Theory for the linewidth of a bad-cavity laser

M.P. van Exter, S.J.M. Kuppens, and J.P. Woerdman
Huygens Laboratory, University of Leiden, P.O. Box 9504, 2300 RA Leiden, The Netherlands
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We derive the quantum-limited linewidth of a bad-cavity laser, where the cavity-loss rate can be comparable to or larger than the gain bandwidth. We allow for incomplete inversion and arbitrary mirror losses and show how the position-dependent gain, refractive index, and populations codetermine the laser linewidth. The result obtained factorizes in parts that have straightforward physical interpretations. In the bad-cavity regime the laser linewidth is shown to be reduced as compared to the Schawlow-Townes expression as a result of anomalous dispersion associated with the optical gain. Saturation is shown to lead to a power-independent contribution to the laser linewidth; the magnitude of this contribution depends critically on the assumed pumping and decay rates.

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I. INTRODUCTION

In every laser the spontaneous emission that inevitably occurs in a gain medium acts as a (quantum) noise source that results in a diffusion of the optical phase and poses a lower limit to the laser linewidth. For a single-mode low-loss fully inverted laser tuned to the center of a homogeneously broadened gain profile this linewidth, as originally derived by Schawlow and Townes [1], is given by

$$\Delta\nu = \frac{\Gamma_0}{4\pi S} = \frac{h\nu}{4\pi}\frac{\Gamma_0^2}{P_{\text{out}}} , \tag{1}$$

where $\Delta\nu$ is the full width at half maximum (FWHM) laser linewidth, $\Gamma_0 = -(c/2L)\ln(R_1R_2)$ is the cold- or empty-cavity-loss rate, with L the cavity length and R_1 and R_2 the mirror reflectivities, S is the average number of photons in the lasing mode, $h\nu$ is the energy per photon, and P_{out} is the total output power through both mirrors.

Nowadays the expression for the laser linewidth has been generalized to include lasers with large losses, incomplete inversion, and strong saturation. Contributions have been made by, e.g., Petermann [2], Henry [3,4], Ujihara [5], Lu [6], Goldberg, Milonni, and Sundaram [7], and Prasad [8]. A limitation of most treatments is that the angular frequency gain bandwidth (FWHM), denoted by $2\gamma = 2/T_2$, is assumed to be very much larger than the cavity-loss rate Γ_0, i.e., $2\gamma \gg \Gamma_0$. Although this assumption is valid for most lasers, it is generally incorrect for short high-gain lasers, where the cavity-loss rate can be comparable to or even larger than the gain bandwidth. We will use the notion "bad cavity" to indicate the latter situation, i.e., $\Gamma_0 \geq \gamma$. Note that a bad-cavity laser does not necessarily have a bad cavity in the sense that the optical loss per round-trip is large. In bad-cavity lasers the polarization cannot be adiabatically eliminated from the laser rate equations, dispersion effects will be important, and the laser linewidth will deviate from the Schawlow-Townes expression [Eq. (1)]. Examples of bad-cavity lasers are the 3.39-μm He-Ne and the 3.51 μm He-Xe gas lasers. Bad-cavity aspects will inevitably also show up in the development of "thresholdless" semiconductor microlasers, as only a bad-cavity laser can be really without threshold [9].

The finite polarization lifetime $T_2 = 1/\gamma$ theoretically leads to two deviations from the usual Schawlow-Townes behavior. First of all, Scully, Süssman, and Benkert [10] have shown that the memory effect of the collective dipole moment colors the spontaneous-emission noise. As a consequence, the evolution of the laser phase on a time scale short as compared to T_2 will be slower than expected for pure phase diffusion and the spectral wings of the emitted laser light will drop faster than Lorentzian. We will not consider this short-time deviation anymore, as it is expected to be very small and only visible in the far wings of the optical spectrum. In the good-cavity limit ($\gamma \gg \Gamma_0$) considered in Ref. [10] the long-time evolution of the optical phase is not affected by the atomic memory and the laser linewidth is still given by Eq. (1). In the opposite (bad-cavity) limit $\gamma \leq \Gamma_0$ the laser linewidth *will* be affected. In fact the atomic memory can lead to a drastic reduction of the laser linewidth as compared to Eq. (1). This has been shown theoretically by Lax [11] and Haken [12] in the mid-1960s and by Kolobov *et al.* [13] in a recent paper, which combines the short-time and long-time evolution.

Recently we have experimentally demonstrated the predicted linewidth reduction for a 3.39-μm He-Ne laser [14]. A comparison with theory is hampered by the fact that Lax, Haken, and Kolobov *et al.* unfortunately only consider cavities with relatively good mirrors, using a mean-field approximation of the intracavity field. In practice, however, the mirror reflectivity must often be rather low to reach the bad-cavity regime, so that the intracavity field shows strong spatial variation. In this paper we will combine the bad-cavity aspect with the situation of large losses, incomplete inversion, and strong saturation mentioned earlier. The aim of the paper is to find how these complications appear in the laser linewidth and, more specifically, to what degree they are

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independent. To reach this goal we will quantify the memory effect of the macroscopic polarization in terms of the anomalous dispersion associated with the optical gain.

We are not the first to address this issue. Important theoretical contributions have been made by Henry [4] and Tromborg, Olesen, and Pan [15]. However, the theory developed by these authors is very general and is therefore not always easy to access. After briefly explaining the key ideas and results of the general theory we will apply it to bad-cavity Fabry-Pérot lasers. We will show that for these lasers the complicated expression for the laser linewidth simplifies to an expression that gives a theoretical basis for the heuristic traveling-wave phase diffusion model introduced earlier by van Exter, Hamel, and Woerdman [16]. It can be solved analytically and leads to a rather elegant equation that factorizes in parts, allowing a straightforward physical interpretation. Crosslinks are made between the theory discussed here and other work on the laser linewidth.

