Canonical and measured phase distributions

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We derive relationships between canonical and measured phase distributions for quantum-oscillator states in the semiclassical regime. First, we extend the formalism for the canonical phase to include external measurement-induced uncertainty. We require that a phase shifter shifts a phase distribution while a number shifter does not change it. These axioms determine pure canonical phase distributions uniquely while a noisy distribution can be interpreted as a weighted average of pure phase distributions. As a second step, we show that measured phase distributions, i.e., s-parametrized phase distributions fulfill approximately the axioms of noisy canonical phase, and we derive simple analytical expressions for the corresponding weight functions. Our analysis thus bridges all three conceptions of quantum-optical phase (canonical phase, s-parametrized phase, phase from measurements) and provides important physical insight into the relationship between them.

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I. INTRODUCTION

Although the quantum-optical phase is still a controversial subject [1,2], recently significant progress has been made in unifying the various different conceptions of phase [3]. First, the results of different formalisms embodying the concept of phase as an observable canonically conjugate to the photon number have been shown to be physically equivalent. In particular, the phase distribution associated with the Helstrom-Shapiro-Shepard probability operator measure [4,5] is equivalent to that derived from the Pegg-Barnett formalism [6] for physical states in the infinite dimensional limit [5,7,8]. Also, the Newton-Barnett-Pegg formalism [9] where the Hilbert space has been doubled gives the same physical results as the formalism mentioned above [10]. Considered purely mathematically, these approaches are quite distinct. However, the phase-distribution function for a given physical state is the same irrespective from which formalism it has been derived. We call this common distribution a *canonical phase distribution* since the Pegg-Barnett formalism [6], for instance, is motivated by the requirement that the physical quantity *phase* be canonically conjugate to the *photon number* similar to definitions of the standard variables *position* and *momentum*.

A second conception of phase [3] is based on examining phase properties via s-parametrized quasiprobability distributions of position and momentum [11]. A phase distribution is obtained from integrating the quasiprobability distribution over the radial coordinate. This conception is not motivated by the complementarity of number and phase nor is it free from arbitrariness. Which s-parametrized phase distribution should be taken? Phase from the Wigner function [12] has at least the merit of coinciding with the canonical phase in the limit of very large intensities [13]. On the other hand, it has the drawback of yielding negative distribution functions for some physical states in the quantum regime [14]. However, it was recently shown [15,16] that s-parametrized phase distributions describe experimentally measured phase probability distributions [17,18] when the parameter s is less or equal to -1. Thus, the motivation for this concept comes *a posteriori* from experiment.

At first glance, the third conception [3], the operational approach to quantum phase [18,19,20] introduces an element of subjectivity into the definition of a physical quantity, especially for weak fields in the quantum regime. Possible experiments are different and depend on the intentions of their designers. So it is highly remarkable that most of them yield the same results under reasonable assumptions [21]. The distributions measured in these experiments [22] can be interpreted as being derived from the Q function [15] or a smoothed Q function in the case of inefficient detection [16]. We call them *measured phase distributions* [23].

As a last step, it remains to be shown how the *measured phase distributions* are related to the *canonical phase distribution.* It would be very surprising indeed if there did not exist a deeper relationship between both, since the experiments are designed as classical phase-measurement schemes and in the classical limit, measured and canonical phase coincides. We note that significant differences occur in the extreme quantum regime of low photon numbers. Here, we are interested in a semiclassical domain of relatively large photon numbers. What would we expect? Since the phase-measurement schemes realize simultaneous yet noisy measurements of position and momentum [24,25], we anticipate that the measured phase distributions should be somewhat broader than the canonical distribution. A comparison of phase variances [26] and calculations for particular states [27] support this assertion. In this paper, we quantify the general asymptotic relationship between canonical and measured phase distributions. Before we address this problem, we provide in Sec. II an alternative and more general theoretical approach to

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canonical phase distributions. It is not motivated by a quantum-estimation problem as in Shapiro's and Shepard's classic paper [5] nor by requiring the construction of Hermitian phase operators. Our approach is based purely on the complementarity of phase and photon number. It has the merit that it can describe phase as a variable influenced by external noise. This element of external statistics and the basic consequences of our axioms of complementarity will give us the key for relating canonical phase to measured phase. We will show in Sec. III that in the semiclassical domain measured phase distributions are averaged pure canonical distributions with respect to a distribution function of reference phases. The latter depends on the mean photon number and the s parameter in a simple way. In the classical limit, it reduces to a δ function, as we would expect. The results are summarized in Sec. IV.

II. CANONICAL PHASE DISTRIBUTIONS

A. Axioms for quantum phase

Preliminaries

In this approach to quantum phase, we consider phase distributions. Since a probability distribution of phase contains all statistical information on the phase properties for a given state it represents the physical quantity *phase* completely. Realistic measurements of phase [24] always involve extra noise beyond that due to the intrinsic quantum phase fluctuations described by the canonical phase distribution [4–6,9]. Consequently, we need a general method that allows the description of phase in the presence of noise. For this we use *probability operator measures* (POM's) [28], as in Shapiro's and Shepard's classic paper [5]. Within the POM formalism, we introduce a probability distribution $\Pr(\varphi)$, $\varphi\in(-\pi,\pi]$, as [29]

$$\Pr(\varphi)=\mathrm{Tr}\{\hat{\rho}\hat{\Pi}(\varphi)\} , \tag{1}$$

where $\hat{\rho}$ is the density matrix and $\hat{\Pi}(\varphi)$ denotes a set of suitable operators parametrized by the phase variable φ. Since probability distributions are real functions, $\hat{\Pi}(\varphi)$ must be Hermitian,

$$\hat{\Pi}(\varphi)^{\dagger}=\hat{\Pi}(\varphi) . \tag{2}$$

A probability distribution is normalized to unity. Consequently, the set of operators $\hat{\Pi}(\varphi)$ must be normalized as well,

$$\int_{-\pi}^{+\pi}d\varphi\,\hat{\Pi}(\varphi)=1 . \tag{3}$$

Lastly, a probability distribution is nonnegative

$$\Pr(\varphi)\geq 0 . \tag{4}$$

This implies that the eigenvalues of $\hat{\Pi}(\varphi)$ must be nonnegative as well. This property together with (2) and (3) is sufficient to identify $\hat{\Pi}(\varphi)$ as a density operator which is often laxly called a *phase state*. One important point is, however, that the operator $\hat{\Pi}(\varphi)$ might be unnormalizable, as the London phase states [30] are, although we note that such states can be represented in larger spaces, e.g., the rigged Hilbert space [31]. Another important point is that a phase state $\hat{\Pi}(\varphi)$ can be a mixed state. This simply means that the measure of φ is not precise and so the probability distribution $\Pr(\varphi)$ represents the results of a noisy measurement of φ. For this reason, we call $\Pr(\varphi)$ a *noisy phase distribution* and $\hat{\Pi}(\varphi)$ a *mixed phase state.*

The basic expression (1) for a probability distribution in quantum mechanics can be also expressed as an overlap of Wigner functions [32,33],

$$\Pr(\varphi)=2\pi\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dp\;W(x,p)W(x,p;\varphi) . \tag{5}$$

Here,

$$W(x,p)\equiv\int_{-\infty}^{+\infty}\frac{dy}{\pi}\,e^{2ipy}\langle x-y|\hat{\rho}|x+y\rangle , \tag{6}$$

and

$$W(x,p;\varphi)\equiv\int_{-\infty}^{+\infty}\frac{dy}{\pi}\,e^{2ipy}\langle x-y|\hat{\Pi}(\varphi)|x+y\rangle \tag{7}$$

are the Wigner functions of the quantum state $\hat{\rho}$ and the mixed phase state $\hat{\Pi}(\varphi)$, respectively [34]. The overlap relation (5) closely resembles the probability overlap in classical statistical physics and hence it helps us to understand quantum mechanics more intuitively. In addition, it is a quite useful tool for finding the semiclassical asymptotics of a given quantum-mechanical problem. Later on we will use the overlap relation for understanding the asymptotic relations between measured phase distributions.

