Mode dynamics in optical cavities

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(Received 15 March 1994; revised manuscript received 5 October 1994)

We consider the dynamical behavior of light in a cavity that contains birefringent elements with time-dependent settings. The evolution of the two-dimensional polarization vector can be described by an equation of the Schrödinger type if the intracavity elements change little on the time scale of the round-trip time. A coupled-mode analysis is given, which models the multimode dynamics of the optical system. We show that a description in terms of isolated two-level systems is justified if the optical band gap is small.

PACS number(s): 32.80.Bx, 41.20.Bt, 03.50.De

I. INTRODUCTION

In a companion paper [1], experimental evidence is provided which confirms the applicability of Landau-Zener [2] and hidden-crossing [3] models to transition rates between modes in a classical optical system. The broader context of the work is that resonators containing optical elements with time-dependent settings provide useful models for quantum systems in demonstrating phenomena closely paralleling Rabi oscillations, Autler-Townes doublet splitting, and other phenomena [4]. The background to the classical optical two-level model has been developed in previous publications; details of the way this classical model is set up and realized experimentally can be found in Refs. [1,4].

An important restriction in the previous work on the model of the cavity has been that the optical elements, which, through their effect on the polarization vector, mimic the interactions in two-level atoms, were limited to the regime of small relative phase shifts. The companion paper [1] demonstrates experimentally that this is unnecessarily restrictive for the application of the classical optical two-level model. It is the purpose of this paper to show theoretically that this restriction is not fundamental to a Schrödinger description of the optical cavity. If the settings of the optical elements change only little on the time scale of the cavity round-trip time, the complete multimode dynamics (polarization and longitudinal modes) is governed by such an equation. The regime in which the system decouples into two-level systems is explored, and we will show that if a two-mode approximation is justified, the Hamiltonian of the system can be expressed in a time-dependent 2×2 Hermitian matrix.

This paper is organized in the following way. In Sec. II a general theoretical framework is presented. In Sec. III the two-mode approximation is discussed. Conclusions are drawn in Sec. IV.

II. DERIVATION OF THE EQUATIONS OF MOTION

In this section we derive a Schrödinger equation for the field evolution in an optical cavity, which contains optical elements with time-dependent settings. In the description we shall ignore losses in the optical elements. The transition from the Maxwell wave equation for the electric field $\vec{E}(z,t)$ in an optical cavity,

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \frac{\partial^2 \vec{E}}{\partial z^2}, \tag{1}$$

to an equation of the Schrödinger type,

$$\frac{\partial \vec{E}}{\partial t} = -i \frac{H}{T} \vec{E}, \tag{2}$$

where H is a Hermitian operator, requires a number of steps (c is the velocity of light and $T = L/c$ is the round-trip time in the cavity; L is the length of the cavity). Since Eq. (1) is second order in time, it allows two counterpropagating waves. For equal amplitudes and polarizations, a superposition of these gives a standing wave, where the field amplitude is zero at the nodes. When looking at time-dependent phenomena in the cavity, these nodes can shift in position, since the resonance frequencies can change. Such a field evolution cannot be described by Eq. (2), since for Hermitian H Eq. (2) implies that $\vec{E}(z,t)$ remains normalized at any fixed position. Obviously this is not demanded by Eq. (1). If an optical system can be described within the slowly varying envelope approximation (SVEA), then the second-order wave equation Eq. (1) is approximated by a first-order equation. In that case a description in terms of Eq. (2) could apply. The reduction to a first-order equation in time implies that we consider light traveling in one direction, which applies to ring cavities. In a linear cavity *two* counterpropagating light waves are necessary to describe the complete field at a certain position in the cavity. This problem for a linear cavity can be overcome by considering only one of the two propagation directions. In an experiment this would correspond to the field which is leaking through one of the mirrors, and which provides the detection signal.

In the SVEA it is assumed that the electric field in the cavity can be written as

$$\vec{E}(z,t) = \vec{\mathcal{E}}(z,t) e^{i(kz-\omega t)}, \tag{3}$$

where $\vec{\mathcal{E}}(z,t)$ is a slowly varying envelope, $k = 2n\pi/L$ with n a (large) integer, and $\omega = ck$. So the field propagates in one direction, along the z axis, and variations of $\vec{\mathcal{E}}$ on the time scale of the optical frequency ω are negligible. The wave equation which governs the evolution of $\vec{\mathcal{E}}(z,t)$ in vacuum is then given by

$$c\frac{\partial \vec{\mathcal{E}}}{\partial z} + \frac{\partial \vec{\mathcal{E}}}{\partial t} = \vec{0}, \tag{4}$$

which is of the form of Eq. (2).