Our theoretical analysis is based upon three assumptions. We assume (i) that spectral hole burning is absent, a condition that is automatically fulfilled for a homogeneously broadened active medium, (ii) that spatial hole burning is unimportant, which is the case for a unidirectional ring laser or a Fabry-Pérot laser in which spatial diffusion of the active molecules is fast enough to effectively erase the induced population grating, and (iii) that the laser frequency is tuned to the center of a symmetric gain profile. Condition (iii) implies that the refractive index is independent of population inversion and that the linewidth enhancement or α parameter [3], which is so important in semiconductor lasers, is zero.

II. GENERAL LINEWIDTH THEORY

The dynamics of a laser is most generally described in terms of four variables: the electric field E, the atomic polarization P, the upper-level population density N_2, and the lower-level population density N_1, each of which depends on position and time [7,11–13]. The interactions are governed by the Maxwell and Bloch equations supplemented with boundary conditions such as the mirror reflectivities and pump rates. The coupling to the outside world is incorporated by the introduction of damping and noise sources. When the mirror reflectivities are high, the intracavity field is almost uniform and it is convenient to introduce cavity modes. In the case of low mirror reflectivities it is still possible to introduce cavity modes, but they will be nonorthogonal [17]. In this case a clearer physical picture is obtained by directly keeping track of the position dependence of E, P, N_2, and N_1.

We will assume that the polarization decay rate is much larger than the population decay rates, but not necessarily larger than the cold-cavity-loss rate. Under this condition saturation by stimulated emission will mainly affect the relatively slow population dynamics, but will hardly change the polarization dynamics, which is dominated by "pure dephasing." The polarization P can then be expressed in terms of the electric field E and a frequency- and population-dependent dielectric constant ϵ, even in the case of saturation. We exclude a possible explicit intensity dependence of ϵ (at constant N_1 and N_2), which in semiconductor lasers is denoted as "nonlinear gain" and which is just a convenient way to parametrize the effects of spectral hole burning. The laser dynamics will thus be described by the wave equation for the optical field, supplemented with rate equations for the populations $N_2(z,t)$ and $N_1(z,t)$.

We start by summarizing the semiclassical results obtained by Henry [4] and Tromborg, Olesen, and Pan [15], making use of the conventions and symbols introduced by the latter authors. For a laser that oscillates in a single fundamental transverse mode $\phi(x,y;z)$, the problem can be reduced from three to one dimension by projection onto this transverse mode. The intracavity electric field is written as

$$E(x,y,z,t) = 1/(2\pi) \int_{-\infty}^{\infty} E_\omega(z)\, \phi(x,y;z) e^{-i\omega t}\, d\omega \ , \quad (2)$$

where the transverse mode is normalized

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi(x,y;z)|^2\, dx\, dy = 1 \ . \quad (3)$$

The propagation of the intracavity field $E_\omega(z)$ is given by the optical wave equation

$$\frac{\partial^2 E_\omega(z)}{\partial z^2} + k^2(z) E_\omega(z) = f_\omega(z) \ , \quad (4a)$$

$$k^2(z) = [n(z,\omega)(\omega/c) - ig(z,\omega)/2]^2 = \epsilon(z,\omega)\, \omega^2/c^2 \ , \quad (4b)$$

where the complex wave number $k(z)$ incorporates the refractive index $n(z,\omega)$ as well as the intensity gain $g(z,\omega)$. The projection of the driving spontaneous-emission noise source onto the considered transverse mode is denoted by $f_\omega(z)$.

Equation (4a) is supplemented with boundary conditions, so that $E_\omega(z)$ satisfies the imposed mirror reflectivities, and with equations that express $k(z)$ in terms of the populations $N_1(z)$ and $N_2(z)$, and the latter in terms of $E_\omega(z)$. For slow processes the noise source $f_\omega(z)$ can be excellently approximated by a (position-dependent) white noise source, often denoted as Langevin noise,

$$\langle f_\omega(z) f^*_{\omega'}(z') \rangle = 2D_{ff*}(z)\, \delta(z-z')\, \delta(\omega-\omega') \ . \quad (5)$$

The inhomogeneous wave equation (4a) can be solved with Green's functions for lasers operating either above or below threshold [15]. We limit ourselves to a single-mode laser operating above threshold. The intracavity electric field can then be factorized in spatial and time-dependent parts as

$$E(z,t) = Z(z)\, a(t)\, e^{i\varphi(t)}\, e^{-i\omega_0 t} + \text{c.c.} \ , \quad (6)$$

where $Z(z)$ is the (complex) intracavity field distribution, $a(t)$ and $\varphi(t)$ are the field amplitude and phase,