So far, we have briefly summarized the general properties of POM's. Now, we turn to two specific requirements for a noisy quantum-phase distribution. Our goal is the definition of quantum-optical phase as canonically conjugate variable with respect to photon number. How can this be achieved? We wish to treat *number* and *phase* similarly to the basic canonically conjugate variables *position* and *momentum.* We could, for instance, extend the canonical commutation relation for position and momentum operators $\hat{q}$ and $\hat{p}$,

$$[\hat{q},\hat{p}]=i \tag{8}$$

to a commutation relation for number $\hat{n}=\hat{a}^{\dagger}\hat{a}$ and phase $\hat{\varphi}$. (As usual, $\hat{a}$ denotes the annihilation operator.) It is well known, however, that a Hermitian phase operator $\hat{\varphi}$ does not exist on the Fock space. Instead, we are considering not phase operators but phase distributions. In defining *phase* as canonically conjugate to *photon number,* we must translate some typical properties of position and momentum distributions into the language of number and phase and regard them as being fundamental. We obtain from the canonical commutation relation that the operator $\hat{S}(q_0)=\exp(-iq_0\hat{p})$ shifts position eigenstates by the amount q_0 while $\hat{T}(p_0)=\exp(ip_0\hat{q})$ shifts momentum eigenstates by p_0, see, for instance, Ref. [35]. Hence the momentum distribution $\mathrm{Tr}\{\hat{\rho}|p\rangle\langle p|\}$ is shifted by p_0 when $\hat{T}(p_0)$ is applied to the quantum state $\hat{\rho}$. On the other hand, the momentum distribution is not changed when $\hat{S}(q_0)$ is applied to $\hat{\rho}$, i.e., when the position is shifted. Position and momentum are strictly independent since shifting one variable does not affect the other. We may regard this mutual independence of the canonically conjugate variables as being fundamental and require the same for *number* and *phase*.

Axioms

We require that a phase-distribution function of a single mode satisfies the following axioms.

(A) A phase shifter shifts the phase distribution.

(B) A number shifter does not change the phase distribution (complementarity). To be explicit, a phase shifter is represented by the unitary transformation operator,

$$\hat{U}(\phi)\equiv\exp(i\phi\hat{a}^{\dagger}\hat{a}) \ . \tag{9}$$

Axiom (A), thus, means that

$$\begin{aligned}\Pr{}'(\varphi)&\equiv\mathrm{Tr}\{\hat{U}(\phi)\hat{\rho}\hat{U}(\phi)^{\dagger}\hat{\Pi}(\varphi)\}\\&=\Pr(\varphi-\phi)\\&=\mathrm{Tr}\{\hat{\rho}\hat{\Pi}(\varphi-\phi)\} \ .\end{aligned}\tag{10}$$

A number shifter is expressed by the operator

$$\hat{E}\equiv\sum_{n=0}^{\infty}|n+1\rangle\langle n| \ . \tag{11}$$

It shifts the photon-number distribution up by one step. ($|n\rangle$ denotes a Fock state.) The operator $\hat{E}$ is nothing but the Susskind-Glogower exponential phase operator $\widehat{\exp}(-i\varphi)$ [36]. As is well known, it is neither a unitary nor a strictly invertible operator, since $\hat{E}^{\dagger}$ annihilates the vacuum state. However, we are not concerned about the problems of $\hat{E}$ as an exponential phase operator here. We only use the number-shifter property of $\hat{E}$. Axiom (B) thus requires

$$\Pr{}'(\varphi)\equiv\mathrm{Tr}\{\hat{E}\hat{\rho}\hat{E}^{\dagger}\hat{\Pi}(\varphi)\}=\Pr(\varphi) \ . \tag{12}$$

Comments

Both axioms (A) and (B) together determine a phase distribution. What do they mean physically? Axiom (A) is almost trivial. We only require that the phase distribution should indeed reflect the basic feature of quantum phase, i.e., that a phase shifter is a phase-distribution shifter. Naturally, many phase-sensitive quantities have the property (A). Axiom (B) is more specific [37]. It means that the distribution function $\Pr(\varphi)$ contains the properties of quantum phase and nothing else. It must not reflect any properties of the canonically conjugate variable, the photon number. Hence (B) means that phase should be complementary to photon number. We also note, however, that if a particular distribution function $\Pr(\varphi)$ satisfies the axioms then so does the weighted average $p_1\Pr(\varphi)+p_2\Pr(\varphi+\delta)$ of this function and the phase-shifted distribution $\Pr(\varphi+\delta)$, which describes uncertainty in the reference phase. We interpret this as the axioms allow for a noisy measure of phase. The nature of this noise is very special in that the resulting distribution still satisfies the axioms of complementarity. Thus our approach here contains, in essence, the basic prescription for describing a noisy measurement of phase without contamination from the complementary observable, photon number. Now we consider the detailed consequences of both axioms.

B. Consequences

Fock representation

We express the noisy phase probability distribution in the Fock basis,

$$\Pr(\varphi)=\sum_{n,m=0}^{\infty}\langle m|\hat{\Pi}(\varphi)|n\rangle\langle n|\hat{\rho}|m\rangle \ . \tag{13}$$

We use the axiom (A),

$$\begin{aligned}\Pr(\varphi)&=\Pr[0-(-\varphi)]\\&=\sum_{n,m=0}^{\infty}\langle m|\hat{\Pi}(0)|n\rangle\langle n|\hat{U}(-\varphi)\hat{\rho}\hat{U}(\varphi)|m\rangle\\&=\sum_{n,m=0}^{\infty}\langle m|\hat{\Pi}(0)|n\rangle e^{i(m-n)\varphi}\langle n|\hat{\rho}|m\rangle \ ,\end{aligned}\tag{14}$$

define the coefficients

$$B_{n,m}\equiv 2\pi\langle m|\hat{\Pi}(0)|n\rangle \ , \tag{15}$$

and obtain

$$\Pr(\varphi)=\frac{1}{2\pi}\sum_{n,m=0}^{\infty}B_{n,m}e^{i(m-n)\varphi}\langle n|\hat{\rho}|m\rangle \ . \tag{16}$$