Consider now a cavity containing N time-dependent optical elements (see Fig. 1). We assume that the optical elements can be described within the Jones matrix formalism [5], and that the optical elements are infinitesimally thin [6]. Therefore the fields at either side of a boundary between two sections are connected by an unitary phase jump. For an electric field described by Eq. (3), hence traveling in the positive z direction, $\vec{\mathcal{E}}$ at the input and output side of an optical element are related by

$$\vec{\mathcal{E}}(z_i^+,t) = S_i(t)\vec{\mathcal{E}}(z_i^-,t), \tag{5}$$

where the time-dependent 2×2 Jones matrix $S_i(t)$ is unitary. The x and y axis form the basis of the S_i, and we choose $\det(S_i)$=1. This restriction can be made without loss of generality. The position of the element i along the optical path is measured by z_i, and the $-$ and $+$ denote the input and output sides of an optical element, respectively (see Fig. 1). The cavity structure imposes periodic boundary conditions,

$$\vec{\mathcal{E}}(L,t) = \vec{\mathcal{E}}(0,t) \tag{6a}$$

and

$$\left.\frac{\partial \vec{\mathcal{E}}}{\partial z}\right|_L = \left.\frac{\partial \vec{\mathcal{E}}}{\partial z}\right|_0 . \tag{6b}$$

To solve the differential equation for $\vec{\mathcal{E}}$ under the conditions (5) and (6) it is convenient to transform the discontinuous $\vec{\mathcal{E}}$ field to a new continuous $\vec{\mathcal{G}}$ field. To derive the equations of motion for the $\vec{\mathcal{G}}$ field we will make use of an intermediate field $\vec{\mathcal{F}}$, which is a continuous solution of the wave equation (4). The transformation from the $\vec{\mathcal{E}}$ field to the $\vec{\mathcal{F}}$ field corresponds to the physical picture that the optical elements are displaced to the end of the interval $[0, L]$, while accounting for the time retardation.

We define $\vec{\mathcal{F}}$ by

$$\begin{aligned}
\vec{\mathcal{F}}(z,t) &= \vec{\mathcal{E}}(z,t), \quad 0 < z < z_1 , \\
\vec{\mathcal{F}}(z,t) &= S_1^{-1}\left(t - \frac{z-z_1}{c}\right)\vec{\mathcal{E}}(z,t), \quad z_1 < z < z_2 , \\
&\vdots \\
\vec{\mathcal{F}}(z,t) &= S_1^{-1}\left(t - \frac{z-z_1}{c}\right) S_2^{-1}\left(t - \frac{z-z_2}{c}\right)\cdots S_N^{-1}\left(t - \frac{z-z_N}{c}\right)\vec{\mathcal{E}}(z,t), \quad z_N < z < L .
\end{aligned} \tag{7}$$

It is easy to check that the phase jumps in $\vec{\mathcal{E}}$ have been compensated by the inverse Jones matrices S_i^{-1}, and that $\vec{\mathcal{F}}$ obeys the wave equation

$$c\frac{\partial \vec{\mathcal{F}}}{\partial z} + \frac{\partial \vec{\mathcal{F}}}{\partial t} = \vec{0}. \tag{8}$$

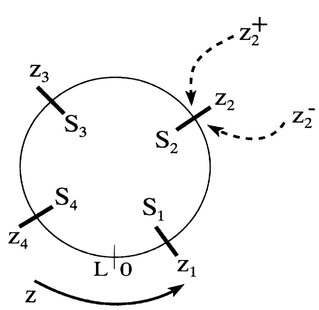

FIG. 1. The configuration of an optical cavity with time-dependent optical elements $S_i(t)$ at positions z_i; $z = 0$ and $z = L$ indicate the origin of the z coordinate.