ω_0 is the average laser frequency, and c.c. stands for the complex conjugate. When amplitude fluctuations are neglected, i.e., $a(t) = a_0$, the FWHM laser linewidth $\Delta\nu$ is determined by the rate of phase diffusion via

$$\frac{d\varphi(t)}{dt} = F_\varphi(t) , \tag{7a}$$

$$\langle F_\varphi(t'+t)F_\varphi(t')\rangle = D_{\varphi\varphi}\,\delta(t) , \tag{7b}$$

$$\Delta\nu = D_{\varphi\varphi}/(2\pi) . \tag{7c}$$

When the laser frequency is tuned to the center of a symmetric gain profile, fluctuations in pump rates and populations have no effect on the optical phase [15]. The noise $F_\varphi(t)$ can then be easily expressed in terms of the projection of the position-dependent noise source $f(z,t)$ on the longitudinal eigenmode $Z(z)$. The phase diffusion constant $D_{\varphi\varphi}$ is found to be [see Eqs. (23), (25), and (36) in Ref. [15]]

$$D_{\varphi\varphi} = \frac{1}{4\pi a_0^2}\,\frac{\int_0^L |Z(z)|^2\, 2D_{ff*}(z)\,dz}{|\,2\int_0^L Z^2(z)\,k(z)\,\partial k(z)/\partial\omega\,dz\,|^2}$$
$$= \frac{c^4}{8\pi a_0^2\omega_0^2}\,\frac{\int_0^L |Z(z)|^2\,D_{ff*}(z)\,dz}{|\int_0^L Z^2(z)\,n(z)\,n_{\rm gr}(z)\,dz\,|^2} , \tag{8}$$

where $\partial k(z)/\partial\omega = 1/v_{\rm gr} = n_{\rm gr}/c$ is the inverse group velocity and $n_{\rm gr}$ is the group refractive index. A calculation of the laser linewidth is now reduced to solving the integrals in Eq. (8), for which one needs the steady-state field distribution $Z(z)$, the phase and group refractive indices $n(z)$ and $n_{\rm gr}(z)$, and the strength of the noise $D_{ff*}(z)$.

There is an ongoing discussion as to whether the noise should be attributed to spontaneous emission inside the cavity, to vacuum fluctuations leaking in from outside, or to a combination of the two [6,7]. Quantum-mechanical treatments show that the choice is related to the ordering of creation and annihilation operators and that the calculated laser linewidth is independent of the chosen ordering [6,7]. For convenience we prefer to lump all noise in spontaneous emission to avoid the introduction of additional localized noise sources at the mirrors. $D_{ff*}(z)$ can then be determined from the fluctuation-dissipation theorem, which relates spontaneous emission to stimulated emission and optical gain. Henry found [4]

$$D_{ff*}(z) = \frac{2\pi\hbar\omega_0^3}{c^3\epsilon_0}\,n(z)\,g(z)\,N_{\rm sp}(z) , \tag{9}$$

where the spontaneous-emission factor $N_{\rm sp} \equiv N_2/(N_2 - N_1)$ measures the degree of inversion. Substitution of Eq. (9) in Eq. (8) gives

$$D_{\varphi\varphi} = \frac{\hbar\omega_0 c}{4\,\epsilon_0 a_0^2}\,\frac{\int_0^L ngN_{\rm sp}|Z(z)|^2\,dz}{|\int_0^L nn_{\rm gr}Z^2(z)\,dz\,|^2} . \tag{10}$$

Note that for a more realistic three-dimensional laser the products $ngN_{\rm sp}$ and $nn_{\rm gr}$ should be averaged over the transverse intensity profile $|\phi(x,y;z)|^2$ and must be interpreted as modally averaged values [15,18]. In case of gain guiding, Eq. (10) must be multiplied by the transverse Petermann factor $K_{\rm trans} \geq 1$ [2,4]; we take $K_{\rm trans} = 1$.

Equation (10) can be recast into a more familiar form [15,18]:

$$D_{\varphi\varphi} = \frac{R_{\rm sp}}{2S} , \tag{11a}$$

$$R_{\rm sp} = c\,\frac{[\int_0^L n\,n_{\rm gr}\,|Z(z)|^2\,dz\,]\,[\int_0^L n\,g\,N_{\rm sp}|Z(z)|^2\,dz\,]}{|\int_0^L n\,n_{\rm gr}\,Z^2(z)\,dz\,|^2} , \tag{11b}$$

$$S = \frac{2\epsilon_0}{\hbar\omega_0}\,a_0^2\int_0^L n\,n_{\rm gr}\,|Z(z)|^2\,dz, \tag{11c}$$

where $R_{\rm sp}$ is the spontaneous-emission rate, which can be interpreted as the number of photons that are spontaneously emitted into the considered cavity mode per unit time, and S is the average number of photons in that mode. Equations (10) or (11a)–(11c) form the basis of our treatment of bad-cavity lasers.

III. APPLICATION TO TRAVELING-WAVE LASERS

We first consider a Fabry-Pérot laser. Figure 1 shows how the intracavity field can be separated in two traveling waves, $Z(z) = Z_+(z) + Z_-(z)$, one moving to the left, with local intracavity power $P_-(z)$, and the other moving to the right, with power $P_+(z)$. As the local intensity inside a dielectric is given by $2\epsilon_0 cn|E|^2$, one easily finds [using Eqs. (2), (3), and (6)]

$$2\epsilon_0 cn\,a_0^2\,Z(z)Z^*(z) = P_-(z) + P_+(z) + \cdots , \tag{12a}$$

$$2\epsilon_0 cn\,a_0^2\,Z(z)Z(z) = 2\sqrt{P_-(z)\,P_+(z)} + \cdots , \tag{12b}$$

where the dots denote (generally complex) terms oscillating at a spatial frequency of $2k(z)$. In the slowly varying envelope approximation, the latter terms disappear when integrated over the cavity length. For completeness we note that, because the intracavity power propagates with the group velocity $v_{\rm gr} = c/n_{\rm gr}$, the total electromagnetic energy contained in the laser cavity is [19]

$$S\,\hbar\omega_0 = \int_0^L (n_{\rm gr}/c)\,\{P_-(z) + P_+(z)\}\,dz . \tag{13}$$

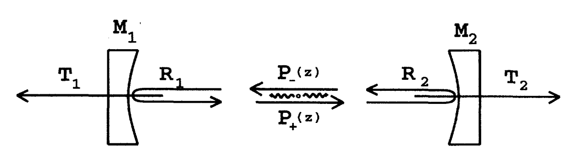

FIG. 1. Laser cavity with intracavity traveling electro-magnetic waves.