Since the operator $\hat{\Pi}(\varphi)$ is Hermitian, the matrix $B_{n,m}$ must be Hermitian as well,

$$B_{n,m}=B_{m,n}^{*} \ . \tag{17}$$

Expressions of the type (16) have been known for several phase-dependent distributions for a long time (cf., for instance, Ref. [19]). As we have seen here, the root of these formulas lies in the phase-shifter axiom (A). Now, we consider the consequences of axiom (B):

$$\begin{aligned}\Pr(\varphi)&=\frac{1}{2\pi}\sum_{n,m=0}^{\infty}B_{n,m}e^{i(m-n)\varphi}\langle n|\hat{E}\hat{\rho}\hat{E}^{\dagger}|m\rangle\\&=\frac{1}{2\pi}\sum_{n,m=1}^{\infty}B_{n,m}e^{i(m-n)\varphi}\langle n-1|\hat{\rho}|m-1\rangle\\&=\frac{1}{2\pi}\sum_{n,m=0}^{\infty}B_{n+1,m+1}e^{i(m-n)\varphi}\langle n|\hat{\rho}|m\rangle \ .\end{aligned}\tag{18}$$

Consequently, the B coefficients should have the number-shift invariance as well,

$$B_{n+1,m+1}=B_{n,m} \ . \tag{19}$$

This simple relation will provide us with the key for relating canonical and measured phase distributions.

Convolution

Because of the invariance relation (19) and the Hermitian condition (17) the B-coefficients depend on a single row of free parameters,

$$B_{n,m}=b_{m-n}, \quad b_\nu=\begin{cases} B_{0,\nu} & \text{for } \nu\geq 0, \\ b^*_{-\nu} & \text{for } \nu<0. \end{cases} \tag{20}$$

These parameters characterize all possible noisy phase distributions satisfying both axioms (A) and (B). Using the definition (20) and the Fock expansion (16) for a noisy phase distribution, we find

$$\Pr(\varphi)=\sum_{\nu=-\infty}^{+\infty} e^{i\nu\varphi} b_\nu c_\nu , \tag{21}$$

with

$$c_\nu=\begin{cases} \frac{1}{2\pi}\sum_{n=0}^{\infty}\langle n|\hat{\rho}|n+\nu\rangle & \text{for } \nu\geq 0, \\ c^*_{-\nu} & \text{for } \nu<0. \end{cases} \tag{22}$$

Here the noisy phase distribution $\Pr(\varphi)$ is expressed as a Fourier series. According to the convolution theorem, we obtain

$$\Pr(\varphi)=\int_{-\pi}^{+\pi}\frac{d\phi}{2\pi} g(\phi)\Pr_p(\varphi-\phi) , \tag{23}$$

with

$$g(\phi)\equiv\sum_{\nu=-\infty}^{+\infty} e^{i\nu\phi} b_\nu \tag{24}$$

and

$$\Pr_p(\varphi)\equiv\sum_{\nu=-\infty}^{+\infty} e^{i\nu\varphi} c_\nu=\frac{1}{2\pi}\sum_{n,m=0}^{\infty} e^{i(m-n)\varphi}\langle n|\hat{\rho}|m\rangle . \tag{25}$$

The function $\Pr_p(\varphi)$ is nothing but the Helstrom-Shapiro-Shepard phase distribution [4,5] and the Pegg-Barnett phase distribution [6] for physical states in the infinite-dimensional limit. We call $\Pr_p(\varphi)$ a *pure canonical phase distribution.* The function $g(\phi)$ is real because of the definition (20) of the b coefficients. Moreover, it must be nonnegative and normalized to unity since the distributions $\Pr(\varphi)$ and $\Pr_p(\varphi)$ are nonnegative and normalized for all states. Hence we can interpret $g(\phi)$ as a probability distribution. Our result (23) thus means that any noisy phase distribution $\Pr(\varphi)$ satisfying both axioms (A) and (B) consists of pure canonical phase distributions $\Pr_p(\varphi)$ averaged with respect to a certain probability distribution $g(\phi)$ of reference phases ϕ which represents the noise.

Pure phase distributions

Finally, it remains to be proved that the pure canonical phase distributions $\Pr_p(\varphi)$ are the only ones that deserve the designation "pure," in the sense that they correspond to pure phase states $\hat{\Pi}(\varphi)=|\varphi\rangle\langle\varphi|$. In fact, they are the only distributions having both properties (A) and (B) and a coefficient matrix $B_{n,m}$ that factorizes according to

$$B_{n,m}=B^*_n B_m . \tag{26}$$

The proof is rather simple. Because of the invariance principle (19), we have

$$B^*_{n+\nu}B_{m+\nu}=B^*_n B_m . \tag{27}$$

Setting $n=m$, we obtain $|B_n|^2=|B_0|^2$, and because of the normalization of the phase distribution $|B_n|^2=1$. We express B_n as $\exp(-i\beta_n)$ and obtain

$$B^*_n B_1=e^{i(\beta_n-\beta_1)}=e^{i(\beta_{n-1}-\beta_0)} \tag{28}$$

and, consequently,

$$e^{i\beta_n}=e^{i\beta_{n-1}}e^{i\phi}, \quad \phi=\beta_1-\beta_0 . \tag{29}$$

Applying this relation n times, we get, finally,

$$B_m=e^{-i\beta_n}=e^{-i\beta_0}e^{-in\phi} . \tag{30}$$

Hence, the phase distribution $\Pr(\varphi)$ is

$$\Pr(\varphi)=\frac{1}{2\pi}\sum_{n,m=0}^{\infty} e^{i(m-n)(\varphi-\phi)}\langle n|\hat{\rho}|m\rangle = \Pr_p(\varphi-\phi) . \tag{31}$$

It corresponds to the well-known phase-distribution first introduced by London [30]. Hence up to a reference phase our basic axioms (A) and (B) determine a canonical phase distribution uniquely when we consider a pure distribution, while, in general, any noisy phase distribution satisfying the complementarity axioms can be seen as a statistical mixture of pure canonical phase distributions.

III. MEASURED PHASE DISTRIBUTIONS

A. *Q*-phase distribution

The phase distribution measured in the Noh-Fougères-Mandel experiment [18] is the radius-integrated Q function (Q phase) [15], provided a strong local oscillator and perfect detectors have been employed [21]. Also, in other operational approaches to quantum phase [19,20], the Q-phase distribution is measured. In this Sec. we show that we can interpret this distribution as a smoothed canonical phase distribution. The Q-phase distribution is defined as

$$\Pr_Q(\varphi)=\int_0^\infty dr\,\frac{r}{\pi}\langle re^{i\varphi}|\hat{\rho}|re^{i\varphi}\rangle . \tag{32}$$

($|re^{i\varphi}\rangle$ is a coherent state of amplitude r and phase φ). It reads in terms of the POM formalism,

$$\Pr_Q(\varphi)=\mathrm{Tr}\{\hat{\rho}\hat{\Pi}_Q(\varphi)\} , \tag{33}$$

with

$$\hat{\Pi}_Q(\varphi)\equiv\int_0^\infty \mathrm{d}r\,\frac{r}{\pi}|re^{i\varphi}\rangle\langle re^{i\varphi}| . \tag{34}$$

This type of phase state for the Q phase has been studied in detail by Paul [19].