The effect of the S_i is now expressed in the boundary conditions for $\vec{\mathcal{F}}$,

$$\vec{\mathcal{F}}(L,t) = M^{-1}(t)\vec{\mathcal{F}}(0,t), \tag{9a}$$

and

$$\left.\frac{\partial \vec{\mathcal{F}}}{\partial z}\right|_L = M^{-1}(t)\left.\frac{\partial \vec{\mathcal{F}}}{\partial z}\right|_0 - \frac{1}{c}\frac{dM^{-1}(t)}{dt}\vec{\mathcal{F}}(0,t), \tag{9b}$$

where Eqs. (6) and (7) have been used. The time-dependent round-trip matrix $M(t)$ is defined by

$$\begin{aligned}
M(t) = S_N\left(t - \frac{L-z_N}{c}\right)\cdots S_2\left(t - \frac{L-z_2}{c}\right) \\
\times S_1\left(t - \frac{L-z_1}{c}\right) .
\end{aligned} \tag{10}$$

The advantage of using the intermediate $\vec{\mathcal{F}}$ field over the $\vec{\mathcal{E}}$ field is that the terms which result from retardation can readily be identified: they disappear in the limit $c \to \infty$. In Eq. (10) retardation is merely a time displacement, which can be easily compensated for. However, the retardation term in the boundary condition for the first derivative, Eq. (9b), cannot be easily compensated for. Retardation effects become important if the elements S_i change on the time scale of the round-trip time T. These effects highly complicate an analytical treatment of the time evolution of the optical field. Therefore we restrict ourselves within the context of the present paper to the case that the modulations of the S_i are all slow on the time scale of T. This implies that the wavelength associated with the time modulation of the optical elements is much larger than the optical path length L. Hence no information as to the position of the optical elements can be obtained from the field leaking out of the cavity.

We take now the limit $c \to \infty$ in Eqs. (9) and (10) and apply the second transformation to remove the remaining discontinuities at the boundaries [7]. This corresponds to smearing out the effective optical element, which was located at the end of the interval $[0, L]$, over the whole cavity. The new field $\vec{\mathcal{G}}$ is related to the field $\vec{\mathcal{F}}$ according to

$$\vec{\mathcal{G}}(z,t) = e^{-i\frac{z}{L}H(t)}\vec{\mathcal{F}}(z,t), \tag{11}$$

where the Hermitian matrix $H(t)$ is defined by

$$e^{-iH(t)} = M(t). \tag{12}$$

Notice that Eq. (12) defines H only mod 2π.

The boundary conditions for $\vec{\mathcal{G}}$ are found from Eqs. (9) and (11),

$$\vec{\mathcal{G}}(L,t) = \vec{\mathcal{G}}(0,t), \tag{13a}$$

and

$$\left.\frac{\partial \vec{\mathcal{G}}}{\partial z}\right|_L = \left.\frac{\partial \vec{\mathcal{G}}}{\partial z}\right|_0 . \tag{13b}$$

So, like the $\vec{\mathcal{E}}$ field, the $\vec{\mathcal{G}}$ field is periodic in z. The differential equation for $\vec{\mathcal{G}}$ can be found by substituting Eq. (11) into Eq. (8),

$$\frac{\partial \vec{\mathcal{G}}}{\partial t} = -i\frac{H_{\text{eff}}}{T}\vec{\mathcal{G}}, \tag{14}$$

where the effective Hamiltonian H_{eff} is given by

$$H_{\text{eff}} = H + \frac{L}{i}\frac{\partial}{\partial z} + \frac{T}{i}e^{-i\frac{z}{L}H}\left\{\frac{\partial}{\partial t}e^{i\frac{z}{L}H}\right\}. \tag{15}$$

For a given H, Eqs. (13)–(15) give a complete description of the optical field in the cavity within the limit $TdH/dt \ll 1$. From Eqs. (6), (7), and (11) it follows that $\vec{\mathcal{G}}(0,t) = \vec{\mathcal{F}}(0,t) = \vec{\mathcal{E}}(0,t)$. Therefore, for $z = 0$ these equations also give the evolution of $\vec{\mathcal{E}}$.