The coupling between the two traveling waves can be neglected in the absence of spatial hole burning and when $k(z)$ varies relatively slowly with position (WKB approximation). In this case we deal effectively with a traveling-wave laser and the physics is the same as that of a unidirectional ring laser. Rigrod [20] has noted that, the gain of the leftward and rightward traveling waves being equal, the product $P_-(z)\,P_+(z)$ should be independent of the position z. When this is used, substitution of Eqs. (12a) and (12b) in Eq. (10) yields a surprisingly simple expression for the laser linewidth:

$$\Delta\nu = \frac{\hbar\omega_0}{4\pi\tau_R^2}\int_0^L \left(\frac{1}{P_-(z)} + \frac{1}{P_+(z)}\right) g(z)\,N_{\rm sp}(z)\,dz \ , \tag{14}$$

where we have introduced the cavity round-trip time τ_R as

$$\tau_R = 2\int_0^L n_{\rm gr}(z)/c\,dz \ . \tag{15}$$

Equation (14) is important as it allows for a simple physical interpretation of the quantum-limited laser linewidth. It shows that one can treat phase diffusion as a local process, being proportional to the local spontaneous-emission rate and inversely proportional to the local traveling-wave power. This gives a formal corroboration of the heuristic traveling-wave model for laser phase diffusion that we introduced previously [16]. Note that the spontaneous-emission noise source $[\propto g(z)N_{\rm sp}(z)]$ is proportional to the local excited-state population $N_2(z)$, since $g(z) \propto [N_2(z) - N_1(z)]$, as one would intuitively expect. Quantum-mechanical treatments show that for antinormal ordering of the operators the noise source would have been proportional to the local ground-state population $N_1(z)$. However, for this ordering, one also has to take into account disturbances due to vacuum fluctuations leaking through the mirrors. Goldberg, Milonni, and Sundaram [7] have shown that such an analysis gives the same result as that for normal ordering.

Equation (14) is an extension of the traveling-wave expression derived in Ref. [21] in the sense that in our work the bad-cavity aspects have been included. The prefactor in Eq. (14) shows that the effects of a position-dependent dispersion on the laser linewidth can be conveniently expressed in terms of an integral over the cavity, yielding the round-trip time τ_R. Possible bad-cavity aspects of the laser show up in this prefactor only. Note that the *phase* refractive index n has dropped out of the equations; only the *group* refractive index $n_{\rm gr}$ is relevant for the laser linewidth, because only $n_{\rm gr}$ enters the expression for τ_R. The simple physical picture is that the spontaneous-emission noise source disturbs the intracavity field by emitting wave packets that travel around with the group velocity.

As mentioned above, a Fabry-Pérot laser without spatial hole burning can theoretically be converted into a unidirectional traveling-wave ring laser, by unfolding the laser cavity. Equation (14) is thus also valid for the latter type of lasers, when (i) the integral is taken over a round-trip, (ii) either $P_-(z)$ or $P_+(z)$ is removed from the integral, and (iii) the factor 2 is omitted from Eq. (15).

IV. LINEWIDTH AS A FUNCTION OF OUTPUT POWER

We will now cast Eq. (14) in a form that is more practical from an experimental point of view, by expressing the intracavity powers in terms of the laser output power $P_{\rm out}$ and by analytically solving the integral. For the laser cavity sketched in Fig. 1, having mirrors $i = 1, 2$ with intensity reflectivities R_i and transmissivities $T_i = 1 - R_i$, one finds

$$\begin{aligned} P_{\rm out} &= \{(1-R_1)/\sqrt{R_1} + (1-R_2)/\sqrt{R_2}\}\,P_c \\ &= -\ln(R_1R_2)\,\sqrt{K}\,P_c \ , \end{aligned} \tag{16a}$$

$$K = \left\{\frac{(1/\sqrt{R_1}) - \sqrt{R_1} + (1/\sqrt{R_2}) - \sqrt{R_2}}{-\ln(R_1R_2)}\right\}^2 \ , \tag{16b}$$

$$P_-(z)\,P_+(z) = P_c^2 \ , \tag{16c}$$

where P_c is the local traveling-wave power at the position where $P_-(z) = P_+(z)$, not necessarily being the center of the cavity. The longitudinal Petermann K factor has been introduced as a measure for the spatial uniformity of the intracavity field [5,17]. For small output coupling ($R_i \approx 1$) we have $K \approx 1$, whereas $K > 1$ for larger output coupling.