Wigner function

To compare the Q-phase distribution with the pure canonical phase distribution, we can use the Wigner-function-overlap relation [32], i.e., according to Eq. (5), we express $\Pr_Q(\varphi)$ in terms of Wigner functions,

$$\Pr_Q(\varphi)=2\pi\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dp\; W(x,p)W_Q(x,p;\varphi)\;, \quad (35)$$

with

$$W_Q(x,p;\varphi)=\int_0^{\infty}dr\frac{r}{\pi}W_{\rm coh}(x,p)\;. \quad (36)$$

Here, $W_{\rm coh}(x,p)$ denotes the Wigner function of the coherent state $|re^{i\varphi}\rangle$

$$W_{\rm coh}(x,p)=\frac{1}{\pi}\exp[-(x-2^{1/2}r\cos\varphi)^2-(p-2^{1/2}r\sin\varphi)^2]\;. \quad (37)$$

We substitute

$$x_\varphi=x\cos\varphi+p\sin\varphi,\quad p_\varphi=-x\sin\varphi+p\cos\varphi\;, \quad (38)$$

and obtain

$$W_Q(x,p;\varphi)=\frac{1}{(2\pi)^2}e^{-p_\varphi^2}\{e^{-x_\varphi^2}+x_\varphi\pi^{1/2}[1+\mathrm{erf}(x_\varphi)]\}\;. \quad (39)$$

The Wigner function associated with the Q-phase distribution grows linearly in the phase direction φ. It has a Gaussian profile which originates physically from the vacuum noise involved in realistic measurements of phase [24], see Fig. 1. In contrast, the Wigner function of a pure phase state (cf. Fig. 3 in Ref [34]) grows quadratically in the phase direction. It is much narrower, shows characteristic oscillations, and becomes negative in certain regions. Here, on the other hand, the Wigner function of the Q-phase state is always positive which already indicates that it represents a statistical mixture. (Only Gaussian pure states have non-negative Wigner functions [33].)

Fock representation

In Fock representation, the Q-phase distribution reads

$$\Pr_Q(\varphi)=\frac{1}{2\pi}\sum_{n,m=0}^{\infty}B_{n,m}e^{i(m-n)\varphi}\langle n|\hat\rho|m\rangle\;, \quad (40)$$

with

$$B_{n,m}=\frac{[\frac{1}{2}(n+m)]!}{(n!m!)^{1/2}}\;, \quad (41)$$

as derived some time ago [19]. [Here and elsewhere, we denote the function $\Gamma(x+1)$ by $x!$.] Obviously, $\Pr(\varphi)$ satisfies axiom (A). Moreover, it shows the number-shift invariance (B) to a rather good approximation,

$$B_{n+1,m+1}=\frac{\frac{1}{2}(n+1+m+1)}{(n+1)^{1/2}(m+1)^{1/2}}B_{n,m}\approx B_{n,m}\;. \quad (42)$$

The B coefficients differ by the ratio of the arithmetic and geometric mean of the photon numbers $n+1$ and $m+1$ [38]. When n and m are large, the invariance principle (19) is fulfilled quite well. This means that the Q-phase distribution is approximately a canonical phase distribu-

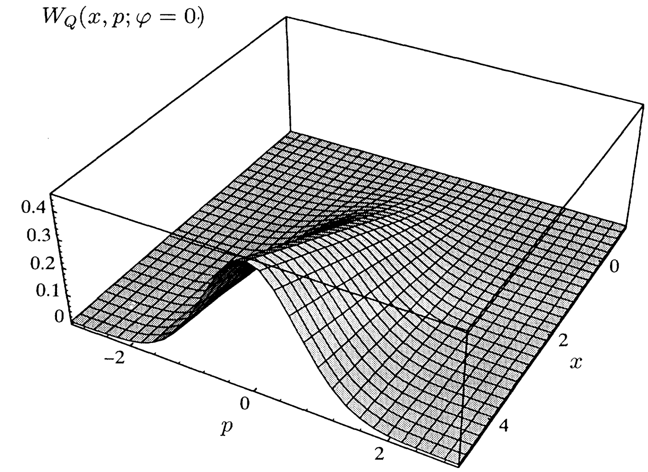

FIG. 1. Wigner function (39) for the Q-phase state (34).

tion. On the other hand, we note that the B coefficients tend to unity in the limit $n,m\to\infty$. Thus, the Q phase coincides with the pure canonical phase for very large photon numbers. Hence, the number-shift invariance cannot be exact. Here, we are interested in an intermediate regime where n and m are relatively large. We will now quantify the asymptotics of the Q-phase distribution.

Convolution

For relatively narrow photon-number distributions compared to the mean photon number, we can approximate

$$B_{n,n+|\nu|}\approx B_{N,N+|\nu|}\;, \quad (43)$$

with N being the mean photon number

$$N=\mathrm{Tr}\{\hat\rho\hat a^\dagger\hat a\}\;. \quad (44)$$

Consequently, the Q-phase distribution reads

$$\Pr_Q(\varphi)=\sum_{\nu=-\infty}^{+\infty}e^{i\nu\varphi}B_{N,N+|\nu|}c_\nu\;, \quad (45)$$

where the c coefficients are defined in Eq. (22). Similarly to noisy canonical phase distributions treated above, we obtain according to the convolution theorem,

$$\Pr_Q(\varphi)=\int_{-\pi}^{+\pi}\frac{d\phi}{2\pi}g(\phi;N)\Pr_p(\varphi-\phi)\;, \quad (46)$$

with

$$g(\phi;N)=\sum_{\nu=-\infty}^{+\infty}e^{i\nu\phi}B_{N,N+|\nu|}\;. \quad (47)$$

Note that the mean photon number N enters the expression (47) for the reference-phase distribution $g(\phi;N)$. We wish to derive an asymptotic expression for $g(\phi;N)$ in case of large N. We replace the Fourier series in Eq. (47) by an integral and use the saddle-point method to evaluate it. According to Stirling's formula [39] [Vol. 1, Eq. 1.18(1)], we obtain from Eq. (41)

$$\ln B_{N,N+|\nu|}\approx\left[N+\frac{|\nu|}{2}+\tfrac{1}{2}\right]\ln\left[N+\frac{|\nu|}{2}+1\right]$$
$$-\tfrac{1}{2}(N+\tfrac{1}{2})\ln(N+1)$$
$$-\tfrac{1}{2}(N+|\nu|+\tfrac{1}{2})\ln(N+|\nu|+1)$$
$$=-\frac{3+2N}{16(N+1)^2}\nu^2+O(|\nu|^3)\ . \quad (48)$$

Hence, we get for the distribution function

$$g(\phi;N)=\int_{-\infty}^{+\infty}d\nu\exp\left[i\nu\phi-\frac{3+2N}{16(N+1)^2}\nu^2\right] \quad (49)$$

and, finally,

$$g(\phi;N)=2\pi\left[\frac{4(N+1)^2}{\pi(2N+3)}\right]^{1/2}\exp\left[-\frac{4(N+1)^2}{2N+3}\phi^2\right]\ . \quad (50)$$

