\mathcal{J}^3\rangle=\frac{1}{V}\int d^3x\,\tilde{e}\{\tfrac{1}{2}\sigma^3-i(x\partial_y-y\partial_x)\}e\ . \tag{3.9}$$

The factor $\frac{1}{2}$ indicates, of course, that the SU(2) parameter λ is half the angular rotation, due to the 2-1 homomorphism between the two groups.

This calculation was done, for simplicity, with only one component. It is clear that if we use the three spatial components we get

$$\langle\mathcal{J}^a\rangle=\frac{1}{V}\int d^3x\,\tilde{e}\{\tfrac{1}{2}\sigma^a-i\epsilon^{abc}(x_b\partial_c)\}e\ , \tag{3.10}$$

where the expression in parentheses is the angular-momentum operator Λ in quantum mechanics. If e is an eigenframe of this operator, we obtain the values of the components of the angular momentum associated with a fluctuation related to the spin-$\frac{1}{2}$ representation.

The result is

$$\langle \Lambda \rangle = \frac{1}{V} \int \tilde{e} \Lambda e \, d^3x = \frac{1}{V} \int \tilde{e} e \lambda \, d^3x \ . \tag{3.11}$$

As before, the product $\tilde{e}e$ gives a unit matrix of scalars, and we may construct the integral in the base space

$$\int \tilde{e} e \, d^3x = IV \ , \tag{3.12}$$

where I is the identity and V is the characteristic volume. It is clear that we may introduce a volume-normalized e by dividing by V. The measured value of the operator Λ coincides with the diagonal matrix formed with its eigenvalues, as indicated above:

$$\langle \Lambda \rangle = \begin{bmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_i \end{bmatrix} = \lambda \ . \tag{3.13}$$

In the case described, the frame states are pure (quantum terminology) with respect to the angular momentum. Both the frame and operator may be diagonalized simultaneously, or equivalently they commute with each other.

In general, the frame is not pure with respect to the operator. Thus the result of the measurements is not the diagonal elements of the operator (eigenvalues). If we designate the columns of e by Φ_α and the files of $\tilde{e}$ by $\overline{\Phi}^{\beta}$, for the measurements we have the matrix

$$\rho_\alpha^\beta = \frac{1}{V} \int \overline{\Phi}^{\beta} \Lambda \Phi_\alpha d^3x \ , \tag{3.14}$$

which corresponds to the density matrix (for the observable operators Λ).

The frame sections play the role of wave functions and the group generators play the role of quantum operators. These similarities between our geometric theory and quantum mechanics provide essentially equivalent results. There are differences; in particular, wave functions have a complex structure and our frame sections have a Clifford structure. Rather than a contradiction, this difference is a generalization, since there are complex structures in different subspaces of the geometric Clifford algebra. It is possible to introduce spaces of sections, but they certainly may have a structure more general than a Hilbert space. The geometrical and groupal elements in the theory actually determine many of its physical features.

IV. GEOMETRIC CHARGE

The geometric source current J is a generalization of electric current. The three compact generators κ_0, κ_5, and $\kappa_1\kappa_2\kappa_3$ are equivalent as electromagnetic generators within the theory because there are automorphisms that transform any one of them into any other. It follows that the set ι^α, that enters into the current, is defined up to an algebra automorphism. This allows us to take the ι^0 element as any of three electromagnetic generators without changing the physical content of the theory.

The generalized source current J is the canonical current $\mathcal{J}$ corresponding to a variation generated by an electromagnetic generator. In order to see this, we choose the set κ^μ for ι^μ, and look for a variation generator that results in an automorphism of the current. In other words, we look for a generator that gives a set equivalent to the set κ^μ by right multiplication. A generator that accomplishes this is κ^5. For a different choice, the automorphism would be a different one, but it would lie in the electromagnetic sector. Thus it is clear that the current J corresponds to variations generated by the electromagnetic sector.

When we make a measurement of this canonical current $\mathcal{J}$, we are measuring the charge associated with the fluctuation of e related to a fundamental irreducible representation of the group. If we neglect the gravitational part, the metric is flat, and the expression for the measurement is

$$\begin{aligned} \langle \mathcal{J} \rangle &= \frac{1}{V} \int \mathcal{J}^\mu d\sigma_\mu = \frac{1}{V} \int \tilde{e} \kappa^\alpha u_\alpha^\mu \dot{e} d\sigma_\mu \\ &= \frac{1}{V} \int \tilde{e} \kappa^0 \kappa^5 e \, d^3x \ , \end{aligned} \tag{4.1}$$

where it is understood that we are working in the bundle SM which is the Whitney sum of the associated spinor vector bundle VM and its conjugate. Explicitly, in terms of elements of VM the last equation is written as

$$\langle J \rangle = \frac{1}{V} \int d^3\kappa \begin{bmatrix} e^{-1} & \\ & \tilde{e} \end{bmatrix} \begin{bmatrix} \kappa^0\kappa^5 & \\ & -\kappa^0\kappa^5 \end{bmatrix} \begin{bmatrix} e & \\ & \tilde{e}^{-1} \end{bmatrix} . \tag{4.2}$$