To solve the integral in Eq. (14) one needs expressions for $P_-(z)$, $P_+(z)$, $g(z)$, and $N_{\rm sp}(z)$. For a uniformly pumped homogeneously broadened laser medium, $g(z)$ and $N_{\rm sp}(z)$ can be easily expressed in terms of the local traveling-wave powers as (see Appendix A)

$$g(z) = g_0\left[1 + \left(\frac{P_-(z) + P_+(z)}{P_{\rm sat}}\right)\right]^{-1} \ , \tag{17a}$$

$$N_{\rm sp} = N_{\rm sp}^{(0)}\left[1 + k_2\left(\frac{P_-(z) + P_+(z)}{P_{\rm sat}}\right)\right] \ , \tag{17b}$$

where g_0 is the (position-independent) unsaturated gain, $P_{\rm sat}$ the saturation power, $N_{\rm sp}^{(0)}$ the unsaturated spontaneous-emission factor, and k_2 a dimensionless parameter ($0 \le k_2 \le 1$), which describes how the spontaneous-emission factor $N_{\rm sp}$ increases upon saturation. The exact values of $N_{\rm sp}^{(0)}$ and k_2 depend on the relative strength of the pump and decay rates of the upper and lower laser level (see Appendix A) [22]. For an ideal four-level laser $N_{\rm sp}^{(0)} = 1$ and $k_2 = 0$.

The laser output power appearing in Eq. (16a) can be expressed as

$$P_{\rm out} = P_{\rm sat}\,\{g_0\,L + \tfrac{1}{2}\ln(R_1R_2)\} \ . \tag{18}$$

It should be realized that $P_{\rm sat}$, being equal to $I_{\rm sat}\mathcal{A}$, where $I_{\rm sat}$ is the saturation intensity and $\mathcal{A}$ is the modal cross section, can be z dependent, since generally $\mathcal{A} = \mathcal{A}(z)$. However, we prefer to stick with the one-dimensional approach, as inclusion of the transverse aspects makes the problem extremely complicated. We will therefore assume that both $\mathcal{A}$ and $P_{\rm sat}$ are independent of z.

By substituting Eq. (17b) in Eq. (14), by expressing $P_-(z)$ and $P_+(z)$ in terms of $P_{\rm out}$ and R_i, and by using $P_+(z)$ as integration variable, one finds

$$\Delta\nu = \frac{h\nu}{4\pi\tau_R^2}\{\ln(R_1R_2)\}^2 N_{\rm sp}^{(0)}\frac{K}{P_{\rm out}} + \Delta\nu_0 \,, \quad (19a)$$

$$\Delta\nu_0 = \frac{-h\nu\ln(R_1R_2)}{2\pi\tau_R^2} k_2 N_{\rm sp}^{(0)}\frac{F}{P_{\rm sat}} \,, \quad (19b)$$

$$F = \left(\frac{(1/R_1 - R_1 + 1/R_2 - R_2)}{-4\ln(R_1R_2)} + \tfrac{1}{2}\right) . \quad (19c)$$

For a nonideal four-level laser ($k_2 \geq 0$) we find a power-independent contribution $\Delta\nu_0$ to the laser linewidth (as we will see below, τ_R is independent of $P_{\rm out}$). The factor F describes the effect of field nonuniformity on this power-independent contribution. The F factor is practically equal to the Petermann K factor when the mirror reflectivities are reasonably large. For example, if $R_1 = R_2 = 1 - \delta$, both K and F can be expanded as $1 + (1/12)\delta^2 + O(\delta^3)$. Only for very large outcoupling do we find that $F \neq K$, more specifically, $F \geq K$. To give some examples: for a laser with $R_1 = R_2 = 0.30$ one finds $K = 1.127$ and $F = 1.130$, when $R_1 = R_2 = 0.10$ one finds $K = 1.528$ and $F = 1.575$, and when $R_1 = R_2 = 0.01$ one finds $K = 4.62$ and $F = 5.93$ [23].

We will now express the cavity round-trip time τ_R, as given by Eq. (15), in terms of the cold-cavity-loss rate Γ_0,

$$\Gamma_0 = (c/L)\int_0^L g(z)\,dz = -\,c/(2L)\,\ln(R_1R_2) \,. \quad (20)$$

The crucial parameter in this transformation is the (position-dependent) group refractive index $n_{\rm gr}(z)$. For a homogeneously broadened (Lorentzian) line with a FWHM γ/π, the Kramers-Kronig relations yield

$$n_{\rm gr}(z) \equiv n + \omega\frac{dn}{d\omega} = n_{\rm gr}^{(0)} + \frac{g(z)\,c}{2\gamma} \,, \quad (21)$$

where $n_{\rm gr}^{(0)}$ is the group refractive index in the unpumped laser and $g(z)c/(2\gamma)$ is the anomalous dispersion associated with the gain of the lasing transition. Note that in a gas laser $n_{\rm gr}^{(0)} \approx 1$, but $n_{\rm gr}$ can still deviate significantly from 1, when $g(z)c \geq 2\gamma$, i.e., when the laser operates in the bad-cavity regime. Other types of bad-cavity lasers will be discussed at the end of this paper.

Substitution of Eq. (21) in Eq. (15) and comparison with Eq. (20) yields

$$\tau_R = \frac{2L}{c}\{n_{\rm gr}^{(0)} + \Gamma_0/(2\gamma)\} \,. \quad (22)$$

Anomalous dispersion thus leads to an increase of τ_R in the active cavity as compared to the unpumped laser. This is related to the phenomenon of mode pulling observed in bad-cavity lasers [24]. Note that τ_R is independent of the laser output power and that the quantity $\Delta\nu_0$ introduced in Eq. (19b) can thus indeed be interpreted as a power-independent contribution to the laser linewidth.