This simple formula describes the asymptotics of the reference-phase distribution $g(\phi;N)$. It is a Gaussian with a width depending roughly inversely on the mean photon number N which reflects the extra noise involved in phase measurements [24]. The distribution $g(\phi;N)$ gets narrower with increasing N and finally tends to a δ function for very large N, since with increasing intensity the influence of this noise decreases. In the macroscopic regime the measurement-induced noise is negligible. This is readily understood from the particular source of extra noise being present in an experimental setup that allows measurement of the Q phase. It is either vacuum noise introducing the apparatus via the unused port of a beam splitter [15,20] or amplification noise [19]. In both cases, the noise becomes negligible when the intensity of the initial field is high. On the other hand, a distribution function $g(\phi)$ that is independent of the field can most easily be interpreted in taking the term "reference-phase distribution" literally, i.e., in identifying the noise with phase instabilities in the reference beam, e.g., a high-intensity laser beam. In contrast, the N-dependent distribution $g(\phi;N)$ shows that the Q-phase distribution is also an (approximative) noisy measure of the pure canonical phase distribution but now the noise relative to the field intensity decreases as the latter increases. In particular, if the field is in a coherent state then both the relative measurement-induced noise and the intrinsic quantum-phase fluctuations of the field vanish at roughly equal rates as the intensity increases. In this respect, the Q phase remains a good measure of the pure canonical phase distribution even for lower values of N. This is, however, a special result for coherent states. For states with narrower phase distributions, e.g., phase-optimized states [40], a reference-phase distribution that decreases at a faster rate is required to preserve the relative accuracy of the noisy phase distribution [26].

Numerical tests

We tested the accuracy of our treatment of the Q phase as a noisy canonical phase for squeezed states [41] having quadrature wave functions,

$$\psi_{\text{sq}}(x)\equiv\langle x|\psi_{\text{sq}}\rangle=(\zeta/\pi)^{1/4}\exp[-(\zeta/2)(x-\sqrt{2}\alpha)^2]\ , \quad (51)$$

see Ref. [12], Eq. (3). Here, ζ is a real and positive parameter which characterizes the squeezing. (It is equivalent to Schleich's s [12]. In order to avoid confusion with the quasiprobability parameter, we denote it by ζ.) The real parameter α characterizes the coherent amplitude of the squeezed state. The mean photon number N of the state $|\psi_{\text{sq}}\rangle$ is given by [41]

$$N=|\alpha|^2+\tfrac{1}{4}(\zeta+\zeta^{-1}-2)\ . \quad (52)$$

We calculated numerically the pure canonical phase distribution (25) using the photon-number probability amplitudes $\langle m|\psi_{\text{sq}}\rangle$ given by Ref. 12, Eq. (4). According to Eq. (46) this distribution was convoluted numerically with the weight function $g(\phi;N)$ of Eq. (50) and compared with the exact Q-phase distribution for squeezed states, as found in Ref. [27], Eqs. (32), (33) with $\mu=\zeta,\alpha_0=\alpha$. Figure 2 shows the exact versus the approximate Q-phase distributions for some squeezed states with mean photon numbers $N=25$. We see that the approximate curve fits quite well the exact one. We observed that for lower mean photon numbers the agreement between both curves becomes worse while for higher intensities they become almost indistinguishable.

B. s-phase distribution

Due to losses in overall detection efficiency the phase from an s-parametrized quasiprobability distribution (s phase) is measured in realistic experiments [16]. The parameter s is, in general, less than -1, which means that the quasiprobability distribution is a smoothed Q function. The measured phase distribution is given by

$$\Pr_s(\varphi)\equiv\int_0^{\infty}\mathrm{d}r\, rW(r\cos\varphi,r\sin\varphi;s)\ . \quad (53)$$

Here, $W(r\cos\varphi,r\sin\varphi;s)$ denotes an s-parametrized quasiprobability distribution [11,16],

$$W(r\cos\varphi,r\sin\varphi;s)=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dp\,W(x,p)\times W_{\text{DT}}(x,p)\ , \quad (54)$$

with

$$W_{\text{DT}}(x,p)=\frac{-1}{\pi s}\exp\left[\frac{1}{s}(x-r\cos\varphi)^2+\frac{1}{s}(p-r\sin\varphi)^2\right]\ . \quad (55)$$

The function $W_{\text{DT}}(x,p)$ can be interpreted as the Wigner function of a displaced thermal state $\hat\rho_{\text{DT}}$. Using some standard expressions for thermal states $\hat\rho_T$ and Ref. [42], we easily obtain the corresponding density operator,

$$\hat\rho_{\text{DT}}=\frac{2}{1-s}\exp\left[-2\,\mathrm{arcoth}(-s)\left[\hat a-\frac{r}{\sqrt{2}}e^{i\varphi}\right]^{\dagger}\times\left[\hat a-\frac{r}{\sqrt{2}}e^{i\varphi}\right]\right]\ . \quad (56)$$