To analyze the evolution of $\vec{\mathcal{G}}$ we give an expansion into longitudinal modes. We write $\vec{\mathcal{G}}(z,t)$ as

$$\vec{\mathcal{G}}(z,t) = \sum_m \vec{g}_m(t)e^{i\kappa_m(z-ct)}, \tag{16}$$

with $\kappa_m = 2m\pi/L$, and m is an integer. Next, the $\vec{g}_m$ are rotated to the eigenvector basis of H,

$$\vec{a}_m = D\vec{g}_m, \tag{17}$$

where D is defined through

$$H = D^\dagger \varphi\sigma_z D, \tag{18}$$

with $\pm\varphi$ are the eigenvalues of H, the dagger denotes Hermitian conjugated, and the Pauli matrix σ_z is given by

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{19}$$

Note that by defining the mode amplitudes $\vec{a}_m$ in this way, $\vec{\mathcal{G}}$ is written at the time-dependent eigenvector basis of H. The dynamical variables are given by the vector components $a_{x,m}$ and $a_{y,m}$,

$$\vec{a}_m = a_{x,m}\hat{\vec{x}} + a_{y,m}\hat{\vec{y}} . \tag{20}$$

Substituting Eqs. (16)–(18) into Eq. (14) yields the evolution equation for the $\vec{a}_m$,

$$\begin{aligned}\frac{d\vec{a}_m}{dt} = -\Bigg[& i\frac{\varphi}{T}\sigma_z\vec{a}_m + \frac{1}{2}i\frac{d\varphi}{dt}\sigma_z\vec{a}_m \\ & + \frac{d\varphi}{dt}\sigma_z\sum_{n\neq m}\frac{\vec{a}_n e^{-ic(\kappa_n-\kappa_m)t}}{2\pi(n-m)} \\ & + \sum_n A_{nm}\vec{a}_n e^{-ic(\kappa_n-\kappa_m)t}\Bigg],\end{aligned} \tag{21}$$

with

$$A_{nm} = \frac{1}{L}\int_0^L dz\, e^{-i\frac{z}{L}\varphi\sigma_z}D\frac{dD^\dagger}{dt}e^{i\frac{z}{L}\varphi\sigma_z}e^{i(\kappa_n-\kappa_m)z}. \tag{22}$$

The first term on the right-hand side (rhs) of Eq. (21) would also be present if H would be time independent; in this case, the time derivatives on the rhs of Eq. (21) are zero. The second term is a correction to the first term due to temporal changes in φ. The third term describes an interaction between two modes with the same polarization. These latter two terms arise from the fact that H has been diagonalized, not H_{eff}. The fourth term describes interactions between modes of different polarization.

The second and third term on the rhs of Eq. (21) can be neglected since we assumed that $TdH/dt \ll 1$. Moreover, the third term displays a rapid oscillation at a frequency $2\pi(n-m)/T$ with $n \neq m$, which vanishes after integration of Eq. (21). The fourth term displays a similar oscillation. However, in this term orthogonal polarizations modes are coupled, where the difference frequency can be close to integer multiples of $2\pi/T$. The oscillation frequency of the polarization modes is determined by the

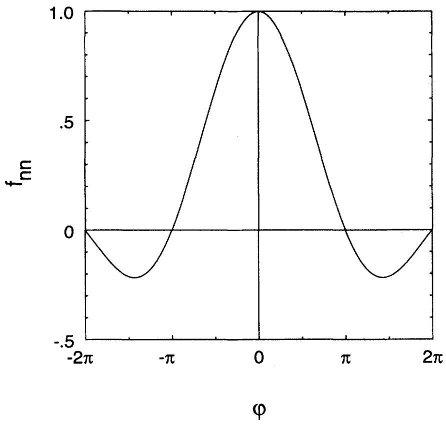

FIG. 2. The function f_{nn} versus φ; $f_{nn}(\varphi)$ describes the resonant transitions between the polarization modes belonging to one longitudinal mode.

first term, where φ can have any value. So the equations of motion for the $\vec{a}_m$ reduce to

$$\frac{d\vec{a}_m}{dt} = -\left[i\frac{\varphi}{T}\sigma_z\vec{a}_m + \sum_n A_{nm}e^{-ic(\kappa_n-\kappa_m)t}\vec{a}_n\right], \quad (23)$$

which is a Schrödinger equation for the optical multilevel system. The integral in the expression for the anti-Hermitian coupling matrix A_{nm} can be directly evaluated, and yields

$$A_{nm} = -i\begin{pmatrix} \Lambda\delta_{nm} & \Gamma^* e^{-i\varphi} f_{nm}(-\varphi) \\ \Gamma e^{i\varphi} f_{nm}(\varphi) & -\Lambda\delta_{nm} \end{pmatrix}, \quad (24)$$