It is instructive to calculate the "dressed"-cavity-loss rate $\Gamma_{\rm dr}$, i.e., the cavity-loss rate of the active laser, which is defined via the ratio of the average number of photons in the mode S over the laser output power $P_{\rm out}$. We find

$$S\,h\nu = \int_0^L (n_{\rm gr}/c)\,[P_-(z) + P_+(z)]\,dz = P_{\rm out}/\Gamma_{\rm dr} \,, \quad (23a)$$

$$\frac{1}{\Gamma_{\rm dr}} = \frac{n_{\rm gr}^{(0)}}{\Gamma_0}\left(\frac{1 + F/\sqrt{K}\,(2P_c/P_{\rm sat})}{1 + \sqrt{K}\,(2P_c/P_{\rm sat})}\right) + \frac{1}{2\gamma} \,. \quad (23b)$$

Equation (23b) shows that the finite polarization lifetime can lead to a drastic reduction of the dressed-cavity-loss rate as compared to the cold-cavity-loss rate. The factor within large parentheses, which is identical to the factor $G_{\rm sat}^{-1} \geq 1$ used in Ref. [25], shows that the dressed-cavity-loss rate exhibits a (generally very small) intensity dependence, due to the change in the intracavity power distribution upon saturation.

Substitution of Eq. (22) in Eq. (19a) finally yields

$$\Delta\nu = \frac{h\nu\,\Gamma_0^2}{4\pi\,P_{\rm out}}\left\{\frac{1}{n_{\rm gr}^{(0)} + \Gamma_0/(2\gamma)}\right\}^2 N_{\rm sp}^{(0)}\,K + \Delta\nu_0 \,, \quad (24a)$$

$$\Delta\nu_0 = \frac{h\nu}{4\pi}\frac{\Gamma_0\,(c/L)}{P_{\rm sat}}\left\{\frac{1}{n_{\rm gr}^{(0)} + \Gamma_0/(2\gamma)}\right\}^2 N_{\rm sp}^{(0)}\,k_2\,F \,. \quad (24b)$$

Equation (24) is the most important result of this paper as it gives an expression for the linewidth of a bad-cavity laser, with arbitrary mirror losses, incomplete inversion, and saturation, in terms of experimentally accessible parameters. Both equations factorize in parts that have easy physical interpretations. A comparison between Eq. (24a) and the Schawlow-Townes expression Eq. (1) shows that (i) the linewidth is reduced by a factor $\{n_{\rm gr}^{(0)} + \Gamma_0/(2\gamma)\}^{-2}$ due to an increase of the cavity round-trip time in the pumped laser as compared to an empty cavity [14]; (ii) the linewidth is increased by a factor $N_{\rm sp}^{(0)} \geq 1$ when the inversion (at zero power) is incomplete; and (iii) the linewidth is increased by a factor $K \geq 1$ to account for the nonuniformity of the intracavity power. Here, we recover the familiar result that a nonuniformity of the intracavity intensity leads to an increase of the laser linewidth by a factor K, basically because the local increase of phase diffusion in the low-power regions is not compensated for by the decrease in the high-power regions. While each of these corrections has been separately discussed in the literature, it is the factorization for the general case of a bad-cavity laser that is new.

The power-independent contribution $\Delta\nu_0$ is a direct result of the buildup of population in the lower laser level and the increase of the spontaneous-emission factor N_{sp} with laser power, as will be shown in Appendixes A and B. In these appendixes we also show that $\Delta\nu_0$ will depend critically on the exact population dynamics and should be absent for a perfect four-level laser: $\Delta\nu_0 = 0$ when $k_2 = 0$.

V. DISCUSSION

The theory developed in this paper is valid for good- as well as bad-cavity lasers. It proved to be essential for the interpretation of linewidth measurements on small high-gain 3.39-μm He-Ne lasers [14], lasers that generally operate in the bad-cavity regime. An obvious question that arises is "Which other types of lasers can operate in the bad-cavity regime?," or phrased in a different way: "For which other types of lasers is the ratio $\beta \equiv g_0 c/(2\gamma n_{gr}^{(0)}) \gg 1$?"

In the visible and near-infrared range there are practically no lasers that fulfill this criterion. Although the gain in solid-state lasers and dye lasers can be enormous, it is, however, practically always associated with a very broad gain bandwidth, making $\beta \ll 1$. For semiconductor lasers bad-cavity aspects and anomalous dispersion will be important only when they operate on spectrally narrow gain profiles; here one may think of, e.g., quantum-dot lasers, or lasers operating at low temperatures on narrow exciton transitions [9]. Another exception in the near infrared might be the miniature stoichiometric Nd laser, which has been predicted to have a gain of as much as 10 dB per optical wavelength [26], which should make $\beta \gg 1$. Indicative of the bad-cavity aspects in stoichiometric Nd lasers is the fact that, as a result of dispersion, the phase and group refractive index can differ substantially [26].

An increase of the wavelength towards the midinfrared generally leads to an increase of the gain ($g_0 \propto \lambda^2$) and a decrease of the gain bandwidth (radiative linewidth proportional to λ^{-2}, Doppler width proportional to λ^{-1}). In the midinfrared bad-cavity aspects of lasers operating on electronic transitions will thus show up relatively quickly. As mentioned in the Introduction, this has been demonstrated for a 3.39-μm He-Ne laser, where for a 1-mm capillary $g_0 = 36$ m^{-1}, $\gamma/\pi = 300$ MHz, and $n_{gr}^{(0)} = 1$, making $\beta = 5.7$ [14]. It applies even more to a 3.51-μm He-Xe laser, where for a 0.25-mm capillary $g_0 = 195$ m^{-1}, $\gamma/\pi = 190$ MHz, and $n_{gr}^{(0)} = 1$, making $\beta \approx 50$ [27].