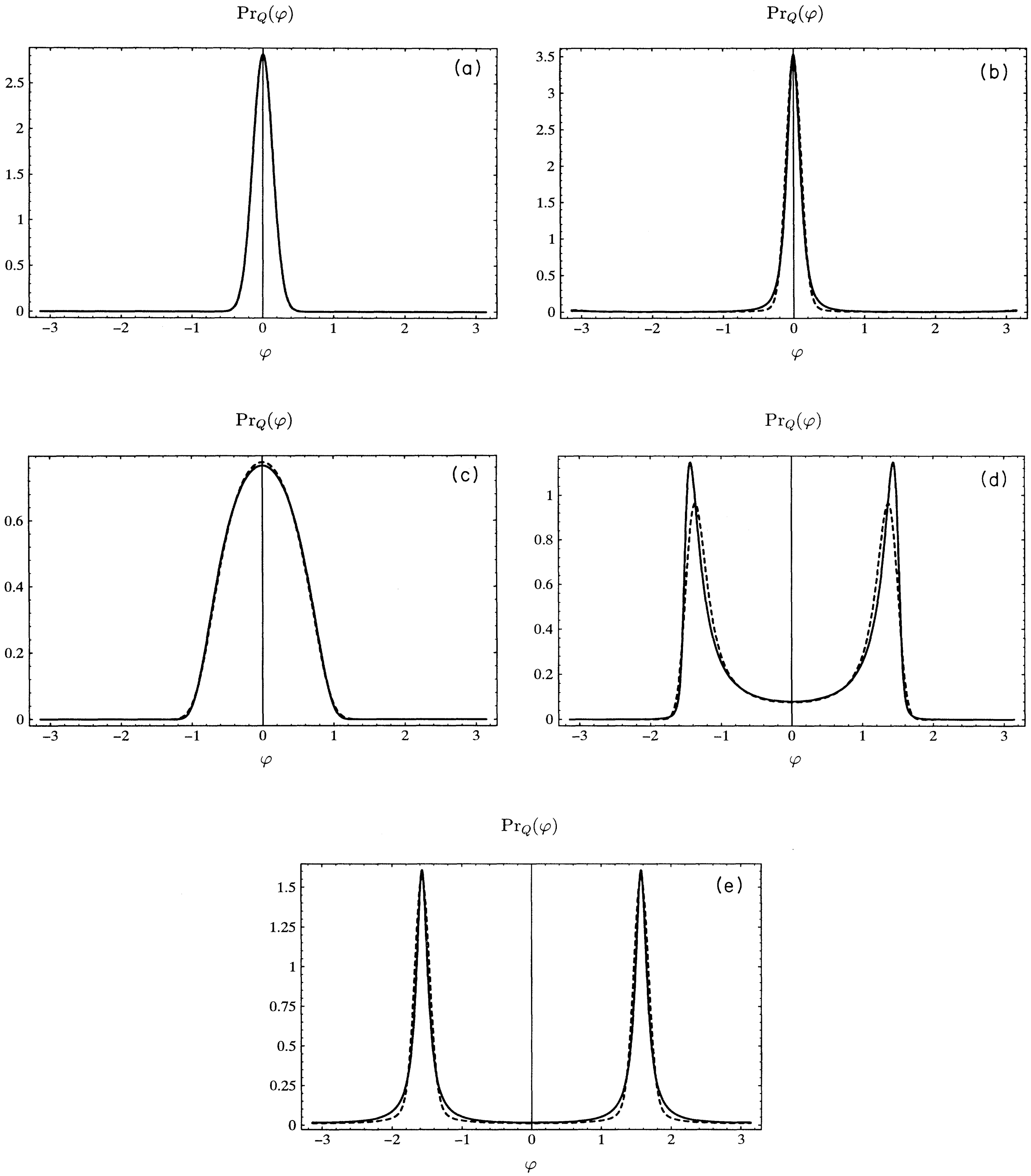

FIG. 2. Comparison of exact Q-phase distributions for squeezed states (51) with the results of our treatment of the Q phase as a noisy canonical phase, i.e., as convolutions (46) of pure canonical phase distributions (25) with weight functions (50). The exact Q-phase distribution (line) and the approximation (dashed line) is plotted for (a) a coherent state with $\alpha=5$ and $\zeta=1$ (no difference between the exact and the approximative distribution is visible); (b) a phase-squeezed state with $\alpha=4.5$ and $\zeta=0.047\,727\,5$; (c) an amplitude-squeezed state with $\alpha=4.5$ and $\zeta=20.9523$ [the inverse squeezing parameter as in (b)]; (d) an amplitude-squeezed state with $\alpha=1$ and $\zeta=97.9898$ showing bifurcation of the phase distribution [12]; and (e) a squeezed-vacuum state with $\alpha=0$ and $\zeta=101.99$. In all cases the mean photon number N was fixed to $N=25$.

According to the Wigner-function-overlap relation [32], we can write Eq. (54) as

$$W(r\cos\varphi, r\sin\varphi;s)=\frac{1}{2\pi}\,\mathrm{Tr}\{\hat{\rho}\hat{\rho}_{\mathrm{DT}}\}\ . \tag{57}$$

We substitute this result in Eq. (53), replace $r2^{-1/2}$ by r, and obtain, finally, the POM form for the s-phase distribution,

$$\mathrm{Pr}_s(\varphi)=\mathrm{Tr}\{\hat{\rho}\hat{\Pi}_s(\varphi)\}\ , \tag{58}$$

with

$$\hat{\Pi}_s(\varphi)=\frac{2}{\pi(1-s)}\int_0^\infty \mathrm{d}r\, r\exp[-2\,\mathrm{arcoth}(-s)(\hat{a}-re^{i\varphi})^\dagger \times(\hat{a}-re^{i\varphi})]\ . \tag{59}$$

In the following paragraphs, we relate the s-phase distribution to the pure canonical phase distribution. First, we calculate the Wigner function for the "s-phase state" $\hat{\Pi}_s(\varphi)$ and then, we apply a similar procedure as for the Q phase. We verify that the number-shift invariance (19) is fulfilled approximately, and calculate the asymptotics of the related reference-phase distribution.

Wigner function

Similar to the Wigner function associated with the Q phase, we obtain from Eqs. (53)–(55) the Wigner function for the "s-phase state" $\hat{\Pi}_s(\varphi)$,

$$W_s(x,p;\varphi)=\frac{-1}{2s\pi^2}\int_0^\infty dr\, r\exp\left[\frac{1}{s}(x-r\cos\varphi)^2+\frac{1}{s}(p-r\sin\varphi)^2\right]. \tag{60}$$

Evidently, $W_s(x,p;\varphi)$ is simply a scaled Q-phase Wigner function,

$$W_s(x,p;\varphi)=\frac{1}{-s}W_Q\left[\frac{x}{\sqrt{-s}},\frac{p}{\sqrt{-s}};\varphi\right]. \tag{61}$$

(Note that $s<-1$.) This indicates that the s-phase distribution is broader than the Q-phase distribution, as we would expect since extra detection noise is involved in inefficient phase-measurement schemes [16]. We now derive an approximative relation which links all s-phase distributions for $s\le 0$. Writing the expression (39) for the Q-phase Wigner function $W_Q(x,p;\varphi)$ in polar coordinates, we find that approximately

$$W_Q(r\cos\theta, r\sin\theta;\varphi)\approx\frac{r}{2\pi^{3/2}}\exp[-r^2(\varphi-\theta)^2] \tag{62}$$

holds for $r\gg 1$. According to the overlap relation (5) written in polar coordinates, and utilizing Eq. (61), we obtain for the s-phase distribution

$$\begin{aligned}\mathrm{Pr}_s(\varphi)&=\int_0^\infty dr\, r\int_{-\pi}^{+\pi}d\theta\, W(r\cos\theta, r\sin\theta)\times\frac{r}{\sqrt{-\pi s}}\exp\left[\frac{r^2}{s}(\varphi-\theta)^2\right]\\&=\int_{-\pi}^{+\pi}d\theta\int_0^\infty dr\, rW[r\cos(\varphi-\theta), r\sin(\varphi-\theta)]\times\frac{r}{\sqrt{-\pi s}}\exp\left[\frac{r^2}{s}\theta^2\right].\end{aligned} \tag{63}$$

We approximate

$$\mathrm{Pr}_s(\varphi)\approx\int_{-\pi}^{+\pi}d\theta\frac{R}{\sqrt{-\pi s}}\exp\left[\frac{R^2}{s}\theta^2\right]\times\int_0^\infty dr\, rW[r\cos(\varphi-\theta), r\sin(\varphi-\theta)]\ , \tag{64}$$

where R is a typical radius of the Wigner function $W(x,p)$ for the particular physical state $\hat{\rho}$, for instance, the mean radius. The integral with respect to r yields the phase distribution derived from the Wigner function ($s=0$). Thus, we find that the relation

$$\mathrm{Pr}_s(\varphi)=\int_{-\pi}^{+\pi}\frac{d\phi}{2\pi}G_s(\phi;R)\mathrm{Pr}_0(\varphi-\phi) \tag{65}$$

between an s-phase distribution and the Wigner-phase distribution holds approximately in the semiclassical regime. Hence, we can interpret $\mathrm{Pr}_s(\varphi)$ as consisting of Wigner-phase distributions averaged with respect to a reference-phase distribution given by

$$G_s(\phi;R)=\left[\frac{4\pi^2R}{-s}\right]^{1/2}\exp\left[\frac{R^2}{s}\phi^2\right]. \tag{66}$$

(Note that $s<0$.) Since the Wigner phase approaches the pure canonical phase in the limit of very large intensities [13], we anticipate that an s-phase distribution is approximately a noisy canonical phase distribution. We may already guess what the reference-phase distribution looks like.