where δ_{nm} is the Kronecker delta, Λ and Γ are functions which depend on the explicit form of H, and

$$f_{nm}(\varphi) = \frac{\sin\varphi}{\varphi + \pi(n-m)}. \quad (25)$$

After a simple base transformation, the phase factors $\exp(\pm i\varphi)$ in the off-diagonal of A_{nm} can be shown to have the same influence on the dynamics as the second term on the rhs of Eq. (21). Therefore we may neglect them. The function f_{nm} describes resonant coupling between mode n and mode m if $\varphi \simeq -\pi(n-m)$. In Fig. 2 $f_{nn}(\varphi)$ is plotted. For φ close to 0 the two polarization modes of $n = m$ are resonantly coupled. For φ close to π, the function $f_{nm}(\pm\varphi)$ describes resonant coupling to the modes with $n = m \mp 1$.

III. TWO-MODE APPROXIMATION

In the discussion so far we only neglected terms which influence the dynamics if H changes at the time scale of T. We will now consider a special limit of Eq. (23), in which the longitudinal modes decouple. In order to neglect the influence of $f_{nm}(\varphi)$ on the field evolution, the function Γ should only effect the dynamics when $\varphi \ll 1$. In that case $f_{nn} \simeq 1$ [8]. To illustrate that this is indeed possible, we give a general form of H

$$H = \begin{pmatrix} \lambda & e^{i\gamma}\Delta \\ e^{-i\gamma}\Delta & -\lambda \end{pmatrix}, \quad (26)$$

where λ, Δ, and γ are functions of time. The function Γ then takes the form

$$\Gamma = \frac{1}{2}e^{-i\gamma}\left(i\frac{\dot{\Delta}\lambda - \dot{\lambda}\Delta}{\Delta^2+\lambda^2} + \frac{\dot{\gamma}\Delta}{\sqrt{\Delta^2+\lambda^2}}\right). \quad (27)$$

The resonant behavior of the denominators in Eq. (27) shows that Γ affects the dynamics if $\lambda \lesssim |\Delta|$. Hence if $\Delta \ll 1$ then Γ influences the dynamics where $\varphi \ll 1$, since $\varphi = \sqrt{\Delta^2+\lambda^2}$. Therefore, in the limit that $\Delta \ll 1$ (small coupling) and $-\pi/2 < \varphi < \pi/2$, we may set $f_{nm} = \delta_{nm}$, in which case the longitudinal modes decouple. At the basis of the $\vec{g}_m$ the equations of motion read

$$\frac{d\vec{g}_m(t)}{dt} = -i\frac{H(t)}{T}\vec{g}_m(t). \quad (28)$$

Hence, if the coupling is small, and if $-\pi/2 < \varphi < \pi/2$ then Eq. (28) describes to a good approximation the field evolution in the cavity.

IV. CONCLUSIONS

A description in terms of a Schrödinger equation has been given for the electric field in an optical cavity, which contains time-dependent optical elements. In an unidirectional ring cavity the complete electric field is modeled by this equation. In a linear cavity, though, only one of the two propagation directions is modeled. This difference is inherent to a description in terms of a Schrödinger equation, since it is a first-order differential equation in t, whereas the Maxwell wave equation is second order in t.

Previous models were limited to a regime where all the phase shifts of the elements were small, and slowly varying (compared to round-trip time T). We have extended this description and have shown that a description in terms of a Schrödinger equation is not restricted to this particular case. The complete multimode evolution is also governed by an equation of the Schrödinger type, provided only that the elements vary slowly on the time scale of T. There is no restriction as to the absolute value of the phase shifts. If, in addition, the band gap is small, then a two-mode description can be given. These results allow for a proper interpretation of experiments in which the phase shifts do not stay close to zero, as is the case, for example, in optical experiments on Landau-Zener transitions [1]. The restriction that the phase shifts change slowly compared to T is necessary, in order to neglect retardation effects. Inclusion of these effects in the description of the optical system is interesting and will be the subject of further examination.

ACKNOWLEDGMENTS

This work is part of the research program of the Foundation for Fundamental Research of Matter (FOM) and was made possible by the financial support from the Netherlands Organization for Scientific Research (NWO).

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