A further increase of the wavelength towards the far infrared, using rovibrational transitions, will not automatically lead to stronger bad-cavity aspects, because (i) the increase in gain will be limited as the population inversion in the molecular medium is necessarily only a fraction of the total population due to the thermal distribution over the various rovibrational states, and (ii) the reduction in gain bandwidth will be limited by collisions. Using typical numbers from the literature [28] we predict that some waveguided CO_2 lasers and waveguided far-infrared molecular lasers actually operate in the bad-cavity regime as defined in this article. However, due to the limited gain in these systems the cavity-loss rate will necessarily be small, cavity lengths are typically about 1 m, and it will be difficult to measure the quantum-limited linewidth of these lasers.

VI. CONCLUSIONS

Starting from the general expression for the laser linewidth as given by, e.g., Tromborg, Olesen, and Pan [15], we obtain a formal corroboration of the traveling-wave phase diffusion model [16]. For a Fabry-Pérot laser without spatial hole burning, the obtained result Eq. (14) can be explicitly solved, resulting in a simple expression for the laser linewidth in terms of its output power [Eq. (24)]. In the derivation we assumed that the polarization decay rate γ was much larger than the relevant population decay rates. No assumptions were made regarding the magnitude of γ as compared to the cold-cavity-loss rate Γ_0, making the result also valid for a bad-cavity laser ($\Gamma_0 \geq \gamma$). We found several corrections to the Schawlow-Townes expression for the laser linewidth. We predict a power-independent contribution $\Delta\nu_0$, which depends critically on the pumping and decay rates and is absent for an ideal four-level laser (see Appendix A). In Appendix B this result has been compared with other predictions found in the literature.

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APPENDIX A: CALCULATION OF STEADY-STATE POPULATION

In this appendix we will calculate how the steady-state populations N_2 and N_1 change with intensity. We consider the most general situation, as depicted in Fig. 2, for which the population rate equations are

$$\frac{dN_2}{dt} = \Lambda_2 - (A + \gamma_2) N_2 - \frac{\sigma I}{h\nu}(N_2 - N_1), \quad \text{(A1a)}$$

$$\frac{dN_1}{dt} = \Lambda_1 + A N_2 - \gamma_1 N_1 + \frac{\sigma I}{h\nu}(N_2 - N_1), \quad \text{(A1b)}$$

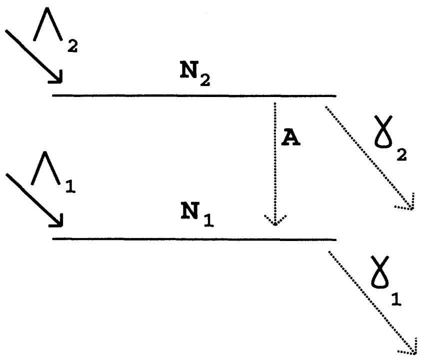

FIG. 2. Level scheme with pump rates and decay rates.

where Λ_2 and Λ_1 are the pump rates, $(A + \gamma_2)$ and γ_1 are the spontaneous-emission decay rates, and $\sigma \equiv dg/dN = \lambda_0^2/(4\pi)AT_2$ is the cross section for absorption and stimulated emission between the laser levels [29]. The spontaneous-emission decay from the upper laser level has been separated in $A\,N_2$ and $\gamma_2\,N_2$, where the product $A\,N_2$ acts as an additional (population-dependent) pump rate for the lower laser level.

The above rate equations are identical to those used by Kolobov *et al.* [13]. They differ from those used by Sargent, Scully, and Lamb [30], and Prasad [8], who omit the term AN_2 in Eq.(A1b), by assuming that $\gamma_1 \gg A$ and $\gamma_2 \gg A$, thus effectively setting $A = 0$. They are also more general than the two-level Haken-Lamb model and the three-level Lax-Louisell model, which have recently been compared by Levien, Collett, and Walls [31]. Whereas these treatments aim at a full quantum-mechanical description of the laser, we have the simpler job of calculating the steady-state populations.

From the steady-state solution of Eqs. (A1) we find

$$N_2 - N_1 = \frac{(\,N_2^{(0)} - N_1^{(0)}\,)}{1 + I/I_{\text{sat}}}\,, \tag{A2a}$$

$$N_2 = N_2^{(0)}\,\frac{1 + k_2\,I/I_{\text{sat}}}{1 + I/I_{\text{sat}}}\,, \tag{A2b}$$

$$N_1 = N_1^{(0)}\,\frac{1 + k_1\,I/I_{\text{sat}}}{1 + I/I_{\text{sat}}}\,, \tag{A2c}$$

$$N_{\text{sp}} = N_{\text{sp}}^{(0)}\,\{\,1 + k_2\,I/I_{\text{sat}}\,\}\,, \tag{A2d}$$

$$I_{\text{sat}} = \frac{h\nu}{\sigma}\,\frac{\gamma_1\,(\,A + \gamma_2)}{\gamma_1 + \gamma_2}\,, \tag{A2e}$$

$$N_2^{(0)} = \frac{\Lambda_2}{A + \gamma_2}\,, \tag{A2f}$$

$$N_1^{(0)} = \frac{\Lambda_1}{\gamma_1} + \frac{A}{A+\gamma_2}\,\frac{\Lambda_2}{\gamma_1}\,, \tag{A2g}$$

$$k_2 = \frac{\Lambda_1 + \Lambda_2}{\Lambda_2}\,\frac{A + \gamma_2}{\gamma_1 + \gamma_2}\,, \tag{A2h}$$

$$k_1\,N_1^{(0)} = k_2\,N_2^{(0)}\,, \tag{A2i}$$

where we have introduced the unsaturated populations $N_2^{(0)}$ and $N_1^{(0)}$ and the saturation intensity I_{sat}. The parameters k_1 and k_2 are equal to those used in Ref. [22]. Inversion can only be reached when $0 \leq k_2 \leq 1$. The unsaturated spontaneous-emission factor $N_{\text{sp}}^{(0)}$ can be calculated from Eqs. (A2f) and (A2g)). The (spatially averaged) ratio $(\,N_2^{(0)} - N_1^{(0)}\,)/(N_2 - N_1) = 1 + I/I_{\text{sat}}$ is the pump parameter of the laser. It quantifies how far the laser is operated above threshold.