Fock representation

Now, we investigate how accurately the axioms (A) and (B) are fulfilled. Tanás, Miranowicz, and Gantsog [27] found the Fock representation of an s-parametrized phase distribution to be given by

$$\mathrm{Pr}_s(\varphi)=\frac{1}{2\pi}\sum_{n,m=0}^{\infty}B_{n,m}e^{i(m-n)\varphi}\langle n|\hat{\rho}|m\rangle\ , \tag{67}$$

with

$$B_{n,m}=\left[\frac{2}{1-s}\right]^{(n+m)/2}(n!m!)^{1/2}\times\sum_{k=0}^{\min(n,m)}\frac{[\frac{1}{2}(n+m)-k]!}{k!(n-k)!(m-k)!}\left[-\frac{1+s}{2}\right]^k. \tag{68}$$

Evidently, the phase-shifter axiom (A) for a canonical phase distribution is satisfied. How accurately is the number-shift invariance (B) fulfilled? In order to test axiom (B) it is convenient to write the expression (68) for the B coefficients in a different way. We recall the definition [39] [Vol. 2, Eq. 10.8(12)] of the Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\sum_{k=0}^{n}\begin{bmatrix} n+\alpha \\ k \end{bmatrix}\begin{bmatrix} n+\beta \\ n-k \end{bmatrix}\times(x-1)^{n-k}(x+1)^k , \quad (69)$$

and find

$$B_{n,m}=\left[\frac{m-n}{2}\right]!\left[\frac{2}{1-s}\right]^{(m-n)/2}\left[\frac{s+1}{s-1}\right]^n\left[\frac{n!}{m!}\right]^{1/2}\times P_n^{[m-n,-(m+n)/2]}[1-4(s+1)^{-1}] \quad (70)$$

for $m \geq n$. (Otherwise, n and m should be interchanged.) We utilize a property [39] [Vol. 2, Eq. 10.8(33)] of the Jacobi polynomials and obtain

$$B_{n+1,m+1}=\frac{\frac{1}{2}(n+1+m+1)}{(n+1)^{1/2}(m+1)^{1/2}}B_{m,m}(1+\varepsilon), \quad (71)$$

with ε given by

$$\varepsilon=\frac{m-n}{n+1+m+1}\left[\frac{s+1}{s-1}\right]\Theta \quad (72)$$

and

$$\Theta=\frac{P_n^{[m-n,-(m+n)/2-1]}[1-4(s+1)^{-1}]}{P_n^{(m-n,-(m+n)/2)}[1-4(s+1)^{-1}]} . \quad (73)$$

The expression (71) looks quite similar to the approximate number-shift invariance (42) of the Q-phase distribution. Apart from the factor $(1+\varepsilon)$, the shifted coefficients $B_{n+1,m+1}$ differ from the initial coefficients $B_{n,m}$ by the ratio of the arithmetic and the geometric mean of $n+1$ and $m+1$, as the Q-phase coefficients do. Evidently, for the Q phase where s equals -1, the correction ε is zero. Now, we prove that ε is small for $s<-1$ [43]. Because of

$$\begin{bmatrix} n+\beta+1 \\ n-k \end{bmatrix} > \begin{bmatrix} n+\beta \\ n-k \end{bmatrix} , \quad (74)$$

we obtain from the definition (69) of the Jacobi polynomials that for $x>1$,

$$0<P_n^{(\alpha,\beta)}(x)<P_n^{(\alpha,\beta+1)}(x) . \quad (75)$$

For $s<-1$, we get $x=1-4(s+1)^{-1}>1$, and, hence,

$$0<\Theta<1 . \quad (76)$$

According to Eq. (72) this implies that the correction ε in the number-shift relation (71) is indeed small for states having narrow photon distributions compared to the mean photon number. Consequently, the number-shift invariance (19) holds approximately in the semiclassical regime. Thus, measured phase distributions, i.e., s-phase distributions with $s \leq -1$, can be regarded as noisy canonical phase distributions.

Convolution

Hence, we can interpret $\mathrm{Pr}_s(\varphi)$ as pure canonical phase distributions $\mathrm{Pr}_p(\varphi)$ averaged with respect to a weight function,

$$\mathrm{Pr}_s(\varphi)=\int_{-\pi}^{+\pi}\frac{d\phi}{2\pi}g_s(\phi;N)\mathrm{Pr}_p(\varphi-\phi) , \quad (77)$$

with

$$g_s(\phi;N)=\sum_{\nu=-\infty}^{+\infty}e^{i\nu\phi}B_{N,N+|\nu|} . \quad (78)$$

As before, N denotes the mean photon number. Similar to the case of the Q phase, we derive an asymptotic expression for $g_s(\phi;N)$. We replace the Fourier series by an integral and use the saddle-point method for evaluating it. It is shown in the Appendix that

$$B_{N,N}=1, \quad \left.\frac{\partial B_{N,N+|\nu|}}{\partial\nu}\right|_{\nu=+0}\to 0 , \quad \left.\frac{\partial^2 B_{N,N+|\nu|}}{\partial\nu^2}\right|_{\nu=+0}\to\frac{s}{4N} \quad (79)$$

for large mean photon numbers N. Hence, the saddle point lies at $\nu=0$ and we can approximate $B_{N,N+|\nu|}$ by

$$B_{N,N+|\nu|}\approx\exp\left[\frac{s\nu^2}{8N}\right] \quad (80)$$

in the Fourier series (78). Replacing the series by an integral, we obtain, finally,

$$g_s(\phi;N)=2\pi\left[-\frac{2N}{\pi s}\right]^{1/2}\exp\left[\frac{2N}{s}\phi^2\right] . \quad (81)$$

(Note that $s<-1$.) This expression describes the intensity-dependent probability distribution of reference phases for inefficiently measured phase distributions considered as noisy canonical phase distributions. Our expression (81) is very similar to Eq. (66), which was motivated by the overlap relation (5). Note that the general formula (81) is less accurate than the specific expression (50) for the Q phase since some more approximations are involved. However, when we set $s=-1$, both formulas converge to the same expression for large mean photon numbers N. The s dependence in Eq. (81) is easy to understand. The more inefficient the phase measurement is the larger is the modulus of the s parameter and the broader is the weight function. We also note that as in the case of the Q phase, the reference-phase distribution for the s-parametrized phase gets narrower with increasing intensity. It approaches a δ function in the macroscopic limit. We tested numerically the accuracy of our treatment of s-parametrized phase as a noisy canonical phase. For $N/|s|>20$, we found a similar accuracy as in the case of the Q phase.

IV. SUMMARY

We have extended the treatment of canonical quantum-optical phase within the formalism of probability distributions to include external measurement-induced uncertainty. We require that quantum phase satisfies two elementary axioms motivated by the complementarity of phase and photon number: *A phase shifter shifts a phase distribution* while *a number shifter does not change it* [37]. These requirements determine a pure canonical phase distribution uniquely as being the Helstrom-Shapiro-Shepard phase distribution [4,5] or the Pegg-Barnett phase distribution [6] for physical states when the limit of infinite Hilbert-space dimension has been taken. A noisy canonical phase distribution can be interpreted as a weighted average of pure canonical phase distributions where the weight function represents uncertainty in the reference phase.