It should be emphasized that the matrices in the last equation are 8×8 real matrices, the double-dimensional representation used in the SM bundle, as indicated in the Appendix. A more complete discussion is given in previous publications. The even generators may be written as 4×4 complex matrices using the isomorphisms of Eq. (3.2).

In Eq. (4.2), it is possible to substitute any equivalent generator for $\kappa^0\kappa^5$. It is possible, to choose a frame section e corresponding to eigenvectors of the fundamental representation of SU(2). In fact, since κ^5 commutes with $\kappa^1\kappa^2$, it is also possible to choose a frame corresponding to common eigenvectors of these two anti-Hermitian generators belonging to the two su(2) subalgebras in sl(4,$\mathbb{R}$). The eigenvalues correspond to the quanta of spin and charge, as follows:

$$\kappa^1\kappa^2 = \begin{bmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{bmatrix} , \tag{4.3}$$

$$\kappa^5 = \begin{bmatrix} -i & & & \\ & -i & & \\ & & i & \\ & & & i \end{bmatrix} , \tag{4.4}$$

which is precisely the explicit form of these matrices when working on the bundle SM. Of course it is usual to work with the associated Hermitian operators obtained by multiplication by i, with real eigenvalues ± 1. Nevertheless the use of the anti-Hermitian expression is natural since they are generators of the two compact su(2) subalgebras.

If e is an eigenframe of the generator, we get

$$\langle J \rangle = \pm i \ . \tag{4.5}$$

In other words, the result of this measurement for the fundamental representation is a quantum number equal to ± 1. This conserved number may be interpreted as the quantum of charge. It is known that the electron charge plays two roles, one as the quantum of charge and the other as the square root of the fine-structure coupling constant α.

In order to be able to reduce the theory to electromagnetism we must account for Coulomb's coupling constant $k/4\pi$. This k may be absorbed into the definition of the current in Eq. (1.1), but at the end it must be identified. It is better to show it explicitly and to keep the frame e separate, as a section in the principal bundle, so that the conjugate $\tilde{e}$ is the dual inverse of e, and the product $\tilde{e}e$ is the unit matrix. The units of k are $ml^3t^{-2}q^{-2}$. The dimensionless fine-structure constant α is given by $ke^2/4\pi\hbar c$. If we set $k=4\pi$, the units correspond to the Gaussian system, where Coulomb's constant is 1. If we set $k=1$, we obtain the Heaviside-Lorentz system where Coulomb's constant is $1/4\pi$. In these cases, the connection Γ corresponds to eA, in terms of the potential A and the electron charge e. On the other hand, it seems better to consider that the geometric theory introduces a natural unit of charge by defining the electromagnetic potential equal to the connection. In this form the new geometric unit of charge (the "electron") equals e coulombs, and Coulomb's constant becomes the fine-structure constant α. With our definition of current and coupling constant $(4\pi\alpha)$ the calculated charge of the electron is ± 1. In arbitrary units, the calculated quantum of charge q is $\pm(4\pi\alpha\hbar c/k)^{1/2}$.

In other words, when there is only a U(1) electromagnetic field in flat space, our field equations reduce to

$$d^*d\,\Gamma = 4\pi\alpha J \ . \tag{4.6}$$

A particular solution for a static spherically symmetric connection Γ is

$$\Gamma_0 = \frac{\alpha q}{r} \ , \tag{4.7}$$

where q is the charge in electrons. If we now change our units to the Gaussian system (1 electron $=e$ coulombs), where $\alpha = e^2$,

$$\Gamma_0/e = \Phi = \frac{\alpha q}{er} = \frac{qe}{r} \ , \tag{4.8}$$

which is Coulomb's law in terms of the charge qe in coulombs.