Equation (A2a) shows how an increase of the (local) intensity leads to a reduction of the (local) population inversion. It demonstrates the well-known saturation behavior of a homogeneously broadened line, which is solely characterized by the ratio I/I_{sat}. For an ideal four-level system, where $(\gamma_1 \gg A, \gamma_2)$ this is all we need to know: the ground state remains empty, $N_1 \approx 0$ and $k_2 \approx 0$. For other systems the description is more complicated, as $N_1 \neq 0$ and saturation results both in a reduction of N_2 as compared to $N_2^{(0)}$ as well as in an increase of N_1 as compared to $N_1^{(0)}$ $(k_1 \geq 1)$. In a typical experiment a laser with a fixed cavity is studied and the laser power is varied via the unsaturated gain g_0 [see Eq. (18)]. The spatially integrated population inversion is then fixed by the gain needed to compensate for the mirror losses, and both N_2 and N_1 increase with laser power. This increase, which leads to the described linear power dependence of $N_{\text{sp}}^{(0)}$ and to a nonzero $\Delta\nu_0$, is quantified by k_2. In general $0 \leq k_2 \leq 1$ quantifies how much the laser's saturation behavior differs from that of an ideal four-level laser, for which $k_2 = 0$.

To get a feeling for the magnitude of k_2 we will finally consider a nonideal laser, where only the upper laser level is pumped $(\Lambda_1 = 0)$. In the limit considered by Sargent, Scully, and Lamb $(A \approx 0)$ one then finds $k_2 = \gamma_2/(\gamma_1 + \gamma_2)$, which is small when $\gamma_1 \gg \gamma_2$, i.e., when the laser resembles an ideal four-level laser, but which can be close to 1 when $\gamma_1 \ll \gamma_2$. In the opposite limit, where the upper laser level decays dominantly towards the lower laser level $(\gamma_2 \ll A)$, one finds $k_2 \approx A/\gamma_1$, which is small when $\gamma_1 \gg A$, but which can be close to 1 when $\gamma_1 \approx A$ $(\gamma_1 < A$ will not produce inversion). Once more we find that k_2 quantifies how much the laser's saturation behavior differs from that of an ideal four-level laser.

APPENDIX B: POWER-INDEPENDENT LINEWIDTH: OTHER THEORIES

In the literature one finds several predictions for the existence of a power-independent contribution $\Delta\nu_0$. We will argue here that these other predictions are also, often implicitly, based on the increase of N_1 and N_{sp} with laser power as originally discussed by Yariv and Vahala [32]. This is true for Ujihara [5], who finds an expression for the laser linewidth [Eq. (31) in Ref. [5]] which exhibits a power-independent contribution only when the position-averaged N_{sp} increases linearly with laser power; we interpret Ujihara's factor $\frac{1}{2}\bar{N}/(N\bar{\sigma}_{th}) + \frac{1}{2}$ as a position-averaged N_{sp}. Goldberg, Milonni, and Sundaram [7] find a similar result. They split the laser linewidth in an external part, which is power independent, and an internal part, which contains the spatially averaged populations. For the Fabry-Pérot laser that we have considered, the latter term will give rise to a power-independent contribution only when the lower-level population increases upon saturation, i.e., when $(P_2 - P_1)_t$ in Eq. (45) of Ref. [7] decreases upon saturation.

Levien, Collett, and Walls [31] have compared the two-level laser model of Haken and Lamb with the three-level model of Lax and Louisell. For the Lax-Louisell model, they find $\Delta\nu_0 = 0$, in agreement with the simplification $N_1 = 0$. For the Haken-Lax model they do find a power-independent contribution $\Delta\nu_0$, which is equal to our result if we set $k_2\, N_{sp}^{(0)} = \frac{1}{2}$. This specific value also occurs in other work [6,8]. In Ref. [6] Lu uses density operators to find that the linewidth of a good-cavity laser is proportional to the sum of the unsaturated gain and the cavity-loss rate, which is equal to saturated gain. This makes $\Delta\nu_0$ equal to the Schawlow-Townes linewidth at an intracavity power of $2P_c = 2P_{sat}$. It equals our result when we set $k_2 = \frac{1}{2}$ and $N_{sp}^{(0)} = 1$. Similarly, Eq. (5.20) of Prasad [8] is equal to our Eqs. (19a) and (19c) only if we set $k_2 N_{sp}^{(0)} = \frac{1}{2}$. This specific value of $k_2\, N_{sp}^{(0)} = \frac{1}{2}$ seems to be related to the chosen laser model. For $T_2^{-1} \gg (\gamma_1 + A), \gamma_2$ we predict a value between $0 \leq k_2 \leq 1$ that depends crucially on the exact pumping and decay rates (see Appendix A).

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