As a second step, we have linked recent *phase measurements* [18] with *canonical phase* in the semiclassical regime. Under reasonable assumptions [21], measured phase distributions [22] are s-parametrized phase distributions, i.e., integrals of smoothed Q functions over the radial coordinate [15,19,20,16]. We have shown that these distributions fulfill approximately the basic axioms of canonical quantum phase for states having a narrow photon distribution compared to the mean photon number. In this case, we can interpret a measured phase distribution as a weighted average of pure canonical phase distributions. The important point is that the measurement is not contaminated (at least to the level of the approximation) by the conjugate variable, photon number. The weight function depends on the s-parameter which comprises the overall detection noise in phase measurements [16]. In contrast to an axiomatically defined noisy phase distribution, the weight function depends weakly on the mean photon number as well. It tends to a δ function for very large intensities since the extra noise involved in phase measurements [24] becomes negligible in the classical domain. Numerical tests illustrate that our treatment of measured phase distributions as noisy canonical phase distributions is well justified in the semiclassical regime for mean photon numbers exceeding roughly twenty times the modulus of the s parameter. Our analysis, thus, bridges all three conceptions of quantum-optical phase [3] (canonical phase, s-parametrized phase, phase from measurements) and provides important physical insight into the relationship between them.

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APPENDIX

We wish to derive asymptotic expressions for $B_{N,N}$,

$$B'_{N,N} \equiv \left. \frac{\partial B_{N,N+|\nu|}}{\partial \nu} \right|_{\nu=+0}, \tag{A1}$$

$$B''_{N,N} \equiv \left. \frac{\partial^2 B_{N,N+|\nu|}}{\partial \nu^2} \right|_{\nu=+0}, \tag{A2}$$

in the limit of large N for the B coefficients being defined by Eq. (68). Since $P_N^{(0,-N)}(x) = [\frac{1}{2}(x+1)]^N$, we obtain immediately from Eq. (70)

$$B_{N,N} = 1 . \tag{A3}$$

Now, we calculate the first derivative $B'_{N,N}$ of $B_{N,N+|\nu|}$ with respect to ν at $\nu=+0$. We get from the number-shift relation (71) the recurrence relation,

$$B'_{N+1,N+1} = B'_{N,N} + \varepsilon' , \tag{A4}$$

with

$$\varepsilon' \equiv \left. \frac{\partial}{\partial \nu} \frac{\nu/2}{N+\dfrac{\nu}{2}+1} \left[\frac{s+1}{s-1} \right] \Theta \right|_{\nu=0} \tag{A5}$$

$$= \frac{1}{2(N+1)} \left[\frac{s+1}{s-1} \right] \Theta_0 . \tag{A6}$$

Here, Θ_0 is given by

$$\Theta_0 = \Theta|_{\nu=0} = \frac{P_N^{(0,-N-1)} \left[1 - \dfrac{4}{s+1} \right]}{P_N^{(0,-N)} \left[1 - \dfrac{4}{s+1} \right]} = \left[\frac{s+1}{s-1} \right]^N . \tag{A7}$$

Hence, the recurrence relation for the $B'_{N,N}$ reads

$$B'_{N+1,N+1} = B'_{N,N} + \frac{1}{2(N+1)} \left[\frac{s+1}{s-1} \right]^{N+1} . \tag{A8}$$

According to the definition (68) of the B coefficients, we have

$$B_{0,\nu} = \left[\frac{2}{1-s} \right]^{\nu/2} \frac{\left[\dfrac{\nu}{2} \right]!}{\sqrt{\nu!}} \tag{A9}$$

and, hence,

$$B'_{0,0} = \tfrac{1}{2} \ln \left[\frac{2}{1-s} \right] . \tag{A10}$$

Solving the recurrence relation (A8) and using the initial value (A10), we obtain

$$B'_{N,N} = \frac{1}{2} \left[\ln \left[\frac{2}{1-s} \right] + \sum_{k=1}^{N} \frac{1}{k} \left[\frac{s+1}{s-1} \right]^k \right] . \tag{A11}$$

For $s=-1$, thus, considering the Q phase, the coefficients $B'_{N,N}$ are always zero. For $s<-1$, they approach zero in the limit of large N,

$$B'_{N,N} \to 0 , \tag{A12}$$

because [44] (Vol. 1, Eq. 5.2.4.4)

$$\sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{s+1}{s-1} \right]^k = -\ln \left[\frac{2}{1-s} \right] . \tag{A13}$$

Now, we derive an asymptotic expression for the second derivative $B''_{N,N}$ of $B_{N,N+|\nu|}$ at $\nu=+0$. First, we obtain from the definition (68) of the B coefficients the differential equation,

$$(1-s)\frac{\partial B_{N,N+|\nu|}}{\partial s}=\left[N+\frac{\nu}{2}\right]B_{N,N+|\nu|}-[N(N+|\nu|)]^{1/2}B_{N-1,N-1+|\nu|}\ . \tag{A14}$$

Consequently,

$$\begin{aligned}(1-s)\frac{\partial B''_{N,N}}{\partial s}&=N(B''_{N,N}-B''_{N-1,N-1})+B'_{N,N}-B'_{N-1,N-1}+\frac{1}{4N}\\&=N(B''_{N,N}-B''_{N-1,N-1})+\frac{1}{2N}\left[\frac{s+1}{s-1}\right]^N+\frac{1}{4N}\ ,\end{aligned} \tag{A15}$$

where the recurrence relation (A8) for $B'_{N,N}$ has been used. Motivated by the result (66) derived using the overlap relation (5), we make the ansatz

$$B''_{N,N}=a_N s\ . \tag{A16}$$

Inserting the ansatz (A16) into the differential equation (A15) we obtain, neglecting the $(2N)^{-1}(s+1)^N(s-1)^{-N}$ term,

$$(1-s)a_N=N(a_N-a_{N-1})s+\frac{1}{4N}\ . \tag{A17}$$

Comparing the powers of s on the left and the right side of this equation, we get

$$a_N=\frac{1}{4N} \tag{A18}$$

and

$$(N+1)a_N=Na_{N-1}\ . \tag{A19}$$

Equations (A18) and (A19) do not contradict each other for large N. Hence the ansatz (A16) is justified, and we obtain, finally,

$$B''_{N,N}\to\frac{s}{4N}\ . \tag{A20}$$

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$$|\langle\psi_1|\psi_2\rangle|^2 = 2\pi \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\; W_1(x,p) W_2(x,p) \;.$$

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$$\begin{aligned}\mathrm{Tr}\{\hat{\rho}_1\hat{\rho}_2\} &= \sum_{ab} p_a p_b |\langle a|b\rangle|^2 \\ &= 2\pi \sum_{ab} p_a p_b \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\; W_a(x,p) W_b(x,p) \\ &= 2\pi \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp\; W_1(x,p) W_2(x,p) \;.\end{aligned}$$

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[38] The replacement of the geometric by the arithmetic mean in the Fock representation was also the basic approximation in analytic expressions for the wave function and the Wigner function of a London phase state [30], see H. Paul, in *International Conference on Quantum Electronics Technical Digest Series 1992* (Institut für Nachrichtentechnik der Tu Wien, Vienna, 1992), Vol. 9, p. 274; H. Paul and W. Schleich (unpublished). Excellent agreement of the approximate with the numerically calculated exact Wigner function was pointed out in Ref. [34].

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