The expression for the frame section in SM indicates that the charge eigenvalues corresponding to $\bar{e}^{\,-1}$, which occupy the lower right submatrix, are the negative of those of e, which occupy the upper left submatrix. The frame section e in VM represents charged matter, and its conjugate frame section represents matter with an opposite charge. We may conjecture that charge conjugation is represented by the involution $\bar{e}^{\,-1}$. The source current may be generalized to include opposite charges by adding a term depending on the involuted fields (conjugation):

$$J^\mu = \tilde{e}\iota^\alpha u^\mu_\alpha e - \overline{e\iota}^\alpha u^\mu_\alpha \tilde{\bar{e}} \ . \tag{4.9}$$

V. CONCLUSIONS

We have shown that it is possible to introduce, in the geometric unified theory, a hypothesis concerning the mathematical representations of measurable properties of geometric excitations. Accordingly, the measurement process of an excitation property around a geometric matter section is defined as a functional of the geometric current that is the generator of the excitation. For point-like excitations (point particles) the functional reduces to the Dirac functional, leading to the expression for expectation values. Because of the properties of the sections e, the results of the measurement are the eigenvalues of the generators (operators) of the excitation.

The angular momentum is the canonical geometric current associated with a variation of the sections generated by a rotation. This leads to the expression of total angular momentum as the differential and matricial operator of quantum mechanics. Similarly, the electric charge is represented by the canonical current associated with a variation of sections generated by the electromagnetic sector.

The measurement of the angular-momentum current of an excitation of a section (the fundamental representation) results in a quantum number $\pm\frac{1}{2}$ $[\pm(\hbar/2)$ in other units]. Similarly, since electromagnetism is related to other SU(2) contained in SL(4,ℝ), using its generators the measurement of charge is ± 1. $[(4\pi\alpha\hbar c/k)^{1/2}$ in other units.] We should emphasize again that the natural unit of electric charge is the one that makes the electromagnetic potential coincide with a component of the geometric connection (e disappears for minimal coupling).

Of course, these ideas apply to the fundamental representation of the group which corresponds to spin $\hbar/2$ and charge e. This representation forms a building block from which higher-dimensional irreducible representations may be constructed. For such fields the matrices are of higher dimensions and should have eigenvalues of $n\hbar/2$ and qe.

The picture that is emerging from this theory is that differential geometry is the germ of quantum physics. Through nonlinear field equations, matter determines the geometry and must obey integrability equations of motion. The equations imply a generalized Dirac equation for the sections e which play the role of wave functions representing matter. An irreducible fluctuation (particle) is an irreducible representation of the group carrying certain discrete numbers. In this manner the discreteness of quantum theory arises in compatibility with the continuity of differential geometry. The response to a microscopic measurements reveals these discrete numbers.

In summary, we have shown that measurements in our theory are not only compatible with some results of standard quantum mechanics, but that additionally they imply the existence of the quantum e of the electric charge. Any geometric measurement of charge gives an integer multiple of e.

APPENDIX

The structure group G acts naturally on the double spinor space S by a representation ρ:

$$\rho\colon G \to {}^2G \ , \tag{A1}$$

$$\rho(g)=\begin{bmatrix} g & \\ & \bar{g}^{-1}\end{bmatrix} . \tag{A2}$$

The map ρ induces its derivative map at the identity

$$\rho_{*I}\colon A \to {}^2A \tag{A3}$$

$$\rho_{*I}(a)=\begin{bmatrix} a & \\ & -\bar{a}\end{bmatrix} , \quad a\in A \ , \tag{A4}$$

which has an inverse.

The Clifford product in A induces a product in ${}^2A$,

$$\begin{bmatrix} a & \\ & -\bar{a}\end{bmatrix}\circ\begin{bmatrix} b & \\ & -\bar{b}\end{bmatrix}\to {}^2A \ , \tag{A5}$$

by means of the expression, for $a',b'\in {}^2A$,

$$a'b'=\rho_{*I}(\rho_{*I}^{-1}(a') \ \ \rho_{*I}^{-1}(b')) \ , \tag{A6}$$

which explicitly gives

$$\begin{bmatrix} a & \\ & -\bar{a}\end{bmatrix}\circ\begin{bmatrix} b & \\ & -\bar{b}\end{bmatrix}=\begin{bmatrix} ab & \\ & -\bar{b}\,\bar{a}\end{bmatrix} . \tag{A7}$$

If we represent the matrices $a'\in {}^2A$ by their A component a, the $\circ$ product may be indicated by the Clifford product in A,

$$a'\circ b'\approx ab \ . \tag{A8}$$

This convention is used throughout the paper. Practically this means that all calculations may be made using elements $a\in A$. At the end, the other component in ${}^2A$ may be obtained by conjugation $-\bar{a}$:

$$a'b'=\rho_{*I}[\rho_{*I}^{-1}(a') \ \ \rho_{*I}^{-1}(b')] \ , \tag{A9}$$

which explicitly gives

$$\begin{bmatrix} a & \\ & -\bar{a}\end{bmatrix}\circ\begin{bmatrix} b & \\ & -\bar{b}\end{bmatrix}=\begin{bmatrix} ab & \\ & -\bar{b}\,\bar{a}\end{bmatrix} . \tag{A10}$$

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